

Thue-Morse at Multiples of an Integer

Johannes F. Morgenbesser¹

*Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner
Hauptstraße 8–10, A–1040 Wien, Austria,*

*Institut de Mathématiques de Luminy, Université de la Méditerranée, 13288 Marseille
Cedex 9, France*

Jeffrey Shallit

School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada

Thomas Stoll

*Institut de Mathématiques de Luminy, Université de la Méditerranée, 13288 Marseille
Cedex 9, France*

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1. Introduction and Preliminaries

Let $s_b(n)$ denote the sum of the digits of n when expressed in base b , and let

$$t_n = s_2(n) \bmod 2, \quad n \geq 0,$$

be the Thue-Morse sequence \mathbf{t} . In an e-mail message dated June 7 2010, Jorge Buescu of the Universidade de Lisboa observed that the Thue-Morse sequence

Email addresses: `johannes.morgenbesser@tuwien.ac.at` (Johannes F. Morgenbesser), `shallit@cs.uwaterloo.ca` (Jeffrey Shallit), `stoll@iml.univ-mrs.fr` (Thomas Stoll)

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can be regarded as a 2-coloring of the integers, and therefore, by van der Waerden's theorem, must contain arbitrarily long monochromatic arithmetic progressions.² (By a *monochromatic arithmetic progression* we mean a series of indices $i, i + j, i + 2j, \dots, i + (n - 1)j$ such that $t_i = t_{i+j} = \dots = t_{i+(n-1)j}$.) He then asked, is it true that \mathbf{t} has no infinite monochromatic arithmetic progressions?

The answer is yes: \mathbf{t} has no infinite monochromatic arithmetic progressions. This is a consequence of the following result of Gelfond [5], which says that the values of $s_b(n)$ are equally distributed in residue classes, even if the residue class of n is fixed. (A weaker result, applicable in the case of the Thue-Morse sequence, had previously been given by Fine [4].)

Theorem 1. *Let b, r, m be positive integers with $\gcd(b - 1, r) = 1$, and let a, c be any integers. Then the number of integers $n \leq x$ congruent to a mod k such that $s_b(n) \equiv c \pmod{r}$ is equal to $\frac{x}{kr} + O(x^\lambda)$ for some $\lambda < 1$ that does not depend on x, k, a , or c .*

Gelfond's theorem, however, concerns the *average* distribution of the values of $s_b(n)$ in residue classes. It suggests the following question: how large can the smallest n be that is congruent to a mod k and satisfies $s_b(n) \equiv c \pmod{r}$?

In this paper we answer the question for the case $a = 0$, k arbitrary, $c = 1$, $b = r = 2$. In other words, we find a bound on the number of terms in a fixed arithmetic progression of the Thue-Morse sequence we have to look at in order to see a "1". We include some weaker results for arbitrary b and give some conjectures.

Remark 1. Jean-Paul Allouche notes that Buescu's original question can also be answered by appealing to a lemma in his paper [1, p. 284]. His lemma states that if a, b, c are integers with $b - c > a$, then $t_{an+b} - t_{an+c}$ cannot be constant for large n . If t_{An+B} were constant for some integers A, B then it would have the same value when replacing n by $n+2$. Thus $t_{An+2A+B} - t_{An+B}$ would be constant and equal to 0, but $2A + B - B = 2A > A$ and we are done.

Remark 2. Dartyge, Luca and Stănică [3] recently investigated another problem on the *pointwise* behavior of s_b on integer multiples, namely, to bound the

²We do not need the power of van der Waerden's theorem to prove this. For example, as we will see later, if $k = 2^r - 1$ for some $r \geq 1$, then $s_2(kn) = r$ for $1 \leq n \leq k$.

smallest nontrivial n that is congruent to $0 \pmod k$ and satisfies $s_b(n) = s_b(k)$. For other distributional properties of s_b on integer multiples we refer the interested reader to the bibliographic list in [3].

To begin with, for $k \geq 1$ we write

$$\mathcal{N}_k = \{n : t_{kn} = 1\}, \quad f(k) = \min\{n : n \in \mathcal{N}_k\}.$$

The first few values of $(f(k))_{k \geq 1}$ are given by

$$1, 1, 7, 1, 5, 7, 1, 1, 9, 5, 1, 7, 1, 1, 19, 1, 17, 9, 1, 5 \dots \quad (1)$$

The function f is of interest because of some old work of Newman [7]. Leo Moser observed that the first 7 multiples of 3 all have an even number of digits in their base-2 expansion. In our notation, this means $f(3) = 7$. Newman showed that among the first multiples of 3, there is always a small preponderance of those with even parity. More precisely, he showed that for all $x \geq 2$,

$$\frac{1}{20} \cdot (3x)^\alpha < |(\mathbb{N}_0 \setminus \mathcal{N}_3) \cap [0, x-1]| - |\mathcal{N}_3 \cap [0, x-1]| < 5 \cdot (3x)^\alpha, \quad (2)$$

where $\alpha = \log_4 3$. Coquet [2] gave a precise expression for the middle term in (2) that involves a continuous periodic fractal function with a completely explicit Fourier expansion.

From Gelfond's theorem we get that $f(k) < \infty$ for all k . Indeed, a simple observation shows that $f(k) = O(k)$. To see this, we need the following result (see [6] for the base b generalization). For the convenience of the reader we here include a full proof for $b = 2$.

