

Decomposition of perturbed Chebyshev polynomials

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Abstract

We characterize polynomial decomposition $f_n = r \circ q$ with $r, q \in \mathbb{C}[x]$ of perturbed Chebyshev polynomials defined by the recurrence

$$f_0(x) = b, \quad f_1(x) = x - c, \quad f_{n+1}(x) = (x - d)f_n(x) - af_{n-1}(x), \quad n \geq 1,$$

where $a, b, c, d \in \mathbb{R}$ and $a > 0$. These polynomials generalize the Chebyshev polynomials, which are obtained by setting $a = 1/4$, $c = d = 0$ and $b \in \{1, 2\}$. At the core of the method, two algorithms for polynomial decomposition are provided, which allow to restrict the investigation to the resolution of six systems of polynomial equations in three variables. The final task is then carried out by the successful computation of reduced Gröbner bases with Maple 10. Some additional data for the calculations are available on the author's web page.

1 Introduction

Let $a, b, c, d \in \mathbb{R}$, $a > 0$ and consider the polynomials $f_n(x)$ of degree n defined by the three-term recurrence

$$\begin{aligned} f_0(x) &= b, \\ f_1(x) &= x - c, \\ f_{n+1}(x) &= (x - d)f_n(x) - af_{n-1}(x), \quad n \geq 1. \end{aligned} \tag{1.1}$$

These polynomials generalize the monic *Chebyshev polynomials of the first kind* $t_n(x)$ and *Chebyshev polynomials of the second kind* $u_n(x)$, which are obtained for $(a, b, c, d) = (\frac{1}{4}, 2, 0, 0)$ and $(\frac{1}{4}, 1, 0, 0)$, respectively. We point out, that there already exists a notion of so-called *generalized Chebyshev polynomials* in several complex indeterminates x_1, \dots, x_k (see [17, Chapter 2, pp.26]) based on a representation involving symmetric functions. In this paper, however, we are concerned with a different type of generalization and will only deal with polynomials in one variable x .

To begin with, we note some known special cases of (1.1). If $b = 1$ and $d = 0$, then (1.1) reflects the definition of so-called *co-recursive* Chebyshev polynomials, which were first studied by

Geronimus [11] and subsequently generalized to other classical orthogonal polynomials by Chihara [3], Slim [24], Dini *et al.* [6], Marcellán *et al.* [19], Ifantis & Siafarikas [14] and Foupouagnigni *et al.* [10]. The additional parameter $b \in \mathbb{R}$ allows some interesting polynomial families. Mention, for instance, co-recursive versions of *Fermat polynomials* for $(a, b) \equiv (2, 1)$ and of *Fermat-Lucas polynomials* for $(a, b) \equiv (2, 2)$ (see [29]). The recurrence (1.1) with generic constant parameters $a, b, c, d \in \mathbb{R}$ with $a, b > 0$ has been treated by several authors mostly from a measure-theoretic point of view, e.g. Cohen & Trenholme [4], Grosjean [13] and Saitoh & Yoshida [22].

The aim of the present paper is to study polynomial decomposition of $\{f_n\}$ of (1.1) with $a, b, c, d \in \mathbb{R}$ and $a > 0$. Polynomial decomposition theory is concerned with characterizing all representations of a given polynomial $f = r_1 \circ r_2 \circ \dots \circ r_l \in \mathbb{C}[x]$, where $r_i \in \mathbb{C}[x]$, $l \geq 2$ and “ \circ ” denotes the usual functional composition. If for each $1 \leq i \leq l$ we have $\deg r_i > 1$, then the decomposition is called a *non-trivial* decomposition. In the particular case of a binary decomposition, i.e. $l = 2$, we call r_1 the *left* and r_2 the *right component* of the decomposition. Two binary decompositions $f = r_1 \circ r_2 = s_1 \circ s_2$ are said to be *equivalent* if there is a linear polynomial κ such that $s_1 = r_1 \circ \kappa$ and $s_2 = \kappa^{-1} \circ r_2$. In general, a non-trivial decomposition can only be determined up to equivalence. We call a polynomial *f decomposable* (over \mathbb{C}) if it has at least one non-trivial decomposition with complex components.

Above all, it is well-known [23, Theorem 6, p.20], that if a polynomial is indecomposable over \mathbb{R} (i.e., with components in $\mathbb{R}[x]$), then it is also indecomposable over any field extension of \mathbb{R} . Hence, regarding the real polynomials $\{f_n\}$ of (1.1), we can safely restrict our attention to decompositions involving components in $\mathbb{R}[x]$ only, since there cannot be any new decomposable polynomial with components in $\mathbb{C}[x]$. For this and other facts from decomposition theory, we refer to the recent monograph of Schinzel [23]. We want to point out, that in our arguments it will be crucial that $f_n \in \mathbb{R}[x]$, while in our main result (Theorem 2.1) we aim for maximal generality allowing complex components.

