Recurrence and periodic billiard orbits in polygons

Serge Troubetzkoy
Centre de physique théorique,
Institut de mathématiques de Luminy et
Federation de Recherches des Unites de Mathematique de Marseille,
CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France.

Abstract

We show that almost all billiard trajectories return parallel to themselves for rank 1, ergodic polygons. Applications are given to the existence of periodic trajectories.

1 Introduction.

Consider the billiard in a polygon. A direct application of the Poincaré recurrence theorem tells us that the orbit of almost every point returns close to itself. A point in the billiard phase space has a base point x in the polygon and a direction θ, thus close means that both the base point and the direction are close. In this article we consider a stronger type of recurrence. We fix a direction θ, and ask does the orbit of (x, θ) return close and parallel to itself. If, for each θ, this happens for almost every x, we call the polygon angularly recurrent.

Billiards in rational polygons have a well known decomposition into directional flows [10]. From this decomposition, a simple application of the Poincaré recurrence theorem shows that billiards in rational polygons are angularly recurrent. Gutkin and Troubetzkoy introduced the notion of directional flows for general polygons [13]. The directional flow is on a compact surface if and only if the polygon is rational. In the noncompact case, one can not apply the Poincaré recurrence theorem. None the less, Gutkin and Troubetzkoy found a class of polygons, generalized parallelograms (all sides of a generalized parallelogram are parallel to one of two fixed vectors), for which a noncompact version of the Poincaré recurrence theorem holds. In this article we generalize this result to a larger class of polygons. This class, consists of rank 1, ergodic polygons, which will be defined precisely in the next section. Unlike, generalized parallelograms, it is dense in the set of all polygons. The mechanism which produces angular recurrence for this class is different than the mechanism which produced it for generalized parallelograms.

An orbit which is perpendicular to a side returns to that side parallel to itself it is automatically periodic. This fact was first noticed by Ruijgrok [21] who conjectured that all triangles have a perpendicular periodic orbit. Thus for an angularly recurrent polygons, perpendicular trajectories are periodic with probability one. This fact was exploited by Boshernitzan [3] and Galperin, Stepin and Vorobetz [7] who independently gave elementary proofs that all rational polygons have perpendicular periodic trajectories. It was used by Cipra, Hanson and Kolan [5] who showed that almost every perpendicular trajectory is periodic in right triangles and by Gutkin and Troubetzkoy [13] who showed the analogous property for generalized polygons. Here, we apply the same idea to show our new class of polygons has this property as well.

Finally we turn to the question of stability of these periodic orbits [7]. We show that every periodic orbit in an arbitrary polygon is stable at least under a co-dimension 1 perturbation. The set $X = X_p$ of all simply connected p-gons can be naturally thought of as a subset of $\mathbf{R}^{2(p-2)}$. For the periodic orbits in rank 1 polygons, in particular those constructed in this article, we explicitly describe $p-2$ of the coordinates of the perturbation for which the periodic orbit persists.

2 Notation and statements of theorems

Let Q be a simply connected polygon. Let ϕ be the billiard flow and T be the first return map to the boundary ∂Q. Let P be the phase space of the billiard ball map, namely P is the boundary ∂Q cross inward pointing vectors. We will use arc length s along the boundary and the angle θ with respect to a fixed reference direction as coordinates of a point $x = (s, \theta) \in P$. The natural (finite) T-invariant measure μ is given by $d\mu = dw\, d\theta$ where w is the perpendicular width measure of a set of parallel rays while the natural (finite) ϕ-invariant measure is given by $d\mu\, dt$ where t is distance in the flow direction [10],[22].

The notion of directional flows for general polygons was introduced by Gutkin and Troubetzkoy [13]. They associated a "flat surface" $S = S(Q)$ to each polygon Q. This surface is noncompact unless the polygon Q is rational. The surface S has a natural cross section, S intersect the boundary of Q, which we will also denote by S. If we consider the billiard flow (resp. map) only for points which start in a given direction θ then the billiard flow (resp. map) lives on a copy of S (called S_θ) and is called the directional billiard flow ϕ_θ (resp. map T_θ). The natural (possibly infinite) T_θ-invariant measure on the cross section S is the width measure w. The aim of introducing the directional billiard flows was to obtain a new tool to analyze polygonal billiards. If Q is a rational polygon then the surface S is compact and w is a finite measure. In this case the directional flows have been used to prove many deep theorems about polygonal billiards [1],[3],[4],[12],[14],[15],[16],[17],[18],[23],[25].

For irrational polygons the surface S is noncompact and the invariant measure is infinite. Thus, for this tool to be useful, one should show that the directional billiard flow ϕ_θ (resp. map T_θ) is recurrent. In [13] Gutkin and Troubetzkoy showed that for a class of irrational polygons called generalized parallelograms and some related polygons such as right triangles the directional billiard flow (resp. map) is indeed recurrent for all θ. The main result of this article is that directional billiard flows are recurrent for wider class of irrational polygons.

We can formulate our results without referring to directional billiards. We will denote by $\theta(x)$ the angular component of a point $x = (s, \theta) \in P$. Let $Q_\theta := \partial Q \times \theta$, i.e. the set of all points in the direction θ. For $B \subset Q_\theta$ we say that a point $x \in B$ is recurrent with respect to B if there is a $k \geq 1$ such that $T^k x \in B$. We call a direction θ *angularly recurrent* if w-a.e. point $x \in B$ is recurrent with respect to B for each measurable set $B \subset Q_\theta$. We call a polygon Q *angularly recurrent*[1] if it is angularly recurrent for each direction θ. In particular, the orbit of μ-almost every point returns parallel to itself for an angularly recurrent polygon. From the decomposition into directional flows, it easily follows that rational polygons are angularly recurrent [10],[22]. In [13] it was shown that generalized parallelograms and right triangles are angularly recurrent.

Consider the abelian group $G_0 \subset \mathbf{S}^1$ generated by the angles of a polygon. This group is a direct product of $\mathbf{Z}^r$ and a finite cyclic group. The number r is called the rank of the polygon, and is equal to the number of independent irrational angles of the polygon[2] [13]. Our main result is:

Theorem 1 *Any rank 1 polygon for which T^2 is ergodic is angularly recurrent.*

As an immediate corollary we have:

Corollary 2 *w-almost every perpendicular orbit is periodic in any rank 1 polygon for which T^2 is ergodic.*

The proof actually uses a fact which seems to be substantially weaker than ergodicity. We believe that ergodicity is an unnecessary assumption in the statement of this theorem, that is any rank 1 polygon is angularly recurrent. The proof of this fact must be along different lines than our proof. Since a triangle which is ergodic must be irrational lemma 1 implies that any triangle for which T^2 is ergodic and which has one rational angle is angularly recurrent.