Proposition 1. *Let $t, p \geq 1$. For all k with $1 \leq k < 2^t$ we have*

$$s_2(p2^t - k) = s_2(p-1) + t - s_2(k-1).$$

Proof. Write $k-1 = \sum_{i=0}^{t-1} \kappa_i 2^i$ with $\kappa_i \in \{0, 1\}$. Then

$$\begin{aligned} s_2(p2^t - k) &= s_2((p-1)2^t + 2^t - k) = s_2(p-1) + s_2((2^t - 1) - (k-1)) \\ &= s_2(p-1) + s_2\left(\sum_{i=0}^{t-1} (1 - \kappa_i) 2^i\right) = s_2(p-1) + \sum_{i=0}^{t-1} (1 - \kappa_i) \\ &= s_2(p-1) + t - \sum_{i=0}^{t-1} \kappa_i = s_2(p-1) + t - s_2(k-1). \end{aligned}$$

□

Now, consider $n = 2^{2r+1} - 1 \geq k$. Then

$$s_2(kn) = s_2(k2^{2r+1} - k) = s_2(k-1) + 2r + 1 - s_2(k-1) \equiv 1 \pmod{2},$$

so that $f(k) < 4k$.

Our main result shows that $k = 3$ is the first of an infinite class of integers that maximize $f(k) - k$.

Theorem 2. *For all $k \geq 1$ we have*

$$f(k) \leq k + 4. \tag{3}$$

Moreover, we have

- (i) $f(k) = k + 4$ if and only if $k = 2^{2r} - 1$ for some $r \geq 1$.
- (ii) There are no k with $f(k) = k + 3$ or $f(k) = k + 2$.
- (iii) $f(k) = k + 1$ if and only if $k = 6$.
- (iv) $f(k) = k$ if and only if $k = 1$ or $k = 2^r + 1$ for some $r \geq 2$.

The proof of Theorem 2 is constructive. It allows to show that for all $k \geq 1$ we can always find a small n with $t_{kn} = 1$ having Hamming weight at most 3.

Corollary 1. *For all k there is an $n \in \mathcal{N}_k$ with $n \leq k + 4$ and $s_2(n) \leq 3$.*

This is optimal in the sense that there are infinitely many k such that all $n \in \mathcal{N}_k$ satisfy $s_2(n) \geq 3$. In the following proposition we give such an infinite family.

Proposition 2. *Let $r \geq 4$ and set $k = 3 \cdot 2^r + 3$. If $s_2(kn) \equiv 1 \pmod{2}$ then $s_2(n) \geq 3$.*

Proof. Suppose that $n = 2^j$, $j \geq 0$. Then $s_2(k \cdot 2^j) = s_2(k) = s_2(2^{r+1} + 2^r + 2 + 1) = 4 \equiv 0 \pmod{2}$. Let $n = (2^j + 1) \cdot 2^{j'}$ for $j \geq 1$ and $j' \geq 0$. Then we have

$$s_2(kn) = s_2(3 \cdot 2^{r+j} + 3 \cdot 2^r + 3 \cdot 2^j + 3). \tag{4}$$

If $j = 1$ then $s_2(kn) = s_2(9 \cdot 2^r + 9) = 4$. Suppose $j > 1$ and without loss of generality assume that $r \geq j$. If $r > j + 1$ then the summands in (4) are noninterfering. This yields $s_2(kn) = 8$. If $r = j + 1$ then $s_2(kn) = s_2(3 \cdot 2^{2j+1} + 9 \cdot 2^j + 3)$ equals 6 or 4 depending on whether $j > 2$ or $j = 2$. The same is true in the case where $r = j$ and $s_2(kn) = s_2(3 \cdot 2^{2j} + 3 \cdot 2^{j+1} + 3)$. \square

The paper is structured as follows. We introduce some useful notation in Section 2 which allows us to perform addition in the binary expansion of integers in a well-arranged way. In Section 3 we shortly outline the idea of the proof of our main result. In Section 4 we state some auxiliary results which are based on a detailed investigation of various cases. Section 5 is devoted to the proof of Theorem 2. We conclude with some results in the general case, where the condition $s_2(kn) \equiv 1 \pmod{2}$ is changed to $s_b(kn) \equiv c \pmod{r}$ (Section 6).

2. Notation

In this section we introduce some notation. If

$$k = \varepsilon_{\ell-1}(k)2^{\ell-1} + \varepsilon_{\ell-2}(k)2^{\ell-2} + \cdots + \varepsilon_0(k)$$

is the canonical base-2 representation of k , satisfying $\varepsilon_j \in \{0, 1\}$ for all $0 \leq j < \ell$ and $\varepsilon_{\ell-1}(k) \neq 0$, then we let $(k)_2$ denote the binary word

$$\varepsilon_{\ell-1}(k)\varepsilon_{\ell-2}(k)\cdots\varepsilon_0(k).$$

Additionally, for each $k \in \mathbb{N}$ we let $\ell(k)$ denote the length of $(k)_2$; for $k \geq 1$ this is $\ell(k) = \lfloor \log_2 k \rfloor + 1$. If w_1 and w_2 are two binary words, then w_1w_2 denotes the binary word obtained by concatenation. The symbol a^n , $n \geq 1$, $a \in \{0, 1\}$ is an abbreviation for the word

$$\overbrace{aa \cdots a}^n,$$

and a^0 is equal to the empty word. For $a \in \{0, 1\}$, we use the notation $\bar{a} = 1 - a$. We define the function s for all binary words $w = \varepsilon_{j-1} \cdots \varepsilon_0$ by

$$s(w) = \#\{i, 0 \leq i < j : \varepsilon_i = 1\},$$

and in particular, we have $s((k)_2) = s_2(k) \equiv t_k \pmod{2}$. If $1 \leq j \leq \ell(k)$, we set

$$L_j(k) = \varepsilon_{j-1} \cdots \varepsilon_0,$$

the j least significant bits of k in base 2, and

$$U_j(k) = \varepsilon_{\ell(k)-1} \cdots \varepsilon_{\ell(k)-j},$$

the j most significant bits of k . For example, if $k = 119759$, then we have $(k)_2 = 11101001111001111$, $\ell(k) = 17$ and

$$\overbrace{1110100111}^{U_{12}(k)} \underbrace{11001111}_{L_8(k)}.$$

Written in short form, this means that $(k)_2 = 1^3 0 1 0^2 1^4 0^2 1^4$,

$$U_{12}(k) = 1^3 0 1 0^2 1^4 0 \quad \text{and} \quad L_8(k) = 1^2 0^2 1^4.$$

In what follows, we use the convention that if we are talking about $L_j(k)$ or $U_j(k)$, we assume that $\ell(k) \geq j$. Note that for all $k \in \mathbb{N}$ and $j < \ell(k)$ we have

$$s(L_{\ell(k)-j}(k)) \equiv s(U_j(k)) + t_k \pmod{2}.$$

Furthermore, this function also satisfies

$$s(w_1 w_2) = s(w_1) + s(w_2)$$

for two binary words w_1 and w_2 .

