APPROXIMATION AND BILLIARDS

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HISTORY

1964 Berezin: Suggests the introducing approximation idea into dynamical systems to his student Stepin.

1967 Katok-Stepin: Introduced the idea of approximation in the theory of dynamical systems. Many results, including for example:
a.e. interval exchange not of rotation type on 3 intervals is weakly mixing.
Remark: very recently Avila and Forni have announced that they can prove this for any number of intervals.

1970 Anosov-Katok: Introduced the approximation by conjugation method to smooth dynamical systems. Among their results they constructed *ergodic area preserving diffeo of the disc.*

This method has been very successful with contributions by Herman, Fathi, Handel, Fayad, ...

1975 Katok-Zemlyakov introduced the method of approximation to polygonal billiards. They proved

Theorem *The billiard (flow/map) in a typical polygon is topological transitive.*

• Topological transitive = there is a dense orbit

• Typical = residual = dense G_δ

• Topology on the space of polygons:
$X = \cup X_n$, where X_n is the space of n-gons.
X_n is identified with a subset of $\mathbf{R^{2n}}$ by listing the corners of each polygon in a cyclic order.

• More precisely, a property hold typically if for each n, and for each compact subset K of X_n the property holds for K.

THE SETUP

Consider a polygon $P \subset \mathbf{R^2}$. A billiard ball, i.e. a point mass, moves inside Q with unit speed along a straight line until it reaches the boundary ∂Q, then instantaneously changes direction according to the mirror law: "the angle of incidence is equal to the angle of reflection," and continues along the new line.

THE PHASE SPACE

The *phase space* of the billiard flow is $P \times \mathbf{S^1}$ with appropriate identification on the boundary.

The *phase space* of the billiard map is $\delta P \times \{$ inward pointing vectors $\}$.

UNFOLDINGS

Instead of reflecting the trajectory, we can reflect the polygon continuing the trajectory in a straight line.

THE SQUARE TO THE TORUS

Consider the billiard flow in a given direction in the square. Via unfolding we see that the billiard is equivalent to the linear flow on the torus.

If the direction θ is rational the flow is purely periodic while if the direction θ is irrational the flow is minimal and uniquely ergodic.

$(\frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{2})$-TRIANGLE TO THE OCTAGON

Via unfolding we see that the billiard flow in a given direction in the $(\frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{2})$-triangle is equivalent to the linear flow on the flat octagon. Note: there is a conic singularity with total angle 6π.

With the proper interpretation of “rational” we have the same dichotomy of purely periodic vs minimal and uniquely ergodic.

A *saddle connection* is an orbit segment which starts and ends in a corner of the polygon.

A *“rational” direction* is then a direction of a saddle connection. Note that for the standard torus $\mathbf{R}^2/\mathbf{Z}^2$ we have “rational” $= \mathbf{Q}$.

RATIONAL POLYGONS

A polygon P is called *rational* if all the angles α_i between sides are rational multiples of π, i.e. $\alpha_i = p_i\pi/q_i$.

The construction of the last two slides generalizes to the rational case. Let $O(2)$ be the group of isometries of the unit circle $\mathbf{S^1}$. Let $G \subset O(2)$ be the group generated by the linear reflections in the sides of P. For the square $|G| = 4$, for the $(\frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{2})$-triangle $|G| = 16$. In general $|G| = 2q$ where q is the least common multiple of q_i.

The set $M := P \times G$, after identifications of parallel copies sides of P has the structure of a *flat surface with conic singularities*.

Given a rational n-gon with angles $\pi p_i/q_i$, with p_i, q_i cop-rime, the *genus* of M is

$$1 + \frac{q}{2}\left(n - 2 - \sum \frac{1}{q_i}\right)$$

The billiard map restricted to a direction is an *interval exchange transformation*, i.e. we cut the interval into a finite number of pieces, the map translates each piece in such a fashion that the dynamics is invertible.

A direction θ is call *irregular* if the direction of a side of P is in $G\theta$. If θ is regular then The billiard flow in the direction θ is conjugate to the linear flow in direction θ on M. (Otherwise the linear flow is a factor.)

Proposition *For all but countable many directions θ the billiard flow/map is minimal on the surface M.*

Theorem (Masur) *For a dense set of θ the billiard flow/map has a periodic orbit.*

STRUCTURE OF G

The subgroup $G_0 := G \cap SO(2)$ has index two in G and is generated by rotations by $2\alpha_k = 2p_i\pi/q_i$. The orbit of a point $\theta \neq k\pi/q$ is:

$$G\theta = \left\{\theta + \frac{i}{q}, -\theta + \frac{i}{q} : i = 0, \ldots, p-1\right\}$$

Theorem (Katok-Zemlyakov 1975) *The billiard*
(flow/map) in a typical polygon is topological transitive.