We return to the space X of all simply connected p-gons. We number the angles $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ and the set of lengths of sides $\{l_1, l_2, \dots, l_p\}$ clockwise. Since the existence of periodic billiard trajectories does not change under similarity we can normalize by setting $l_1 = 1$. The angle α_1 is determined by the other angles and the lengths l_2, l_3 are determined by the remaining angles and lengths. Thus we see that X is $2(p-2)$ dimensional. Let $X_{\alpha,l} = X_{\alpha,l}(i_1, \dots, i_n, j_1, \dots, j_m)$ be the set of all polygons for which a fixed set

[1] this concept was called strong recurrence in [13].

[2] For interval exchange transformations, the number $r+1$ is usually called the rank [2].

of angles and lengths are prescribed, i.e. if $\alpha \in \pi \cdot \mathbf{Q}^n$ and $l \in Q^m$ are fixed vectors then $Q \in X_{\alpha,l}$ if and only if $\{\alpha_{i_k} : k = 1, \ldots, n\} = \alpha$ and $\{l_{j_k} : k = 1, \ldots, m\} = l$. Let $G = G_Q \subset O(2)$ be the group generated by reflections in the sides of Q. To show that the set of polygons considered in theorem 1 is dense we will need a special case of the following theorem.

Theorem 3 *There is a dense G_δ subset in X for which all powers T^k, $k \geq 1$ of the billiard ball map are ergodic. This also holds with X replace by $X_{\alpha,l}$ for any α all of whose components are rational multiples of π or more generally for X replaced by any subset of X with the property that for any number M the set of rational tables Q in the subset with $card(G_Q) \geq M$ is dense.*

The theorem implies that there is a dense set of rank 1 polygons for which T^2 is ergodic. Periodic trajectories in rank 1 polygons are not necessarily stable under arbitrary perturbations. However, they are stable under a subclass of perturbations which keep one within rank at most 1. The set of rational polygons together with rank 1 polygons consists of a dense family of $p-2$ dimensional manifolds. Remember that α_1, l_1, l_2, l_3 are either fixed or determined by the other variables. If the polygon Q has $p-2$ rational angles and $\hat{\alpha}$ is one of the irrational angles, then the manifold $M(Q)$ is given by an open ball in the coordinates: $\hat{\alpha}, l_4, \ldots, l_p$. If Q is rank 1 with more than 2 irrational angles (which have some rational relations between them) then $M(Q)$ is still a $p-2$ dimensional open ball, whose angular component is restricted to the appropriate hyperplane.

Theorem 4 *If Q is rank 1, then for each periodic orbit in Q there is an $\epsilon > 0$ so that the periodic orbit persists under any perturbation in the ϵ-ball $B \subset M(Q)$ centered at Q.*

In particular we can applying the theorem to the periodic orbits which we constructed in rank 1, T^2-ergodic polygons. For triangles the parameter space is α_2, α_3. The set of polygons for which a periodic orbit exists then consists of three dense sets of interval, a dense set of vertical intervals, a dense set of horizontal intervals and a dense set of intervals with slope -1 which correspond to $\hat{\alpha}$ being α_3, α_2 and α_1 respectively.

If Q is rank r then one can define an analogous explicit set $M(Q)$ such that each periodic orbit in Q is stable under any perturbation in an ϵ-ball in

$M(Q)$. The set $M(Q)$ is $p-3+r$ dimensional. Perturbations of the lengths $l_4, \dots, l_p$ and an explicit r dimensional submanifold of the angular parameters are allowed.

Using the implicit function theorem one can show that periodic orbits are stable under more general perturbations.

Theorem 5 *Suppose Q is an (arbitrary) p-gon with a periodic orbit. Then there is a co-dimension 1, open submanifold $N(Q) \subset X_p$ with $P \in N(Q)$ such that the periodic orbit is stable under any perturbation in $N(Q)$.*

Remark: The submanifold $N(Q)$ is thus not explicit since it is given by the implicit function theorem. Under the assumptions of theorem 4 we have $M(Q) \subset N$. Thus, theorem 4 is more explicit than theorem 5.

Remark: I learnt the argument of theorem 5 from an anonymous referee (in the case the periodic orbit is perpendicular) who attributed it to A.M. Stepin.

Remark: It is known that if T^k is ergodic for all $k \geq 1$ and there is a c such that $\limsup_{n\to\infty} \mu(T^n A \cap B) \leq c\mu(A)\mu(B)$ for all measurable sets A, B then T is strongly mixing [19]. If, as commonly believed, polygonal billiards are never mixing [14],[12], then in fact the following is true

$$\sup_{A,B} \limsup_{n\to\infty} \frac{\mu(T^n A \cap B)}{\mu(A)\mu(B)} = \infty.$$

3 Polygons with one irrational parameter.

Proof of theorem 1: The proof consists of two lemmas.

Lemma 6 *Suppose Q is rank 1 and T^2 is ergodic. Then Lebesgue almost every θ the direction θ is angularly recurrent.*

Proof of lemma 6: Suppose Q has p sides. The irrational angles of the polygon will be called $\alpha_1, \alpha_2, \dots, \alpha_k$, the rational angles will be called $\alpha_{k+1}, \dots, \alpha_p$. We introduce some more notation which will enable us to give an explicit encoding of the rank 1 structure.

Let $O(2)$ be the group of isometries of the unit circle $\mathbf{S}^1$. Let $G \subset O(2)$ be the group generated by the reflections $\sigma_i, 1 \leq i \leq p$ in the sides of Q. Any $g \in G$ has a representation $g = \sigma_{i_1} \cdots \sigma_{i_l}, l \geq 0$. We write $\rho_l = \sigma_i \sigma_j$ so that ρ_l is the rotation by $2\alpha_l$ where α_l is the angle between sides i and j. Let x be a point in Q. We can code the orbit segment $\{x, T^2 x, \dots T^{2n} x\}$ by $g = \sigma_{i_1} \cdots \sigma_{i_{2n}} = \rho_{l_1} \cdots \rho_{l_n}$.

The subgroup $G_0 = G \cap SO(2)$ has index two in G, and G_0 is generated by rotations by $2\alpha_k$ for $k = 1, 2, \dots, p$. The group G_0 is a direct product of $\mathbf{Z}^r$ and a finite cyclic group. The finite group is generated by all rational solutions s/t of the equations $\sum_{i=1}^p n_i \alpha_i = s/t$ where $n_i \in \mathbf{Z}$. In particular, by choosing $n_i = 1$ for some $i > k$ and all other $n_i = 0$, we see that the rational angles $\alpha_{k+1}, \dots, \alpha_p$ all belong to the finite group. The number $r \leq k$ of independent irrational angles of the polygon is called the rank of the polygon [13].