3. Idea of proof

It is relatively easy to show that $f(k) = k + 4$ if $k = 2^{2^r} - 1$ for some $r \geq 1$ and $f(k) = k$ if $k = 2^n + 1$ for some $n \geq 2$ (see the proofs of Theorem 2 and Lemma 1). Moreover, since $f(k) = f(2k)$ for all $k \geq 1$, in order to prove Theorem 2, it suffices to show that $f(k) < k$ for all odd integers k other than those stated above. Thus, we assume in Section 4 that k is an odd integer.

We use two different ideas in order to succeed, depending on the base-2 representation of k . We show for a large set of integers k that there exists an integer $n < k$ with Hamming weight 2 such that $t_{kn} = 1$. To be more precise, we find for such integers k a positive integer $a < \ell(k)$ such that $t_{kn} = 1$ with $n = 2^a + 1 \leq 2^{\ell(k)-1} + 1 < k$. For the remaining odd integers k we show that there exist positive odd integers $m < k$ with Hamming weight 2 and $n < k$ with Hamming weight 3 such that

$$t_{kn} \equiv 1 + t_k + t_{km} \pmod{2}.$$

This implies that $f(k) < k$ since at least one of the three numbers t_k , t_{km} and t_{kn} has to be equal to 1.

4. Auxiliary results

We have to distinguish several cases according to the beginning and the ending part of the binary expansion of k .

Lemma 1. *Let $k \in \mathbb{N}$ be such that there exists an odd integer $u \geq 1$ with $L_{u+1}(k) = 01^u$. Then we have $f(k) \leq k$. Furthermore, $f(k) = k$ if and only if $k = 2^r + 1$ for some $r \geq 2$.*

Proof. Let $\ell = \ell(k)$ and set $n = 2^{\ell-1} + 1$. In what follows we show that $t_{kn} = 1$. We have

$$(kn)_2 = U_{\ell-(u+1)}(k) 1 0^u L_{\ell-1}(k).$$

The following figure explains this fact:

$$\begin{array}{r} \dots 0 1^{u-1} 1 \\ 1 \dots \\ \hline \dots 1 0^{u-1} 0 \dots \end{array}.$$

The first line $(\dots 0 1^{u-1} 1)$ corresponds to the expansion of $k2^{\ell-1}$ and the second line $(1 \dots)$ to the expansion of k . By “ \dots ” we refer to digits that are not important for our argument. Since

$$s(U_{\ell-(u+1)}(k)) \equiv s(L_{u+1}(k)) + t_k \equiv u + t_k \pmod{2},$$

and

$$s(L_{\ell-1}(k)) \equiv s(U_1(k)) + t_k \equiv 1 + t_k \pmod{2},$$

we obtain

$$t_{kn} \equiv s((kn)_2) \equiv u + t_k + 1 + 1 + t_k \equiv u \equiv 1 \pmod{2},$$

which shows that $t_{kn} = 1$. The definition of $\ell = \ell(k)$ implies that $2^{\ell-1} + 1 \leq k$. If $k = 2^{\ell-1} + 1$, we have $t_{km} = 0$ for all $1 \leq m < k$. Indeed, if $1 \leq m < 2^{\ell-1}$, then the 2-additivity of the binary sum-of-digits function s_2 implies

$$s_2(km) = s_2(2^{\ell-1}m + m) = s_2(m) + s_2(m).$$

Thus we have $t_{km} \equiv s_2(km) \equiv 0 \pmod{2}$ for all $1 \leq m < 2^{\ell-1}$. If $m = 2^{\ell-1}$, then we clearly have $t_{km} = 0$. This finally proves that $f(k) = k$ if $k = 2^{\ell-1} + 1$ and $f(k) < k$ if k satisfies the assumptions of Lemma 1 but $k \neq 2^{\ell-1} + 1$. \square

Lemma 2. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geq 2$ with $L_{u+2}(k) = 101^u$, then we have $f(k) < k$.*

Proof. Set $\ell = \ell(k)$. First, we show that if there exist a positive integer $r \neq u$ such that $U_{r+1}(k) = 1^r 0$, then $f(k) < k$. If $r < u$, we set $n = 2^{\ell-(r+1)} + 1 < k$. Then we have

$$(kn)_2 = U_{\ell-(u+1)}(k) 1 0^{u-(r+1)} 1^{r-1} 0 1 L_{\ell-r-1}(k),$$

as illustrated below:

$$\begin{array}{r} \dots 0 1^{u-(r+1)} 1^{r-1} 1 1 \\ \phantom{0 1^{u-(r+1)}} 1^{r-1} 1 0 \dots \\ \hline \dots 1 0^{u-(r+1)} 1^{r-1} 0 1 \dots \end{array}.$$

Since $s(U_{\ell-(u+1)}(k)) \equiv s(L_{u+1}(k)) + t_k \equiv u + t_k \pmod{2}$ and $s(L_{\ell-r-1}(k)) \equiv s(U_{r+1}) + t_k \equiv r + t_k$, we get

$$t_{kn} \equiv u + t_k + 1 + (r - 1) + 1 + r + t_k \equiv u + 1 \pmod{2}. \quad (5)$$