Proof:

1) Identify the phase space of the billiard flow in each n-gon with $\mathbf{D^2} \times \mathbf{S^1}$ and assume that this identification depends continuously on the polygon. Let B_i be a countable basis for the topology of $\mathbf{D^2} \times \mathbf{S^1}$.

2) Let $X \subset \mathbf{R^{2n}}$ be a compact subset of n-gons. Let $X_k \subset X$ be the set of n-gons P such that for each open set U in the phase space, there exists a billiard trajectory starting in U that visits all (the images of) the sets $B_1, \ldots, B_k$ in the phase space of the billiard flow in P. Each X_k is open, and their intersection is a G_δ set.

To see denseness let Y_q be the set of rational n-gons with angles $\pi p_i/q_i$, with p_i, q_i cop-rime the l.c.m. of the q_i *at least* q.

3) For every $P \in Y_q$ each invariant surface M_θ is $1/q$ dense in the phase space. Therefore for every k there exists q such that for every $P \in Y_q$ the surface M_θ (for any/all θ) intersects all the (images of) the sets $B_1, \ldots, B_k$ in the phase space of the billiard flow in P. Since M_θ has a dense trajectory for all but countably many θ the we have $Y_q \subset X_k$. Thus since Y_q is dense in the space of n-gons, the set X_k is as well. It follows that $\cap X_k$ is dense.

4) Let P be a polygon in $\cap X_k$. Suppose we have inductively chosen a nonempty compact neighborhood $U_{k-1} \subset U$. Since $P \in \cap X_k$ we can find a billiard trajectory starting in U_{k-1} that visits each $B_1, \ldots, B_k$. By continuity there is an compact neighborhood $U_k \subset U_{k-1}$ such that each trajectory in U_k visits $B_1, \ldots, B_k$ within a bounded time T_k. Thus any $x \in \cap U_k$ has a dense trajectory, possibly singular.

5) By varying the order of the B_i we can produce an uncountable collection of such dense trajectories. Since there are only countable many saddle connections, i.e. orbits segments starting and ending at a vertex, most of the dense trajectories are not saddle connections. If such a trajectory is singular, i.e. the nth iterate arrives at a vertex, then the orbit starting at time $n+1$ is dense and nonsingular. ■

ERGODICITY

Let (X, β, μ) be a probability space and $T : X \to X$ a measure preserving transformation. T is *ergodic* iff $\mu(T^{-1}A \triangle A) = 0$ implies $\mu(A) = 0$ or $\mu(A) = 1$.

T is ergodic iff

$$\forall A, B \in \beta : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B)$$

iff for each $f \in L^1(X, \beta, \mu)$ the time mean equals the space mean of f a.e., i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_X f \, d\mu \quad \text{a.e.}$$

Theorem (Kerckhoff, Masur, Smillie 1986): *For any rational polygon, for a.e. direction* θ *the billiard flow* F_θ *is uniquely ergodic on* M.

Theorem (KMS, Katok, Pospiech & Stepin) *The billiard (flow/map) in the typical polygon is ergodic.*

Proof: Similar to the proof of topological transitivity: for a rational polygon the orbit on a good invariant surface is well distributed and the invariant manifolds are also approximately well distributed.

Stronger version: *Let* X *be a close subset of the space of* n*-gons with the property that for any* q *the space of rational* $P \in X$ *with* $|G(P)| \geq q$ *is dense. Then ergodic tables are typical in* X.

Example: Right triangles = the system of two elastic point masses on the interval.

CONSTRUCTIVE ERGODICITY

Definition: Let $\phi : \mathbf{N} \to \mathbf{R}^+$ such that $\lim_{q\to\infty} \phi(q) = 0$. Let P be a n-gon with angles $\alpha_1, \ldots, \alpha_n$. We say that P admits approximation by rational polygons at rate $\phi(q)$ if for every $q_0 > 0$ there is $q > q_0$ and positive integers $p_1, \ldots, p_n$ each cop-rime with q such that $|\alpha_i - \pi p_i/q| < \phi(q)$ for all i.

Theorem: (Vorobets 1997) *Let P be a planar polygon that admits approximation by rational polygons at the rate*

$$\phi(q) = \left(2^{2^{2^{2^q}}}\right)^{-1}$$

Then the billiard flow in P is ergodic.