We choose $\beta_1, \dots, \beta_r$ generators of $\mathbf{Z}^r$ and α a generator of the finite group. Suppose $\theta_i \in S^1$ for $i = 1, 2$. We write $\theta_1 \sim \theta_2$ if $\theta_1 - \theta_2 = 0 \mod 2\alpha$ and we say θ_1 is weakly parallel to θ_2. The equivalence class $\{\theta\}$ of θ is finite since α is rational.

For $x \in P$ we define a function $f(x)$ to keep track of whether the orbit segment $\{x, T^2 x\}$ is coded by a rational rotation, or a rotation by $2\alpha_1, \dots, 2\alpha_k$ as follows. Since the polygon Q is rank 1 there is an irrational number $\beta = \beta_1$ and nonzero integers $m_i, i = 1, 2, \dots, k$ such that $\alpha_i \sim m_i \beta$. We define

1. $f(x) = 0$ if $\theta(x) \sim \theta(T^2 x)$.
2. $f(x) = m_i$ if $\theta(x) \sim \theta(T^2 x) + 2\alpha_i$ for some $i \in \{1, 2, \dots, k\}$.

Let *inv* be the natural involution define by turning a particle around. The function f is measurable, bounded and thus integrable. The integral of f is 0 since $f(x) = -f(inv(T^2 x))$.

We consider $Q_{\{\theta\}} := Q \times \{\theta\}$. We can now explicitly define the encoding of the rank 1 structure. We keep track only of the $\mathbf{Z}$ part of this structure, it is given by $Q_{\{\theta + 2\beta n\}}$ where $n \in \mathbf{Z}$. The set $S_\theta = \bigcup_{n \in \mathbf{Z}} Q_{\{\theta + 2\beta n\}}$ is T^2 invariant. It is the natural cross section of the flat surface introduced in [13]. The function

$$A_n(x) := \sum_{n=0}^{n-1} f(T^{2i}x)$$

keeps track of which level $T^{2n}x$ is on, that is $T^{2n}x \in Q_{\{\theta+2\beta m\}}$ if and only if $A_n(x) = m$. The measure w on S_θ is T and thus T^2 invariant. It is an infinite measure. If we restrict the measure w to $Q_{\{\theta\}}$ then it is finite and (once we have shown that the first return map of T^2 to $Q_{\{\theta\}}$ is well defined) it is $T^2_{Q_{\{\theta\}}}$-invariant.

The idea of the proof is now as follows. The phase space P is the union of the invariant surfaces S_θ. The integral of f is zero, thus by ergodicity the average level on which an orbit is on converges to zero for almost every point. If for a set of positive measure of the θ's, a set of positive w-measure never returns to level zero, we will get a contradiction.

By the ergodic theorem for μ-a.e. point $x = (s, \theta)$ we have

$$\frac{1}{n} A_n(x) \to \int f(x) d\mu(x) = 0.$$

Applying the Fubini theorem we have that $A_n((s, \theta))/n \to 0$ for w-a.e. s for a.e. θ. If θ is typical in this sense, then for each $\epsilon > 0$ for w-a.e. s there is a $N_{(s,\theta)} = N((s, \theta), \epsilon)$ such that $|A_n((s, \theta))| < \epsilon n$ for all $n \geq N_{(s,\theta)}$. For fixed typical θ we choose N so large that the set $G_N(\theta) := \{(s, \hat{\theta}) \in Q_{\{\theta\}} : N_{(s,\hat{\theta})} \leq N\}$ has w-measure greater than $(1 - \epsilon)w(Q_{\{\theta\}})$. For convenience we normalize $w(Q_{\{\theta\}}) = 1$, thus we have

$$w(G_N(\theta)) > (1 - \epsilon). \tag{1}$$

Let $B_{\{\theta\}} \subset Q_{\{\theta\}}$ be the set of all points in $Q_{\{\theta\}}$ which never return to $Q_{\{\theta\}}$ under the action T^2, that is $B_{\{\theta\}} := \{x \in Q_{\{\theta\}} : \theta(T^{2i}x) \not\sim \theta(x) \ \forall i > 0\}$. We claim that $T^{2i}B_{\{\theta\}} \cap T^{2j}B_{\{\theta\}} = \emptyset$ for all $j > i \geq 0$. Otherwise $B_{\{\theta\}} \cap T^{2(j-i)}B_{\{\theta\}} \neq \emptyset$ which contradicts the definition of $B_{\{\theta\}}$. Thus since T^2 preserves the measure w we have

$$w(\bigcup_{i=0}^{n-1} T^{2i}B_{\{\theta\}}) = nw(B_{\{\theta\}}). \tag{2}$$

Now by equation (1) we immediately have

$$w\left(\bigcup_{i=0}^{n-1} T^{2i}(G_N^c \cap B_{\{\theta\}})\right) \leq w\left(\bigcup_{i=0}^{n-1} T^{2i}(G_N^c)\right) \leq \epsilon n. \tag{3}$$

Next if $x \in G_N \cap B_{\{\theta\}}$ then by the choice of N we have $|A_n(x)| < \epsilon n$ and thus $T^{2n}x \in \bigcup_{|m|\leq\epsilon n} Q_{\{\theta+2\beta m\}}$ for all $n \geq N$. Thus

$$w\left(\bigcup_{i=N}^{n-1} T^{2i}(G_N \cap B_{\{\theta\}})\right) \leq \epsilon n \sup_{\phi} w(Q_{\{\phi\}}) := \epsilon n C_1 \tag{4}$$

where C_1 is a positive constant defined by the last equality. Finally

$$w(\bigcup_{i=0}^{N-1} T^{2i}(G_N \cap B_{\{\theta\}})) \leq C_1 \cdot N \tag{5}$$

holds for all $\theta \in \mathbf{S}^1$. Combining equations (2)-(5) yields

$$nw(B_{\{\theta\}}) \leq C_1 N + \epsilon n C_1 + \epsilon n \tag{6}$$

for typical θ for all $n \geq N$. Since $\epsilon > 0$ is arbitrary we have shown that $w(B_{\{\theta\}}) = 0$ for a.e. θ. We have shown that the first return map $T^2_{Q_{\{\theta\}}}$ to the set $Q_{\{\theta\}}$ is well defined w-almost everywhere. Since $w(Q_{\{\theta\}}) < \infty$ the Poincaré recurrence theorem implies that Lebesgue a.e. direction θ is angularly recurrent. □

The above argument does not preclude the existence of bad directions $\hat{\theta}$'s which are not angularly recurrent. To show this we need an additional argument. Let $M = \max\{m_i\}$ where the m_i where defined by $\alpha_i \sim m_i\beta$.