This shows that $f(k) < k$ if $r < u$ since u is even. If $r > u$, we set $n = 2^{\ell-u} + 2^{\ell-u-1} + 1$. Since $\ell - u < \ell - 1$ we have $n < k$. We get

$$(kn)_2 = U_{\ell+2-(u+2)}(3k) 1 0 1^{u-2} 0^2 L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{r} \dots 1 0 1 1^{u-2} 1 \\ 1^{u-2} 1 1 \\ \phantom{1^{u-2}} 1 1 \dots \\ \hline \dots 1 0 1^{u-2} 0 0 \dots \end{array}.$$

Noting that $3k$ has $\ell + 2$ digits, i.e., $\ell(3k) = \ell + 2$, we obtain

$$\begin{aligned} t_{kn} &\equiv s(L_{u+2}(3k)) + t_{3k} + 1 + (u - 2) + s(U_{u+1}(k)) + t_k \\ &\equiv s(L_{u+2}(3k)) + t_{3k} + u - 1 + u + 1 + t_k \\ &\equiv s(L_{u+2}(3k)) + t_{3k} + t_k \pmod{2}. \end{aligned}$$

Since

$$L_{u+2}(3k) = 0^2 1^{u-2} 0 1,$$

we have $t_{kn} \equiv 1 + t_k + t_{3k} \pmod{2}$. As we have seen in Section 3, this implies $f(k) < k$.

For the rest of the proof we assume that $U_{u+1}(k) = 1^u 0$ for some integer $u \geq 2$. If $(k)_2 = 1^u 0 1^u$, then it is easy to see that $f(k) = 3$. Thus we can assume that there exists a positive integer v such that $L_{v+u+2}(k) = 0 1^v 0 1^u$. If v is odd, then we set $n = 2^{\ell-(u+1)} + 1 < k$. We get

$$(kn)_2 = U_{\ell-(v+u+2)}(k) 1 0^{v+1} 1^{u-2} 0 1 L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{ccccccc} \dots & 0 & 1^v & 0 & 1^{u-2} & 1 & 1 \\ & & & & 1 & 1^{u-2} & 1 & 0 & \dots \\ \hline \dots & 1 & 0^v & 0 & 1^{u-2} & 0 & 1 & \dots \end{array}.$$

This implies

$$\begin{aligned} t_{kn} &\equiv s(L_{v+u+2}(k)) + t_k + 1 + (u-2) + 1 + s(U_{u+1}(k)) + t_k \\ &\equiv u + v + u \equiv 1 \pmod{2}. \end{aligned}$$

If v is even, we have two cases to consider: $u \geq 4$ and $u = 2$.

If $u \geq 4$, we set $n = 2^{\ell-u} + 2^{\ell-u-1} + 1 < k$. Then we have

$$(kn)_2 = U_{\ell+2-(v+u+2)}(3k) 0 1^v 0 1^{u-3} 0 1^2 L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{ccccccc} \dots & 0 & 1 & 1^{v-1} & 0 & 1 & 1^{u-3} & 1 & 1 \\ & & & & \dots & 0 & 1^{v-1} & 1 & 0 & 1^{u-3} & 1 & 1 & 1 \\ & & & & & & & & & 1 & 1^{u-3} & 1 & 1 & 0 & \dots \\ \hline \dots & 0 & 1^{v-1} & 1 & 0 & 1^{u-3} & 0 & 1 & 1 & \dots \end{array}.$$

Note that

$$L_{v+u+2}(3k) = 0 1^{v-1} 0^2 1^{u-2} 0 1. \tag{6}$$

Thus we get

$$\begin{aligned} t_{kn} &\equiv s(L_{v+u+2}(3k)) + t_{3k} + v + (u-3) + 2 + s(U_{u+1}(k)) + t_k \\ &\equiv (v-1) + (u-2) + 1 + t_{3k} + v + (u-3) + 2 + u + t_k \\ &\equiv 1 + t_{3k} + t_k \pmod{2}, \end{aligned}$$

and we obtain $f(k) < k$.

Now we consider the case $u = 2$. In order to complete the proof of the lemma, it remains to show that $f(k) < k$ for integers k with $U_3(k) = 1^20$ and $L_{v+4}(k) = 01^v01^2$ for an even positive integer v . If $U_4(k) = 1^201$, then we set $n = 2^{\ell-4} + 1$. Here we get

$$(kn)_2 = U_{\ell-(v+4)}(k) 1 0^{v-1} 1 0^3 L_{\ell-4}(k),$$

as illustrated below:

$$\begin{array}{r} \dots 0 1^{v-1} 1 0 1 1 \\ \phantom{0 1^{v-1}} 1 1 0 1 \dots \\ \hline \dots 1 0^{v-1} 1 0 0 0 \dots \end{array},$$

and we obtain

$$\begin{aligned} t_{kn} &\equiv s(L_{v+4}(k)) + t_k + 1 + 1 + s(U_4(k)) + t_k \\ &\equiv (v+2) + 3 \equiv 1 \pmod{2}. \end{aligned}$$

If $U_5(k) = 1^20^3$, we set $n = 2^{\ell-4} + 2^{\ell-5} + 1$. It follows that

$$(kn)_2 = U_{\ell+2-(v+4)}(3k) 1 0^{v-1} 1 0^2 1 L_{\ell-5}(k),$$

as illustrated below:

$$\begin{array}{r} \dots 0 1 1^{v-2} 1 0 1 1 \\ \dots 0 1^{v-2} 1 1 0 1 1 \\ \phantom{0 1^{v-2}} 1 1 0 0 0 \dots \\ \hline \dots 1 0^{v-2} 0 1 0 0 1 \dots \end{array},$$

and we obtain

$$\begin{aligned} t_{kn} &\equiv s(L_{v+4}(3k)) + t_{3k} + 1 + 1 + 1 + s(U_5(k)) + t_k \\ &\equiv v + t_{3k} + 1 + 2 + t_k \equiv 1 + t_{3k} + t_k \pmod{2}. \end{aligned}$$

Here we used Eq. (6) and we get $f(k) < k$.