TOTAL ERGODICITY

Theorem: (T. 2004) *The billiard map in the typical polygon is totally ergodic, i.e.* T^n *is ergodic for all* $n \in \mathbf{N}$.

This was used to verify the assumptions of the following theorem

Theorem: (T. 2004) If the angles of P depend on only one irrational parameter and T^2 is ergodic, then the billiard has periodic orbits.

Both the constructive ergodicity and the total ergodicity theorems are proven by using an alternative version of the proof of the result of KMS.

For any integer $n \geq 1$, denote by $V_P(n)$ the set of saddle connections of length n, that is the set of trajectories which connect a pair of vertices of P, which avoid the vertices of P at the intermediate points and which involve $(n-1)$ reflections. Let $W_P(n) = \bigcup_{j=1}^{n} V_P(j)$.

Boshernitzan has shown that if P is a rational polygon satisfying

$$card(W_P(n)) = O(n^2)$$

then the billiard flow/map are uniquely ergodic for Lebesgue a.e. direction.

Masur verified equation this assumption for any rational polygon P.

Vorobets has made a constructive version of Masur's theorem.

INFINITE POLYGONS

T. 1999 and then Degli Esposti, Del Magno and Lenci 2000 applied this method to infinite polygons.

Theorem: (DDL 2000) Suppose p_n is a monotonically decreasing sequence of positive numbers satisfying $\sum p_n = 1$. Let

$$P = \cup_{n \geq 0}[n, n+1] \times [0, p_n]$$

Then the typical P is ergodic for a.e. direction.

The topology on the space of such P is given by the metric

$$d(P,Q) := \sum |p_n - q_n| = Area(P \triangle Q)$$

To prove the result we approximate a polygon P by finite polygons from the class, i.e. $q_n \equiv 0$, for $n \geq N$.

CONVEX BILLIARDS

Let C be a strictly convex billiard table. A compact convex set K is a *caustic* of C if the boundary of C is obtained by wrapping a string around K, pulling it tight at a point and moving the point around K while keeping the string tight.

Interpretation: if a billiard orbit is tangent to a caustic once, then it is tangent to it inbetween every pair of bounces.

Theorem: (Lazutkin 1979) If C is a sufficiently smooth strictly convex billiard table then the table contains "many" caustics.

Many = the union of the caustics has positive area.

Corollary: The billiard in a sufficiently smooth strictly convex table is not ergodic, not even topologically transitive.

The proof is based on KAM theory.

Lazutkin's original proof need the table to be C^{553}.

Rüssmann's version of KAM can be used to replace 553 by 8, and finally R. Douady's version of KAM shows that 7 is sufficient.

LOW SMOOTHNESS

Theorem: (Gruber 1990) The C^0-typical convex billiard table has the following properties:

- it contains no caustics,
- it is strictly convex,
- it is of class C^1,
- it is topologically transitive.

To prove the topological transitivity we approximate a polygon P by finite polygons with increasing number of sides.

Using the approximation teachings and the results of KMS one can easily conclude that the C^1-typical C^1-convex billiard table is strictly convex and ergodic.

Infact one can show that the C^1-typical C^1-billiard table is ergodic.

WEAK MIXING

Let (X, β, μ) be a probability space and $T : X \to X$ a measure preserving transformation. T is weak mixing iff

$$\forall A, B \in \beta : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| = 0$$

iff $\forall A, B \in \beta$: there exists $J = J(A, B) \subset \mathbf{N}$ of full density (i.e. $(\#n \in J : n \leq N)/N \to 1$) such that

$$\lim_{n \in J \to\infty} \mu(T^{-k}A \cap B) = \mu(A)\mu(B)$$

$J \subset \mathbf{N}$ is called *full* if J contains arbitrarily long runs, i.e. $\forall N, \exists k$ s.t. $\{k, k+1, \ldots, k+N\} \subset J$.

Theorem: (Stepin, T.) T is weak-mixing iff $\forall A \in \beta$ there exists J full such that

$$\lim_{n \in J \to\infty} \mu(T^{-k}A \cap A) = \mu(A)^2$$

Proof:

• T ergodic. Suppose $T^{-1}(A) = A$ mod 0. By invariance we have $\mu(T^{-n}A \cap A) = \mu(A)$, but by mixing along full sequences this quantity converges to $\mu(A)^2$ for $n \in J$. Thus $\mu(A) = 0$ or $\mu(A) = 1$.

• T weak-mixing.