Lemma 7 *Suppose that Q is a rank 1 polygon for which there is a direction θ such that all the directions $\theta + 2\beta n$ are angularly recurrent for $|n| \leq M$. Then every direction $\hat{\theta}$ is angularly recurrent.*

Ergodicity is not used in the proof of 7. Using lemma 6 for any rank 1, T^2-ergodic polygon we can find a direction θ such that the directions $\theta + 2\beta n$ are angularly recurrent for all $n \in \mathbf{Z}$. Thus theorem 1 follows from the lemmas 6 and 7. □

Proof of lemma 7: For $x = (s_1, \theta_1)$ let $U_\delta(x) := \{(s_2, \theta_2) : \max(|s_1 - s_2|, |\theta_1 - \theta_2|) < \delta\}$. For fixed $Q_{\{\theta\}}$ and for any $x \in Q_{\{\theta_1\}}$ which returns to $Q_{\{\theta_1\}}$ let $n(x)$ be the first return time of T^2, i.e. $T^{2n(x)} \in Q_{\{\theta_1\}}$ and $T^{2m}x \notin Q_{\{\theta_1\}}$ if $0 < m < n(x)$. By local continuity (consider the unfolding[3] of length $2n$) there is a δ-ball $U_\delta(x)$ in the phase space $P(Q)$ centered at x such that for any point $y \in U_\delta(x)$ the orbit segments $\{y, T^2y, \ldots, T^{2n(x)}y\}$ and $\{x, T^2x, \ldots, T^{2n(x)}x\}$ hit the same sequence of sides and thus we have $\theta(y) \sim \theta(T^{2n(x)}y)$.

We call a point $x \in Q_{\{\theta_1\}}$ angularly recurrent if there exists an integer $n(x)$ such that $T^{2n(x)} \in Q_{\{\theta_1\}}$. Let $R_{\{\theta_1\}}$ be the set of points $x \in Q_{\{\theta_1\}}$ which are angularly recurrent. Let

$$M_{\{\theta_1\}}(\delta) = \Big\{ \bigcup U_\delta(x) \ : \ x \in Q_{\{\theta_1\}}, \text{ such that all points } y = (s, \theta(y)) \in U_\delta(x) \text{ satisfy } T^{2n(x)}y \in Q_{\{\theta(y)\}} \Big\}. \tag{7}$$

All $y \in M_{\{\theta_1\}}(\delta)$ are angularly recurrent points. Thus, if $|\theta_1 - \theta_2| < \delta$ then since $U_\delta(x)$ is an l^∞ ball in the (s, θ) coordinates we have

$$Q_{\{\theta_2\}} \cap M_{\{\theta_1\}}(\delta) \subset R_{\{\theta_2\}}. \tag{8}$$

Clearly, for any θ_1 whenever $\delta_2 \leq \delta_1$ we have

$$Q_{\{\theta_1\}} \cap M_{\{\theta_1\}}(\delta_1) \subset Q_{\{\theta_1\}} \cap M_{\{\theta_1\}}(\delta_2). \tag{9}$$

Thus from the definition of $M_{\{\theta_1\}}(\delta)$ it is clear that

$$Q_{\{\theta_1\}} \cap \bigcup_{\delta > 0} M_{\{\theta_1\}}(\delta) = R_{\{\theta_1\}}. \tag{10}$$

Now let $\theta_1 = \theta$ where θ is the angularly recurrent direction in the statement of the lemma. We will Using equations (9) and (10) for each $\epsilon > 0$, we can choose $\delta = \delta(\epsilon, \theta) > 0$ so small that if $|\hat{\theta} - \theta| < \delta$ then

$$w(Q_{\{\theta\}} \cap M_{\{\theta\}}(\delta)) \geq (1 - \frac{\epsilon}{3}) w(R_{\{\theta\}}) = (1 - \frac{\epsilon}{3}) w(Q_{\{\theta\}}) \tag{11}$$

and

$$\max \left(\frac{w(Q_{\{\theta\}})}{w(Q_{\{\hat{\theta}\}})}, \frac{w(Q_{\{\theta\}} \cap M_{\{\theta\}}(\delta))}{w(Q_{\{\hat{\theta}\}} \cap M_{\{\theta\}}(\delta))} \right) \geq 1 - \frac{\epsilon}{3}. \tag{12}$$

[3] See [10],[22] for the definition of unfolding.

Combining inequalities (12) and (11) implies

$$w(Q_{\{\hat{\theta}\}} \cap M_{\{\theta\}}(\delta)) \geq (1-\epsilon) w(Q_{\{\hat{\theta}\}}). \tag{13}$$

Equations (8) and (13) imply that then

$$\frac{w(R_{\{\hat{\theta}\}})}{w(Q_{\{\hat{\theta}\}})} \geq 1-\epsilon. \tag{14}$$

Since $\theta + 2\beta n$ is angularly recurrent for $|n| \leq M$ we can choose $\delta_n = \delta(\epsilon, \theta, n)$ so large that

$$\frac{w(R_{\{\hat{\theta}+2\beta n\}})}{w(Q_{\{\hat{\theta}+2\beta n\}})} \geq 1-\epsilon \tag{15}$$

for all $\hat{\theta}$ satisfying $|\hat{\theta} - \theta| < \delta_n$. Let $\delta = \min_{|i| \leq M} \delta_i$.

Consider an arbitrary direction $\hat{\theta}$. Since β is irrational we can choose $n_+ > M$ and $n_- < -M$ such that the angles $\hat{\theta}_\pm := \hat{\theta} + 2n_\pm \beta$ satisfy $|\hat{\theta}_\pm - \theta| < \delta$. Thus applying equation (15) for each 0f $\hat{\theta}_\pm$ we have at least $(1-\epsilon)\%$ of the points on level $n_\pm + n$ ($|n| \leq M$) are angularly recurrent:

$$\frac{w(R_{\{\hat{\theta}+2\beta(n_\pm+n)\}})}{w(Q_{\{\hat{\theta}+2\beta(n_\pm+n)\}})} > 1-\epsilon. \tag{16}$$