If $U_5(k) = 1^20^21$, we set $n = 2^{\ell-3} + 2^{\ell-1} + 1$. It is easy to see that $\ell(5k) = \ell + 2$ or $\ell(5k) = \ell + 3$. We have

$$(kn)_2 = U_{\ell(5k)-(v+4)}(5k) 1 0^{v+3} L_{\ell-5}(k),$$

as illustrated below:

$$\begin{array}{r}
\dots 0 1 1 1^{v-2} 0 1 1 \\
\dots 0 1^{v-2} 1 1 0 1 1 \\
\phantom{\dots 0 1 1 1^{v-2} 0 1 1} 1 1 0 0 1 \dots \\
\hline
\dots 1 0^{v-2} 0 0 0 0 0 \dots
\end{array}$$

Since

$$L_{v+4}(5k) = 0 1^{v-2} 0^2 1^3,$$

we obtain

$$\begin{aligned}
t_{kn} &\equiv s(L_{v+4}(5k)) + t_{5k} + 1 + s(U_5(k)) + t_k \\
&\equiv (v+1) + t_{5k} + 1 + 3 + t_k \equiv 1 + t_{5k} + t_k \pmod{2}.
\end{aligned}$$

Again, the considerations of Section 3 show that $f(k) < k$. \square

Lemma 3. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geq 2$ and a positive integer $r \neq u$ such that $L_{u+2}(k) = 0^2 1^u$ and $U_{r+1}(k) = 1^r 0$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$. If $r < u$, we set $n = 2^{\ell-r-1} + 1 < k$. In exactly the same manner as at the beginning of the proof of Lemma 2 (see Eq. (5)), we see that $t_{kn} = 1$ and thus, $f(k) < k$.

If $r > u$, we set $n = 2^{\ell-u-1} + 1 < k$. Then we get

$$(kn)_2 = U_{\ell-(u+2)}(k) 1 0 1^{u-1} 0 L_{\ell-(u+1)}(k),$$

as illustrated below:

$$\begin{array}{r}
\dots 0 0 1^{u-1} 1 \\
\phantom{\dots 0 0 1^{u-1} 1} 1 1^{u-1} 1 \dots \\
\hline
\dots 1 0 1^{u-1} 0 \dots
\end{array}$$

Similarly as before, we have $s(U_{\ell-(u+2)}(k)) \equiv s(L_{u+2}(k)) + t_k \equiv u + t_k \pmod{2}$ and $s(L_{\ell-(u+1)}(k)) \equiv s(U_{u+1}) + t_k \equiv u + 1 + t_k$ (note that $u + 1 \leq r$). We obtain

$$t_{kn} \equiv u + t_k + 1 + (u - 1) + (u + 1) + t_k \equiv u + 1 \equiv 1 \pmod{2}.$$

This shows the desired result. \square

Lemma 4. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geq 2$ and a positive integer $s < u - 1$ such that $L_{u+2}(k) = 0^2 1^u$ and $U_{u+s+1}(k) = 1^u 0^s 1$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$ and set $n = 2^{\ell-1} + 2^{u-1} + 1$. Since k is odd and starts with at least two 1's, we see that $n < k$. We have

$$(nk)_2 = U_{\ell-(u+2)}(k) 1 0^{u+s+1} L_{\ell(km)-(s+u+1)}(mk),$$

where $m = 2^{u-1} + 1 < k$ and $\ell(km) = \ell + u$, as illustrated below:

$$\begin{array}{r} \dots 0 0 1^{u-1} 1 \\ 1^{u-1} 1 0^s 1 \dots \\ 1 1^s 1 \dots \\ \hline \dots 1 0 0^{u-1} 0 0^s \dots \end{array}.$$

We have

$$U_{s+u+1}(km) = 1 0^{u-1} 10^s.$$

In particular, we obtain

$$s(L_{\ell(km)-(s+u+1)}(mk)) \equiv s(U_{s+u+1}(km)) + t_{km} \equiv t_{km} \pmod{2},$$

and we get

$$t_{nk} \equiv s(L_{u+2}(k)) + t_k + 1 + t_{km} \equiv u + 1 + t_k + t_{km} \equiv 1 + t_k + t_{km} \pmod{2}.$$

As before, we get $f(k) < k$, which proves Lemma 4. \square

Lemma 5. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geq 2$ and a positive integer $t \geq 2$ such that $L_{u+t+2}(k) = 010^t 1^u$ and $U_{2u-1}(k) = 1^u 0^{u-1}$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$ and let us first assume that $2 \leq t \leq u - 1$. We set $n = 2^{\ell-(u+t)} + 1 < k$. Then we get

$$(kn)_2 = U_{\ell-(u+t+2)}(k) 1 0^{t+1} 1^{u-(t+1)} 0 1^t L_{\ell-(u+t)}(k),$$

as illustrated below:

$$\begin{array}{r} \dots 0 1 0^t 1^{u-(t+1)} 1 1^t \\ 1^t 1^{u-(t+1)} 1 0^t \dots \\ \hline \dots 1 0 0^t 1^{u-(t+1)} 0 1^t \dots \end{array}.$$

Since $u < u + t \leq 2u - 1$, we have

$$s(L_{\ell-(u+t)}(k)) \equiv s(U_{u+t}(k)) + t_k \equiv u + t_k \pmod{2}.$$

This implies

$$t_{kn} \equiv s(L_{u+t+2}(k)) + t_k + 1 + (u - (t + 1)) + t + u + t_k \equiv u + 1 \equiv 1 \pmod{2}.$$