Suppose not, then there exists a Kronecker factor $(\hat{X}, \hat{\beta}, \hat{\mu}, \hat{T})$, i.e. a rotation $\hat{T}$ on a compact Abelien group $\hat{X}$ such that the following diagram commutes:

$$X \xrightarrow{T} X$$

$$\downarrow\pi \qquad \downarrow\pi$$

$$\hat{X} \xrightarrow{\hat{T}} \hat{X}$$

Fix $\hat{A} \in \hat{\beta}$ such that $0 < \hat{\mu}(\hat{A}) < 1$ and let $A := \pi^{-1}(\hat{A})$.

Then since $\hat{T}$ is a rotation, there is a quasi-periodic sequence n_i such that $\hat{\mu}(\hat{T}^{-n_i}\hat{A} \cap \hat{A}) \approx \hat{\mu}(\hat{A})$ and thus

$$\mu(T^{-n_i}A \cap A) = \hat{\mu}(\hat{T}^{-n_i}\hat{A} \cap \hat{A}) \approx \hat{\mu}(\hat{A}) = \mu(A)$$

which contradicts fullness. ■

Recently I learned of several other characterizations of weak-mixing which can be successfully applied to approximation arguments.

(Fayad-Saprykina) T is weak-mixing iff there exists a sequence $m_n \in \mathbf{N}$ such that for any $A, B \in \beta$ we have

$$\mu(T^{-m_n}A \cap B) \to \mu(A)\mu(B)$$

(Furstenberg?) T is weak-mixing iff for all $A, B, C \in \beta$ such that $\mu(A)\mu(B)\mu(C) > 0$ then there exists $n \in \mathbf{N}$ such that

$$\mu(A \cap T^{-n}B)\mu(A \cap T^{-n}C) > 0$$

HYPERBOLIC BILLIARDS

Consider a polygon P. Replace each vertex with a (small) circular arc such that 1) the resulting table is C^1 and 2) the focusing circles lie inside the table.

Theorem: (Bunimovich) For such tables each ergodic component is open (mod 0) and the billiard on each ergodic component is mixing, K-mixing, Bernoulli.

Theorem: (Chernov, T. 1998) For convex P, such tables are ergodic, hence mixing, K-mixing, Bernoulli.

BACK TO C^1 BILLIARDS

Theorem: (Stepin, T.) The billiard flow/map in a C^1-typical C^1 billiard table is weak-mixing.

Idea of proof: approximation by hyperbolic billiard tables.

If the table is convex we approximate by billiards from the CT-class introduced on the last slide. Fix such a table and a finite collection of sets. For all tables sufficiently close to the fixed table. we can find an arbitrarily long run of times such that these sets approximately mix along this run.

If the table is not convex then we do not have a nice class of ergodic tables like the CT-class. To prove our theorem we approximate the table by polygons to obtain generic ergodicity. Then we approximate by B-tables to get the weak-mixing. ■

CONVEX TABLES

We can slightly improve the smoothness is the table is convex, or piecewise concave-convex. Let PCC be the class of C^1 piecewise convex-concave billiard domains. Because of the piecewise monotonicity of the derivative, locally the second derivative exists almost everywhere.

Consider the Hausdorff distance ρ_H between the second derivative of the boundaries of such tables.

Let B^2 be the closure of the class of Bunimovich tables introduced above in the topology induced by the metric $\rho = \rho_H + \rho_{C^1}$.

Theorem: Weak-mixing is typical in B^2

Remark: PCC $\subset B^2 \subset C^1$

BACK TO POLYGONAL BILLIARDS

Recently Avila and Forni have announced that they can prove that a.e. interval exchange not of rotation type is weak-mixing. Furthermore, for almost every translation surface they can show that the directional flow is weak-mixing.

A priori, these results do not hold for polygonal billiards. Thus the results here are conditional.

Let $WMix_c$ be the set of rational polygons for which at least c% of the ergodic components are weak-mixing. Note that the square and the equilateral triangle are not in $WMix_c$ for any $c > 0$. The result of Avila-Forni gives hope that all but finitely many rational polygons are in $WMix_1$.

Theorem: (Stepin, T.)

1) If there exists $c > 0$ such that $WMix_c$ is dense in the set of polygons then weak-mixing is generic for polygonal billiards.

2) Either there exists m such that $WMix_{\frac{1}{m}}$ is somewhere dense in the set of polygons (and hence weak-mixing is on 2nd category for polygonal billiards) or $\cup_m WMix_{\frac{1}{m}}$ is nowhere dense.