Next we show that for every $\hat{\theta}$ w-a.e. point recurs to some level (not necessarily the level it is on) infinitely often. Let $C'_{\{\hat{\theta}\}} \subset Q_{\{\hat{\theta}\}}$ be the set of $x \in Q_{\{\hat{\theta}\}}$ which do not recur to any level infinitely often. Suppose $w(C'_{\{\hat{\theta}\}}) > 0$. Let $C_{\{\hat{\theta}\}} := \{T^{2l(x,0)}x : x \in C'_{\{\hat{\theta}\}}\}$ where $2l(x,0)$ defines the time of the last visit of x to level 0. Since $x \to T^{2l(x,0)}x$ defines a measure preserving embedding we have $w(C_{\{\hat{\theta}\}}) = w(C'_{\{\hat{\theta}\}})$. Just as with $B_{\{\theta\}}$ above we have all forward images of $C_{\{\hat{\theta}\}}$ are disjoint, i.e. $T^{2i}C_{\{\hat{\theta}\}} \cap T^{2j}C_{\{\hat{\theta}\}} = \emptyset$ for all $j > i \geq 0$. For all levels m which a point $x \in C_{\{\hat{\theta}\}}$ reaches define $2l(x,m)$ to be the last time x reaches level m. Let $C^{\pm,n}_{\{\hat{\theta}\}}$ be the set of $x \in C_{\{\hat{\theta}\}}$ whose orbit visits the level $n_\pm + n$. Since M is the maximal number of levels a point can jump in one step, and $x \in C_{\{\hat{\theta}\}}$ must visit infinitely many levels the orbit of each $x \in C_{\{\hat{\theta}\}}$ must visit at least one of the levels $n_\pm \pm n$ for some $|n| \leq M$, thus we have

$$C_{\{\hat{\theta}\}} = \bigcup_{|n| \leq M} C^{\pm,n}_{\{\hat{\theta}\}}. \tag{17}$$

For $n \in \mathbf{Z}$ the map $x \to T^{2l(x,n_\pm+n)}x$ defines a measure preserving embedding of $C^{\pm,n}_{\{\hat\theta\}}$ into the set of points which on level $n_\pm + n$ which never recur to that level. Thus, for $|n| \leq M$ using equation (16) we have $w(C^{\pm,n}_{\{\hat\theta\}}) < C_1\epsilon$. Since ϵ was arbitrary using equation (17) we have $w(C_{\{\hat\theta\}}) = 0 = w(C'_{\{\theta\}})$.

Finally let $B_{\{\hat\theta\}}$ be defined as before, the points in $Q_{\{\hat\theta\}}$ which never recur to $Q_{\{\hat\theta\}}$. Let $B^m_{\{\hat\theta\}} \subset B_{\{\hat\theta\}}$ be the set of points which return to level m infinitely often. By the previous paragraph $B_{\{\hat\theta\}} = \bigcup_{m\neq 0} B^m_{\{\hat\theta\}}$ mod 0. Thus if $w(B_{\{\hat\theta\}}) > 0$ then there is some level, call it m, for which the set $B^m_{\{\hat\theta\}} \subset B_{\{\hat\theta\}}$ has positive width. But since the forward images of $B^m_{\{\hat\theta\}}$ are also disjoint from one another it can not hit level m infinitely often since the map is measure preserving. Thus $w(B^m_{\{\hat\theta\}}) = 0$ for all m. Thus, just as before, the first return map to $Q_{\{\hat\theta\}}$ is defined w-a.e., but now for all $\hat\theta$. Since $w(Q_{\{\hat\theta\}}) < \infty$ the theorem follows by applying the Poincaré recurrence theorem. □

4 Generic ergodicity of all powers of the billiard map.

Although the results of this section are new, they can be derived by combining results of several other authors with only small changes. We include the proofs for completeness. Kerckhoff, Masur, and Smillie have shown that if Q is a rational polygon then the billiard flow ϕ_θ is uniquely ergodic for Lebesgue a.e. θ. Using this result, by approximating general polygons by rational ones they showed that for a dense G_δ in X the billiard flow is ergodic [16]. Their proof also shows that for each α all of whose components are rational multiples of π the billiard flow is ergodic for a dense G_δ in $X_{\alpha,l}$. In both cases it follows that there is a dense G_δ of triangles for which the billiard ball map T is ergodic.

An alternative proof of this fact is derived by combining results of Masur and Boshernitzan. For any integer $n \geq 1$, denote by $V_Q(n)$ the set of generalized diagonals of length n, that is the set of trajectories which connect a pair of vertices of Q, which avoid the vertices of Q at the intermediate points and which involve $(n-1)$ reflections. Let $W_Q(n) = \bigcup_{j=1}^n V_Q(j)$. Boshernitzan has shown that if Q is a rational polygon satisfying

$$card(W_Q(n)) = O(n^2) \tag{18}$$

then the billiard flow ϕ_θ and billiard transformation T_θ are uniquely ergodic for Lebesgue a.e. θ [1]. This proof can be simplified by applying the Veech criterion [2, 24]. Later Masur verified equation (18) for any rational polygon Q [18].

Lemma 8 *If Q is a rational polygon then for Lebesgue almost every θ the billiard transformations T_θ^k are ergodic for all $k \geq 1$.*

Proof: It is well known that the billiard transformations T_θ and their powers T_θ^k are interval exchange maps for all $k \geq 1$ [10],[11],[14],[22]. Let $m_n = m(T^n)$ be the minimal length of the intervals defining the interval exchange T^n.

Veech has shown that if T is minimal and $\limsup_{n\to\infty} nm_n(T) > 0$ then T is uniquely ergodic [2, 24]. Using Masur's quadratic growth estimate [18] Boshernitzan verified that for almost every θ Veech's criterion holds for T_θ. Since the m_n are monotonically decreasing this implies that for each fixed $k \geq 1$ we have $\limsup_{n\to\infty} nm_n(T^k) > 0$.

To conclude the lemma we need to know that for almost every θ the interval exchanges T_θ^k are minimal for all $k \geq 1$. There is a well known sufficient condition for minimality: if T_θ has no orbit starting from a singular point and ending in a singular point (called a saddle connecting in the terminology of interval exchanges and a generalized diagonal in the terminology of polygonal billiards) then T_θ is minimal [2]. Once this condition is satisfied for T_θ it is clearly satisfied for T_θ^k for all $k \geq 1$. There are only a countable number of directions θ for which T_θ has a saddle connection, completing the proof. □

Proof of lemma 3: The proof the approximation argument is similar to the one given in [16]. They prove the generic ergodicity of the billiard flow while we need to prove the generic ergodicity of the powers of the billiard map.

It suffices to prove the result for any compact subset of X, thus without loss of generality we assume that X is compact. Each $x \in X$ corresponds to a polygon Q_x. Assume the arc length of the boundary of each Q_x is one.

All notation from before will now carry the subscript x, i.e. P_x is the phase space of the billiard ball map for Q_x (not flow as in [16]), μ_x is the invariant measure, etc. Let PX be the bundle with base space X so that the fiber, P_x over x is the phase space P_x.