Much motivation for uniformly decomposing polynomial families stems from an application to Diophantine equations. In fact, due to a powerful result of Bilu & Tichy [1], a complete decomposition result for some given polynomial family $\{p_n\}$ is intimately related to a finiteness statement about solution pairs $(x, y) \in \mathbb{Z}^2$ of the Diophantine equation $p_k(x) = p_l(y)$, where $k > l \geq 2$ are fixed integers. We refer the reader to the bibliography list of [7] for decomposition results concerning Bernoulli polynomials, power-sum polynomials, binomial polynomials etc. and to their corresponding Diophantine problems. Mention also, the recent work of Dujella *et al.* [8], where a indecomposability criterion has been established which involves divisibility properties of the degree and the uppermost coefficients of the polynomial under consideration.

2 Main result

In the present work, we restrict to a uniform decomposition result of f_n , which – in our opinion – is of own interest. To start with, set

$$g_n(x) := \frac{f_n(2\sqrt{a}x + d)}{(2\sqrt{a})^n}, \quad n \geq 0, \quad (2.1)$$

which obviously satisfies the perturbed Chebyshev recurrence

$$g_0(x) = b, \quad g_1(x) = x - e, \quad g_{n+1}(x) = xg_n(x) - \frac{1}{4}g_{n-1}(x), \quad (2.2)$$

where $e := (c - d)/(2\sqrt{a}) \in \mathbb{R}$. We want to point out, that $\{g_n\}$ are *not* co-modified polynomials in the usual sense [10], because co-dilation (parameter b) and co-recursion (parameter c) refer to different levels of perturbation (compare with (3.3) and (3.4)). From now on, assume that $a > 0$, such that $\{g_n\}$ denotes real polynomials.

Note that since each decomposition $g_n = r \circ q$ is related to a decomposition of f_n via

$$f_n = (2\sqrt{a})^n x \circ g_n \circ \left(\frac{x-d}{2\sqrt{a}} \right) = ((2\sqrt{a})^n x \circ r) \circ \left(q \circ \frac{x-d}{2\sqrt{a}} \right),$$

it is the same problem to characterize decomposition of g_n or decomposition of f_n . In what follows, denote by $T_n(x)$ the standard *non-monic* Chebyshev polynomial of the first kind of degree n , i.e. $T_n(x) = 2^n t_n(x)$.

Our main result is

Theorem 2.1. *Let f_n, g_n be as defined in (1.1), (2.1) with $a, b, c, d \in \mathbb{R}$, $a > 0$ and set $e = (c - d)/(2\sqrt{a})$. Then f_n is decomposable over \mathbb{C} if and only if we are in one of the following cases:*

1. $n = mk$ with $m, k \geq 2$, $b = 2$, $e = 0$:

$$g_{mk} = 2^{-mk} T_m(x) \circ T_k(x).$$

2. $n = 2k$, $e = 0$:

$$g_{2k} = \hat{g}_k \circ x^2 \quad \text{with} \quad \hat{g}_k(x) := g_{2k}(\sqrt{x}) \in \mathbb{R}[x].$$

3. $n = 8$, $b = -2$, $e = 0$:

$$g_8 = \left(x^2 - \frac{1}{4}x - \frac{1}{128} \right) \circ \left(x^2 - \frac{1}{2}x \right) \circ x^2.$$

4. $n = 6$, $b = -\frac{11}{2}$, $e = \pm \frac{3\sqrt{3}}{2}$:

$$g_6 = \left(x^3 - \frac{15}{8}x^2 + \frac{9}{16}x + \frac{11}{128} \right) \circ \left(x^2 \mp \frac{\sqrt{3}}{2}x \right).$$

5. $n = 6$, $b = -\frac{10}{3}$, $e = \pm \frac{2\sqrt{3}}{3}$:

$$g_6 = \left(x^2 \pm \frac{\sqrt{3}}{2}x + \frac{5}{96} \right) \circ \left(x^3 \mp \frac{\sqrt{3}}{3}x^2 - \frac{1}{4}x \right).$$

6. $n = 4$, $b = 2 - e^2$:

$$g_4 = \left(x^2 - x + \frac{1}{16}b \right) \circ \left(x^2 - \frac{1}{2}ex \right).$$

First, a few remarks are in order. The case (i) holds due to the well-known property [27],

$$T_{mk}(x) = T_m(T_k(x)) = T_k(T_m(x)),$$

where $m, k \geq 1$. The case (ii) is again trivial, since $g_n(x)$ is an even polynomial if $e = 0$, $n = 2k$. Moreover, it is easy to retrieve from the proof of Corollary 3.5 that $\hat{g}_k(x)$ is indecomposable, except for $k = 4$, $b = -2$, where the cases (ii) and (iii) merge. The case (iii) has already been observed by Dujella & Gusić [7], while studying decomposition of so-called *Dickson polynomials of the second kind* (also termed *generalized Fibonacci polynomials* [9]). In a future work [25], we indeed obtain that for (1.1) with $c = d = 0$ and arbitrary $a, b \in \mathbb{R}$, $a \neq 0$, $b \neq 2$ the only sophisticated decomposition comes from $b = -2$ and $n = 8$, namely,

$$f_8 = (x^2 - 4a^2x - 2a^4) \circ (x^2 - 2ax) \circ x^2. \quad (2.3)$$

Finally, all of the cases (iv)–(vi) denote additional sporadic decomposable polynomials.