If $t = u$ and $U_{2u}(k) = 1^u 0^{u-1} 1$, we again set $n = 2^{\ell-(u+t)} + 1 < k$. This time we can write

$$(kn)_2 = U_{\ell-(u+t+2)}(k) 1 0^{t+u+1} L_{\ell-(u+t)}(k),$$

as illustrated below:

$$\begin{array}{r} \dots 0 1 0^t 1^{u-1} 1 \\ 1^t 0^{u-1} 1 \dots \\ \hline \dots 1 0 0^t 0^{u-1} 0 \dots \end{array},$$

and we get

$$t_{kn} \equiv s(L_{u+t+2}(k)) + t_k + 1 + s(U_{u+t}(k)) + t_k \equiv (u+1) + 1 + (u+1) \equiv 1 \pmod{2}.$$

Alternatively, if $t = u$ and $U_{2u+1}(k) = 1^u 0^u a$ for some $a \in \{0, 1\}$, then we set $n = 2^{\ell-(t+u+1)} + 1 < k$. Since we have (recall that $\bar{a} = 1 - a$)

$$(kn)_2 = U_{\ell-(u+t+2)}(k) 1 0 1^{t-1} a \bar{a}^u L_{\ell-(u+t+1)}(k),$$

as illustrated below,

$$\begin{array}{r} \dots 0 1 0^{t-1} 0 1^{u-1} 1 \\ 1 1^{t-1} 0 0^{u-1} a \dots \\ \hline \dots 1 0 1^{t-1} a \bar{a}^{u-1} \bar{a} \dots \end{array},$$

and we finally obtain

$$\begin{aligned} t_{kn} &\equiv s(L_{u+t+2}(k)) + t_k + 1 + (t - 1) + a + au + s(U_{u+t+1}(k)) + t_k \\ &\equiv (u + 1) + 1 + (t - 1) + a(u + 1) + (u + a) \\ &\equiv 1 \pmod{2}. \end{aligned}$$

This shows the desired result. \square

Lemma 6. *Let $k \in \mathbb{N}$. If there exist an even integer $u \geq 2$ and positive integers $t \geq 2$ such that $L_{u+t+2}(k) = 110^t 1^u$ and $U_{2u-1}(k) = 1^u 0^{u-1}$, then we have $f(k) < k$.*

Proof. Let $\ell = \ell(k)$. First, we consider the case $2 \leq t \leq u - 1$. Set $n = 2^{\ell-(t+u-1)} + 2^{\ell-(t+u)} + 1 < k$. Then we have

$$(kn)_2 = U_{\ell+2-(t+u+2)}(3k) 1 0^{t-1} 1 0 1^{u-(t+1)} 0 1^{t-2} 0 1 L_{\ell-(u+t)}(k),$$

as illustrated below:

$$\begin{array}{r} \dots 1 1 0 0^{t-2} 0 1 1^{u-(t+1)} 1 1^{t-2} 1 \\ \dots 1 1 0^{t-2} 0 0 1^{u-(t+1)} 1 1^{t-2} 1 1 \\ \quad \quad \quad 1^{t-2} 1 1 1^{u-(t+1)} 1 0^{t-2} 0 0 \dots \\ \hline \dots 1 0 0^{t-2} 1 0 1^{u-(t+1)} 0 1^{t-2} 0 1 \dots \end{array}.$$

Noting that $\ell(3k) = \ell + 2$, we obtain

$$\begin{aligned} t_{kn} &\equiv s(L_{u+t+2}(3k)) + t_{3k} + (u - (t + 1)) + (t - 2) + s(U_{u+t}(k)) + t_k + 3 \\ &\equiv s(L_{u+t+2}(3k)) + t_{3k} + u + u + t_k \\ &\equiv s(L_{u+t+2}(3k)) + t_{3k} + t_k \pmod{2}. \end{aligned}$$

Since

$$L_{u+t+2}(3k) = 0 1 0^{t-2} 1 0 1^{u-2} 0 1, \quad (7)$$

we have $t_{kn} \equiv 1 + t_{3k} + t_k \pmod{2}$ and consequently $f(k) < k$.

If $t = u$, then we have to consider three different cases. If $U_{2u}(k) = 1^u 0^{u-1} 1$ we again set $n = 2^{\ell-(t+u-1)} + 2^{\ell-(t+u)} + 1 < k$. This time we get

$$(kn)_2 = U_{\ell+2-(t+u+2)}(3k) 1 0^t 1^u 0 L_{\ell-(2u)}(k),$$

as illustrated below,

$$\begin{array}{r} \dots 1 1 0 0^{t-1} 1 1^{u-1} \\ \dots 1 1 0^{t-1} 0 1^{u-1} 1 \\ \quad \quad \quad 1^{t-1} 1 0^{u-1} 1 \dots \\ \hline \dots 1 0 0^{t-1} 1 1^{u-1} 0 \dots \end{array},$$

which yields

$$\begin{aligned} t_{kn} &\equiv s(L_{u+t+2}(3k)) + t_{3k} + 1 + u + s(U_{2u}(k)) + t_k \\ &\equiv s(L_{u+t+2}(3k)) + t_{3k} + 1 + u + (u + 1) + t_k \\ &\equiv s(L_{u+t+2}(3k)) + t_{3k} + t_k \pmod{2}. \end{aligned}$$

As above (see Eq. (7)), we get that $f(k) < k$. If $t = u$ and $U_{2u+1}(k) = 1^u 0^u 1$, then we set $n = 2^{\ell-(t+u)} + 2^{\ell-(t+u)-1} + 1 < k$. We have

$$(kn)_2 = U_{\ell+2-(t+u+2)}(3k) 1^2 0^t 1^{u-1} 0 L_{\ell-(2u+1)}(k),$$

as illustrated below.