Choose a sequence of continuous functions $f_1, f_2, \ldots$ on PX which, when restricted to P_x, for any $x \in X$, are dense in $L^2(P_x)$. Let $E_k(i,n,N)$ be the set of $x \in X$ for which

$$\int_{z\in P_x} \left\{ \frac{1}{N} \sum_{l=0}^{N-1} f_i(T^{kl}z) - \int_{P_x} f_i \, d\mu_x \right\}^2 d\mu_x < \frac{1}{n}.$$

Let $E_k(i,n) = \cup_{N=1}^{\infty} E_k(i,n,N)$, let $E = \cap_{k=1}^{\infty} \cap_{i=1}^{\infty} \cap_{n=1}^{\infty} E_k(i,n)$. The set of $x \in X$ for which $T^k|P_x$ is ergodic for all $k \geq 1$ is precisely E [20]. We prove that:

1. The sets $E_k(i,n,N)$ are open and
2. For a given k, i, n there is an M such that $E_k(i,n)$ contains all rational tables x for which $card(G_x) > M$.

Lemma 3 follows from the two statements. The statements are consequences of the following two lemmas. □

Lemma 9 *Let $N > 0$ and $k \geq 1$ be fixed. Let f be a continuous function on PX. Then*

$$\int_{z\in P_x} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} f(T^{ki}z) - \int_{P_x} f \, d\mu_x \right\}^2 d\mu_x$$

depends continuously on x.

Proof: For $y \in X$ let $a(y) = \int_{P_y} f \, d\mu_y$. We begin by proving that $a(y)$ is a continuous function of y. Let $U_1 \subset X$ be a neighborhood of x. For each $y \in U_1$, there is a homeomorphism h_y between the boundary ∂Q_y and $\mathbf{S}^1$ and this homeomorphism is continuous as y varies in X. Let vert(y) denote the set of vertices of Q_y. Let $\partial Q_1 \subset \mathbf{S}^1$ be the set of $s \in \mathbf{S}^1$ such that $s \notin \{h_y^{-1}(\mathrm{vert(y)}) : \mathrm{y} \in \mathrm{U}_1\}$. Let $F = \sup|f|$. By choosing U_1 sufficiently small we can assume that the length of ∂Q_1 is greater than $1 - \epsilon/(3F)$. Let

P_1 be ∂Q_1 cross $[0, \pi]$ (here we use a different angle coordinate, namely the angle is measured with respect to the clockwise pointing tangent). Let μ_1 be the product of the length measures on P_1. Using the homeomorphisms h_y we can naturally view $U_1 \times P_1$ as a subset of PX.

Let $b(y) = \int_{p \in P_1} f(y, p)\, d\mu_1$. The continuity of b follows from the continuity of f. Furthermore, for $y \in U_1$, b is close to a:

$$|a(y) - b(y)| \leq \int_{P_y - P_1} f\, d\mu \leq \mu(P_y - P_1) \cdot F \leq \frac{\epsilon}{3}.$$

Let $U_2 \subset U_1$ be a neighborhood of x consisting of points y for which $|b(x) - b(y)| < \epsilon/3$. Then for $y \in U_2$ we have

$$|a(x) - a(y)| \leq |a(x) - b(x)| + |b(x) - b(y)| + (b(y) - a(y)| \leq \epsilon.$$

This completes the proof of the continuity of the function a.

We replace the function f by the function $\bar{f}$ as follows: for $z \in P_y$, $\bar{f}(z) = f(z) - a(y)$. Then the proof of the lemma reduces to the proof of the continuity of the following function(s):

$$c(x) = c^{(k)}(x) = \int_{z \in P_x} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} \bar{f}(T^{ki} z) \right\}^2 d\mu_x.$$

Let π be the projection from P_y to Q_y. For $z \in P_y \subset PX$ define

$$l(z) = l^{(k)}(z) = \inf_{\substack{0 \leq i \leq N-1 \\ v \in \mathrm{vert(y)}}} d(\pi(T^{ki} z), v).$$

If $T^{ki} z$ is not defined for some i between 0 and N then we define $l(z)$ to be zero. Clearly $l(z)$ is a continuous function of z.

To prove the continuity of the function c we fix $x \in X$ and let $\epsilon > 0$ be given. Choose a neighborhood U_1 of x such that the length of the set ∂Q_1 is at least $1 - \epsilon/(10F)$.

The gradient of l at $z \in P_x$ where $l(z) = 0$ is non-zero. So, by the implicit function theorem, the set of $z \in P_x$ for which $l(z) = 0$ has codimension 1 in P_x. Thus, it has measure zero. Let $C_\delta = \{z \in P_x : l(z) \geq \delta\}$. Choose $\delta > 0$ so small that the measure of C_δ is at least $1 - \epsilon/(10F)$. Let $D_\delta = \{(p, v) \in$

$C_\delta : p \in \partial Q_1\}$. We can identify the set $U_1 \times D_\delta$ with a subset of PX in a natural way via $(y,p,v) \to (y, h_y^{-1} h_x(p), v)$. Let $\bar{l}(y)$ be the infimum of $l(z)$ for $z \in P_y \cap U_1 \times D_\delta$. Now $\bar{l}$ is continuous and $\bar{l}(x) = \delta$. We can find a neighborhood $U_2 \subset U_1$ so that for $y \in U_2$, $\bar{l}(y)$ is positive. For each $y \in U_2$ let $d(y)$ denote the integral

$$d(y) = d^{(k)}(y) = \int_{z \in y \times D_\delta} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} \bar{f}(T^{ki} z) \right\} d\mu_y.$$

Since, for each $k \geq 1$ the function $f(T^{ki}z)$ is a continuous function of z for each $i \in \{0, 1, \ldots, N-1\}$ and $y \in U_2 \times D_\delta$, the integral varies continuously. We can find a neighborhood U_3 of x on which d varies by less than $\epsilon/3$. The difference between $d(y)$ and $c(y)$ is less than $\epsilon/3$. Thus for $y \in U_3$,

$$|c(x) - c(y)| \leq |c(x) - d(x)| + |d(x) - d(y)| + |d(y) - c(y)| \leq \epsilon.$$

This completes the proof of the continuity of c and the proof of the lemma. $\square$

Lemma 10 *Fix $n > 0$ and a continuous function f on PX. Choose $\delta > 0$ so that if the distance between two points θ_1 and θ_2 is less than δ then $|f(\theta_1) - f(\theta_2)| < 1/2n$. (Recall that PX is compact.) Let M be greater than $2/\delta$. Let Q_x be a rational polygon with $|G_x| \geq M$. Then for each $k \geq 1$ for each N sufficiently large*

$$\left\{ \int_{z \in P_x} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} f(T^{ki} z) - \int_{P_x} f \, d\mu_x \right\}^2 d\mu_x \right\}^{1/2} < \frac{1}{2n}.$$

Proof: Since x is fixed we well drop the subscripts from P, Q, G and μ. For $\theta \in \mathbf{S}^1$ be $u(\theta) = 1/|G| \int_{z \in X_\theta} f(z) dm$ where dm is the arc length measure on X_θ and X_θ is the collection of intervals forming the interval exchange map associated to T_θ. For $z \in P$ let $u'(z) = u(\theta)$ when $z \in X_\theta$. For $z \in P$ and $k \geq 1$ let $v_N^{(k)}(z) = 1/N \sum_{i=0}^{N-1} f(T^{ki} z)$. Let $\text{int} f$ denote $\int_{z \in P} f(z) \, d\mu$. The quantity which appears in the lemma is the norm in the space $L^2(P)$ of the function $v_N^{(k)} - \text{int} f$.