$$\begin{array}{cccccccc} \dots & 1 & 1 & 0 & 0^{t-1} & 1 & 1^{u-1} & \\ & \dots & 1 & 1 & 0^{t-1} & 0 & 1^{u-1} & 1 \\ & & & & 1 & 1^{t-1} & 0 & 0^{u-1} & 1 & \dots \\ \hline & \dots & 1 & 1 & 0^{t-1} & 0 & 1^{u-1} & 0 & \dots & \end{array}.$$

Note that the dots in the first and second line of the figure have to be erased if $k = 51$. (The binary representation of 51 is given by $(51)_2 = 110011$.) We get

$$\begin{aligned} t_{kn} &\equiv s(L_{u+t+2}(3k)) + t_{3k} + 2 + (u-1) + s(U_{2u+1}(k)) + t_k \\ &\equiv s(L_{u+t+2}(3k)) + t_{3k} + 2 + (u-1) + (u+1) + t_k \\ &\equiv s(L_{u+t+2}(3k)) + t_{3k} + t_k \pmod{2}. \end{aligned}$$

Using Eq. (7), we obtain $f(k) < k$. If $t = u$ but $U_{2u+1}(k) = 1^u 0^{u+1}$, then we choose $n = 2^{\ell-1} + 2^u + 1$. Since k is odd and starts with at least two 1's, we again obtain that $n < k$. This leads us to

$$(kn)_2 = U_{\ell-(u+2)}(k) 1 0 1^{u-1} 0 1^{u-1} L_{\ell+u-(2u)}(km),$$

where $m = 2^u + 1 < k$, as illustrated below:

$$\begin{array}{cccccccc} \dots & 0 & 0 & 1^{u-1} & 1 & & & \\ & & 1 & 1^{u-1} & 0 & 0^{u-1} & 0 & \dots \\ & & & & 1 & 1^{u-1} & 0 & \dots \\ \hline & \dots & 1 & 0 & 1^{u-1} & 0 & 1^{u-1} & \dots & \end{array}.$$

Note that $\ell(km) = \ell + u$ and

$$U_{2u}(km) = 1^{2u}.$$

We obtain

$$\begin{aligned} t_{kn} &\equiv s(L_{u+2}(k)) + t_k + 1 + (u-1) + (u-1) + s(U_{2u}(km)) + t_{km} \\ &\equiv u + t_k + 1 + 2u + t_{km} \\ &\equiv 1 + t_k + t_{km} \pmod{2}, \end{aligned}$$

which shows $f(k) < k$ in this case, too.

In order to prove the lemma, it remains to consider the case $t > u$. We set $n = 2^{\ell-1} + 2^{\ell-u-1} + 1 < k$ and we get

$$(kn)_2 = U_{\ell+u-(2u+1)}(km) 1 0^{u-1} 1^{u-1} 0 1 L_{\ell-(u+1)}(k),$$

as illustrated below,

$$\begin{array}{r} \dots 0 1^{u-1} 1 \\ \dots 0 0^{u-1} 0 1^{u-2} 1 1 \\ \qquad \qquad \qquad 1 1^{u-2} 1 0 \dots \\ \hline \dots 1 0^{u-1} 1 1^{u-2} 0 1 \dots \end{array},$$

where $m = 2^u + 1 < k$. Since

$$L_{2u+1}(km) = 0 1^{2u},$$

we obtain

$$\begin{aligned} t_{kn} &\equiv s(L_{2u+1}(km)) + t_{km} + 1 + (u-1) + 1 + s(U_{u+1}(k)) + t_k \\ &\equiv 2u + t_{km} + u - 1 + u + t_k \\ &\equiv 1 + t_k + t_{km} \pmod{2}. \end{aligned}$$

The same argument as before finally shows the desired result. \square

5. Proof of Theorem 2

Proof of Theorem 2. As already noted in Section 3, we have $f(k) = f(2k)$ for all $k \geq 1$. Consequently, it suffices to show that $f(k) \leq k + 4$ for odd integers k .

If $k = 2^{2r+1} - 1$, $r \geq 0$, then $(k)_2 = 1^{2r+1}$ and we trivially have $f(k) = 1$. If $k = 2^{2r} - 1$, $r \geq 1$, then we will show that $f(k) = k + 4$. In order to do this, we note that the binary sum-of-digits function s_2 satisfies the relation

$$s_2(a2^k - b) = s_2(a - 1) + k - s_2(b - 1)$$

for all positive integers a, b, k with $1 \leq b < 2^k$. Thus we have for all $1 \leq m \leq k$,

$$t_{km} \equiv s_2(2^{2r}m - m) = s_2(m - 1) + 2r - s_2(m - 1) \equiv 0 \pmod{2}.$$

If $m = 2^{2r}$ or $m = 2^{2r} + 2$, then we clearly have $t_{km} = 0$ since $t_{km/2} = 0$. If $m = 2^{2m} + 1$, then $km = 2^{4r} - 1$ and consequently $t_{km} = 0$. If $m = 2^{2r} + 3$, then

$$t_{km} \equiv s_2(2^{4r} + 2^{2r+1} - 3) \equiv 1 + (2r + 1) - s_2(2) \equiv 1 \pmod{2},$$

which finally proves $f(k) = k + 4$ for $k = 2^{2r} - 1$, $r \geq 1$. If k is a positive integer different from ones already considered, then there exist positive integers r, s, t and u such that

$$U_{r+s+1}(k) = 1^r 0^s 1 \quad \text{and} \quad L_{t+u+1} = 1 0^t 1^u.$$

If u is odd, then Lemma 1 implies that $f(k) \leq k$ where equality occurs if and only if $k = 2^r + 1$ for some $r \geq 2$. If u is even but $t = 1$ or $r \neq u$, then Lemma 2 and Lemma 3 imply $f(k) < k$. Let us assume that u is even, $t \geq 2$ and $r = u$. Then there exists $a \in \{0, 1\}$, such that