Claim. $lim_{N\to\infty}||v_N^{(k)} - u'|| = 0 \quad \forall k \geq 1.$

The intervals X_θ are parameterized by $\theta \in \mathbf{S}^1/G$. We evaluate the norm by integrating first with respect to X_θ, then with respect to θ, that is for each $k \geq 1$ we have

$$||v_N^{(k)} - u'|| = \left\{ |G| \int_{\theta\in\mathbf{S}^1/G} \frac{1}{|G|} \int_{z\in X_\theta} (v_N^{(k)}(z) - u'(z))^2 dm\, d\theta \right\}^{1/2}.$$

Let $w_N^{(k)} = \left\{ \frac{1}{|G|} \int_{z\in X_\theta} (v_N^{(k)}(z) - u'(z))^2 dm \right\}^{1/2}$. Then for each $k \geq 1$ we have

$$||v_N^{(k)} - u'|| = \left\{ |G| \int_{\theta\in\mathbf{S}^1/G} w_N^{(k)}(\theta)^2\, d\theta \right\}^{1/2}.$$

For each θ, k the ergodicity of $T^k|X_\theta$ implies that $\lim_{N\to\infty} w_N^{(k)}(\theta) = 0$. By lemma 8 we have ergodicity for almost every θ. Since the functions $w_N^{(k)}$ are bounded and converge pointwise almost everywhere to 0 they converge to 0 in norm. This completes the proof of the claim.

Claim. $||u' - \mathrm{int} f|| \leq 1/2n$.

The norm of the function $u' - \mathrm{int} f$ in $L^2(P)$ is equal to the norm of the function $u - inff$ in $L^2(\mathbf{S}^1)$. Let θ_1 be a point in the circle at which u assumes its maximal value B. Let θ_2 be a point at which u assumes in minimum value b. Note that u is constant on the orbits of G. The distance between neighboring points in a G orbit is less than $2/|G| < 2/M < \delta$. By replacing θ_2 by some $\gamma\theta_2$ where $\gamma \in G$, we may assume that the distance between θ_1 and θ_2 is less than δ. It follows from the continuity assumption on f that since θ_1 and θ_2 are closer than δ then $|B - B| = |u(\theta_1) - u(\theta_2)| < 1/2n$. Since u is defined by averaging f, the integral of f over P is equal to the integral of u over $\mathbf{S}^1$. Thus $b \leq \mathrm{int} f \leq B$; hence for each $\theta \in \mathbf{S}^1$ we have $|u(\theta) - \mathrm{int} f| \leq 1/2n$ and thus $||u - \mathrm{int} f|| \leq 1/2n$. This completes the proof of the claim.

We now complete the proof of the lemma. For each fixed $k \geq 1$ choose N sufficiently large that $||v_N^{(k)} - u'|| \leq 1/2n$. Then

$$||v_N^{(k)} - \mathrm{int} f|| \leq ||v_N^{(k)} - u'|| + ||u' - \mathrm{int} f|| \leq 1/2n + 1/2n = 1/n.$$

□

5 Proof of theorem 4

Fix a rank one polygon Q. Note that $M(Q)$ consists of rank 0 and rank 1 polygons. Let $\alpha(Q)$ be the generator of the finite group as in theorem 1. The main observation is that since Q is rank 1 then the number $\alpha(\hat{Q}) = \alpha(Q)$ for all rank 1 $\hat{Q} \in M(Q)$. On the other hand if $\hat{Q} \in M(Q)$ is rank zero then the generator is a divisor of $\alpha(Q)$. For notational purposes let $\alpha = \alpha(Q)$.

To be more explicit about the space of p-gons we put one of the vertices of Q at the origin (say the vertex with angle α_1) and put the side which connects vertex one to vertex two (i.e. the vertex with angle α_2) along the x-axis for all our polygons. Without loss of generality, we will only consider orbits which hit the side lying on the x-axis (otherwise we simply need to think of another side a being fixed by the perturbation). If the period is odd then the orbit is stable for arbitrary small perturbations [7]. Fix the periodic point x in the phase space of Q with period $2n$. Then we have $\theta(T^{2n}x) = \theta(x) + \sum_{i=1}^{p} 2n_i\alpha_i = \theta(x) \mod 2\pi$. Since Q is rank 1 the only way this can happen is if $\theta(T^{2n}x) = \theta(x) + 2m\alpha$ and $2m\alpha = 0 \mod 2\pi$.

Consider the unfolding of length $2n$ and let l denote x's orbit. The bottom and top polygons in the unfolding are parallel (figure 1(a)). Consider small perturbations, $\hat{Q} \in M(Q)$. If we think of the unfolding as a collection of glued copies of Q (resp. $\hat{Q}$), then the above calculation shows that the top and bottom of the unfolding are parallel for any $\hat{Q} \in M(Q)$. The orbit l stayed a certain distance away from the vertices. Thus there will be a small open set O (in X_p) of polygons such that for all $\hat{Q} \in O$ the orbit of the point $(x, \theta) \in \hat{Q}$ will have one and the same unfolding of length $2n$.

In particular, if $\hat{Q} \in M(Q)$ then the orbit $\hat{l}$ of (x, θ), after $2n$ reflections, returns parallel to itself to a point (y, θ) (figure 1(b)). For perpendicular periodic orbits this completes the proof. For other periodic orbits, note that by continuity $y = y(\hat{Q}) \to x$ as $\hat{Q} \to Q$. In particular the orbit segment $\hat{l}$ which denotes the first $2n$ bounces of (x, θ)'s orbit in $\hat{P}$ will approach l.

Consider the segment $\hat{S}$ which connects the point x on the bottom of the unfolding to the point x at the top of the unfolding. Note that $\hat{S}$ is not in the direction θ (unless $\hat{l}$ is already periodic). If $\hat{S}$ lies in the unfolding then it is a periodic orbit. Clearly $\hat{S} \to l$ as $\hat{Q} \to Q$. Thus, since l stays some definite distance away from the vertices, $\hat{S}$ will stay in the same corridor as

l if the perturbation is sufficiently small. Thus $\hat{S}$ is a periodic orbit. □

6 Proof of theorem 5

Fix a polygon Q with a periodic orbit. If the period is odd then the periodic orbit is stable for arbitrary small perturbations [7]. Thus we denote the period by $2n$. Consider the unfolding of length $2n$ and let l denote the "central" periodic orbit of the unfolding. The bottom and top polygons in the unfolding are parallel. Let the sides of the polygon be labelled $1, 2, \ldots, p$ and $\alpha_i \in \mathbf{S}^1$ be the angle that side i makes with respect to some fixed ray in the plane. Let $i_1 i_2 \ldots i_{2n}$ be the sequence of sides that the orbit hits, we call this the orbit's code word of length $2n$. Let θ be the initial direction of l measured with respect to the tangent to the initial side) and x be l's initial point (figure 1(a)).

Consider small perturbations of Q. If we think of the unfolding as a collection of glued copies of Q, then it will change in a continuous way. Similar polygons have the same billiard dynamics, thus we can assume that the initial side remains fixed. The orbit l (in the original unfolding) stayed a certain distance away from the vertices. Thus there will be a small open set O (in X_p) of polygons such that for all $\hat{Q} \in O$ the orbit of the point $(x, \theta) \in \hat{Q}$ will have one and the same code word of length $2n$ or equivalently the same unfolding of length $2n$. In general the top and bottom of the unfolding need not be parallel. One can explicitly compute the angle between the top and bottom copies of Q, it is $2(\alpha_{i_1} - \alpha_{i_2} + \alpha_{i_3} - \ldots - \alpha_{i_{2n}})$. Since this function is linear we can find a co-dimension 1 open submanifold $N(Q) \subset O$ for which the top and bottom are parallel.

Thus if $\hat{Q} \in N(Q)$ then the orbit $\hat{l}$ of (x, θ), after $2n$ reflections, returns parallel to itself to a point (y, θ) (figure 1(b)). By continuity $y = y(\hat{Q}) \to x$ as $\hat{Q} \to Q$. In particular the orbit segment $\hat{l}$ which denotes the first $2n$ bounces of (x, θ)'s orbit in $\hat{P}$ will approach l.

Consider the segment $\hat{S}$ which connects the point x on the bottom of the unfolding to the point x at the top of the unfolding. Note that $\hat{S}$ is not in the direction θ (unless $\hat{l}$ is already periodic). If $\hat{S}$ lies in the unfolding then it is a periodic orbit. Clearly $\hat{S} \to l$ as $\hat{Q} \to Q$. Thus, since l stays some definite distance away from the vertices, $\hat{S}$ will stay in the same corridor as l is the perturbation is sufficiently small. Thus $\hat{S}$ is a periodic orbit. $\square$

Acknowledgements: Thanks to Eugene Gutkin, Tolya Katok and Yaroslav Vorobets for useful discussions.

References

[1] M. Boshernitzan, *A condition for minimal interval exchange maps to be uniquely ergodic,* Duke Math. J., 52 (1985) 723–752.

[2] M. Boshernitzan, *Rank two interval exchange transformations* Erg. Th. Dyn. Sys., 8 (1988) 379–394.

[3] M. Boshernitzan, *Billiards and rational periodic directions in polygons* Amer. Math. Monthly June-July (1992) 522–529.

[4] M. Boshernitzan, G. Galperin, T. Krüger, and S. Troubetzkoy, *Density of periodic orbits in rational polygons,* preprint.

[5] B. Cipra, R. Hanson, and A. Kolan, *Periodic trajectories in right triangle billiards,* Phys. Rev. E 52 (1995) 2066-2071.

[6] G. Galperin, T. Krüger, and S. Troubetzkoy, *Local instability of orbits in polygonal and polyhedral billiards,* Comm. Math. Phys., 169 (1995) 463–473.

[7] G. Galperin, A. Stepin, and Ya. Vorobets, *Periodic billiard trajectories in polygons: generating mechanisms,* Russ. Math. Surv., 47 (1992) 5–80.

[8] G. Galperin, D. Zvonkin, *Periodic billiard trajectories in right triangles,* preprint.

[9] E. Gutkin, *Billiards on almost integrable polyhedral surfaces* Erg. Th. Dyn. Sys., 4 (1984) 569–584.

[10] E. Gutkin, *Billiard in polygons,* Physica D, 19 (1986) 311-333.

[11] E. Gutkin, *Billiard in polygons: survey of recent results,* J. Stat. Phys., 83 (1996) 7–26.

[12] E. Gutkin, and A. Katok, *Weakly mixing billiards*, Holomorphic Dynamics, SLNM 1345, Springer-Verlag, Berlin (1989) 163–176.

[13] E. Gutkin, and S. Troubetzkoy, *Directional flows and strong recurrence for polygonal billiards,* to appear in Proceedings of the International Congress of Dynamical Systems, Montevideo, Uruguay.

[14] A. Katok, *Interval exchange transformations and some special flows are not mixing*, Isr. J. Math. 35 (1980) 301–310.

[15] A. Katok and A. Zemlyakov, *Topological transitivity of billiards in polygons,* Math. Notes 18 (1975) 760–764.

[16] S. Kerckhoff, H. Masur, and J. Smillie *Ergodicity of billiard flows and quadratic differentials* Annals Math. 124 (1986) 293–311.

[17] H. Masur, *Closed trajectories for quadratic differentials with an application of billiards*, Duke Math. J. 53 (1986) 307–314.

[18] H. Masur, *The growth rate for trajectories of a quadratic differential*, Ergod. Theor. Dyn. Sys. 10 (1990) 151–176.

[19] D. Ornstein, *On the root problem in ergodic theory*, Proc. Sixth Berkeley Symposium (1970), Univ. California Press, Berkeley and Los Angeles, 345–356.

[20] J.C. Oxtoby and S.M. Ulam, *Measure preserving homeomorphisms and metrical transitivity*, Ann. Math. 115 (1941) 874–920.

[21] Th. Ruijgrok, *Periodic orbits in triangular billiards*, Acta Phys. Pol. 22 (1991) 955–981.

[22] S. Tabachnikov, *Billiards,* "Panoramas et Syntheses", Soc. Math. France (1995).

[23] W.A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. Math. 115 (1982) 201–242.

[24] W.A. Veech, *Boshernitzan's criterion for unique ergodicity of an interval exchange transformation* Ergod. Theor. Dyn. Sys. 7 (1997) 149–153.

[25] W.A. Veech, *Teichmuller curves in modular space, Eisenstein series, and an application to triangular billiards*, Inv. Math. 97 (1989) 553–583.