$$L_{t+u+2} = a 1 0^t 1^u.$$

If $s < u - 1$, then Lemma 4 implies $f(k) < k$. Contrarily, if $s \geq u - 1$, then Lemma 5 (if $a = 0$) or Lemma 6 (if $a = 1$) yields $f(k) < k$. Hence we have for all positive integers k ,

$$f(k) \leq k + 4,$$

where equality occurs if and only if $k = 2^{2r} - 1$, $r \geq 1$. Note that an even positive integer $2m$ cannot satisfy $f(2m) = 2m + 4$, since we then would get $f(m) = f(2m) = 2m + 4 \leq m + 4$ and consequently $m \leq 0$. Moreover, we see that for odd integers k there exist no solutions to the equation

$$f(k) = k + \alpha$$

for $\alpha = 0, 1, 2, 3$, except in the case $\alpha = 0$ where we have $f(k) = k$ if and only if $k = 2^r + 1$ for some $r \geq 2$ or $k = 1$. If $k = 2m$ is even, then $f(2m) = 2m + \alpha$ implies $f(m) = 2m + \alpha \leq m + 4$. Hence this can only happen if $m \leq 4 - \alpha$. We see that there exist no solutions to $f(k) = k + \alpha$ for $\alpha = 2$ and $\alpha = 3$, there are no even solutions for $\alpha = 0$ and the only solution to $f(k) = k + 1$ is $k = 6$ (compare with Eq. (1)). This finally proves Theorem 2. \square

Remark 3. By a similar case analysis it might be possible to prove that

$$\min\{n : t_{kn} = 0\} \leq k + 2.$$

However, it does not seem possible to obtain this bound in a direct way from the bound (2).

6. Some weak general results

Given the generality of Gelfond's theorem, it is natural to try to bound the minimal n such that $n \equiv a \pmod{k}$ and $s_b(n) \equiv c \pmod{r}$. Here we only get a weaker upper bound.

Proposition 3. *Let b, r, k be positive integers with $\gcd(b-1, r) = 1$, and let c be any integer. Then there exists a non-negative integer $n < b^r k$ such that $s_b(kn) \equiv c \pmod{r}$.*

Proof. We claim that if $1 \leq k \leq b^t$, then $s_b(k(b^t - 1)) = (b-1)t$. To see this, note that for $p, t \geq 1$ and all k with $1 \leq k < b^t$ we have

$$s_b(pb^t - k) = s_b(p-1) + (b-1)t - s_b(k-1).$$

This is a direct generalization of Proposition 1, and a complete proof can be found in [6]. Let s be the smallest integer such that $k \leq b^s$. Then $b^{s-1} < k$. Choose $t \in \{s, s+1, \dots, s+r-1\}$ such that $(b-1)t \equiv c \pmod{r}$. This is possible since $\gcd(b-1, r) = 1$. Then $s_b(k(b^t - 1)) = (b-1)t \equiv c \pmod{r}$, as desired. Furthermore, $b^t \leq b^{s+r-1} \leq b^r b^{s-1} < b^r k$. Thus we can take $n = b^t - 1$. \square

Corollary 2. *Let b, r, k be positive integers with $\gcd(b-1, r) = 1$, and let a, c be any integers. Then there exists an integer $n < b^{r+1}k^3$ such that $n \equiv a \pmod{k}$ and $s_b(n) \equiv c \pmod{r}$.*

Proof. Without loss of generality we can assume $0 \leq a < k$. As in the proof of Proposition 3 let s be the smallest integer such that $b^s \geq k$, so $b^{s-1} < k$. From Proposition 3 we know that there exists an integer t such that $s_b(k(b^t - 1)) \equiv (c - a) \pmod{r}$, and $b^t < b^r k$. Then clearly $s_b(kb^s(b^t - 1) + a) \equiv c \pmod{r}$, so we can take $n = kb^s(b^t - 1) + a$. Then $n < b^{r+1}k^3$. \square

In the setting of Proposition 3 we conjecture that a similar phenomenon takes place as we have seen in the case of the classical Thue-Morse sequence.

Conjecture 3. *Let b, r be positive integers with $\gcd(b-1, r) = 1$, and let c be any integer. There exists a constant C , depending only on b and r such that for all $k \geq 1$ there exists $n \leq k + C$ with $s_b(kn) \equiv c \pmod{r}$. Furthermore, we can take $C \leq b^{r+c}$.*

We guess that this conjecture is hard to prove. In the case of the Thue-Morse sequence we used a detailed case study to succeed. In principle, in each of our lemmas we make use of a new idea to get a parity change. We do not see how this extends to the general setting where b , r and c all vary freely.

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References

- [1] J.-P. Allouche. Somme des chiffres et transcendance. *Bull. Soc. Math. France* **110** (1982), 279–285. Electronically available at <http://archive.numdam.org>.
- [2] J. Coquet. A summation formula related to the binary digits. *Invent. Math.* **73** (1983), 107–115.
- [3] C. Dartyge, F. Luca, P. Stănică. On digit sums of multiples of an integer. *J. Number Theory* **129** (2009), 2820–2830.
- [4] N. J. Fine. The distribution of the sum of digits (mod p). *Bull. Amer. Math. Soc.* **71** (1965), 651–652. Electronically available at <http://www.ams.org/publications/journals/journalsframework/bull>.
- [5] A. O. Gelfond. Sur les nombres qui ont des propriétés additives et multiplicatives données. *Acta Arith.* **13** (1967/1968), 259–265. Electronically available at <http://matwbn.icm.edu.pl>.
- [6] K. G. Hare, S. Laishram, T. Stoll. Stolarsky’s conjecture and the sum of digits of polynomial values. *Proc. Amer. Math. Soc.* **139** (2011), 39–49.
- [7] D. J. Newman. On the number of binary digits in a multiple of three. *Proc. Amer. Math. Soc.* **21** (1969), 719–721.