The paper is organized as follows. In Section 3 we establish a second order differential equation satisfied by $g_n(x)$. From this, we can bound the degree of the right component q of a binary decomposition $g_n = r \circ q$. Section 4 is devoted to a concrete implementation of two decomposition algorithms. The first algorithm computes the coefficients of the single normed candidate $\hat{q}(x)$ for a right component of fixed degree. The second algorithm then proves or disproves a decomposition involving $\hat{q}(x)$. We implemented these algorithms with Maple 10 [18]; as an important feature, we are able to control the computing time by some precision parameter N_{par} . Finally, Section 5 shows the concrete application to the perturbed Chebyshev polynomials. For the most involved computations, the reader is referred to the data sheet [26], available on the author's web page.

3 Differential equation of second order

There are several ways to identify the polynomials g_n of (2.2). We first recall several definitions of perturbed classical polynomials from [10]. Denote by $\{P_n\}$ a polynomial sequence of classical continuous orthogonal polynomials (Jacobi, Laguerre, Hermite), which satisfies $P_0(x) = 1$, $P_1(x) = x - \beta_0$ and the recurrence

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad (3.1)$$

where β_n , γ_n are specific rational functions in n (see the Askey-scheme [15]). If for $r \in \mathbb{N}$ we replace β_n and γ_n by β_{n+r} and γ_{n+r} , respectively, we get the so-called *rth associated* polynomial family denoted by $\{P_n^{(r)}\}$, i.e., $P_0^{(r)}(x) = 1$, $P_1^{(r)}(x) = x - \beta_r$ and

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r}P_{n-1}^{(r)}(x). \quad (3.2)$$

Obviously, in the case of classical Chebyshev polynomials of the second kind, we have $\beta_n \equiv 0$ and $\gamma_n \equiv \frac{1}{4}$, such that the associated polynomial sequence is again the original Chebyshev sequence. Furthermore, let $\{P_n^{[e]}\}$, $e \in \mathbb{R}$ be the *co-recursive* polynomial sequence of $\{P_n\}$ which is obtained by (3.1), where β_0 is replaced by $\beta_0 - e$, i.e.,

$$P_0^{[e]}(x) = 1, \quad P_1^{[e]}(x) = x - \beta_0 - e$$

and

$$P_{n+1}^{[e]}(x) = (x - \beta_n)P_n^{[e]}(x) - \gamma_n P_{n-1}^{[e]}(x). \quad (3.3)$$

We also recall the notion of *co-dilated* classical orthogonal polynomials $\{P_n^{[b]}\}$, where in (3.1) we replace γ_1 by $b\gamma_1$, i.e.,

$$P_0^{[b]}(x) = 1, \quad P_1^{[b]}(x) = x - \beta_0, \quad P_2^{[b]}(x) = (x - \beta_0)(x - \beta_1) - b\gamma_1, \quad (3.4)$$

and

$$P_{n+1}^{|b|}(x) = (x - \beta_n)P_n^{|b|}(x) - \gamma_n P_{n-1}^{|b|}(x), \quad n \geq 2.$$

Co-recursive and co-dilated classical polynomials are related by the following formulas to the original sequences (see [10, formulas (19) and (27)]),

$$P_n^{[e]}(x) = P_n(x) - eP_{n-1}^{(1)}(x), \quad (3.5)$$

$$P_n^{|b|}(x) = P_n(x) + (1 - b)\gamma_1 P_{n-2}^{(2)}(x), \quad (3.6)$$

where $n \geq 2$. Recall also the well-known formula [27] for $u_n(x)$,

$$u_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^j \binom{n-j}{j} x^{n-2j}. \quad (3.7)$$

Proposition 3.1. *We have*

$$g_n(x) = u_n(x) - eu_{n-1}(x) + \frac{1}{4}(1 - b)u_{n-2}(x), \quad (3.8)$$

thus

$$\begin{aligned} g_n(x) = & x^n - ex^{n-1} - \frac{n+b-2}{4}x^{n-2} + \frac{e(n-2)}{4}x^{n-3} + \frac{(n-3)(n+2b-4)}{32}x^{n-4} \\ & - \frac{e(n-3)(n-4)}{32}x^{n-5} - \frac{(n-4)(n-5)(n+3b-6)}{384}x^{n-6} \pm \dots, \end{aligned}$$

where $u_n(x)$ denotes the monic Chebyshev polynomial of the second kind of degree n .

Proof. The first part follows from (2.2) joined with (3.5), (3.6) for $P_n(x) = u_n(x)$, namely,

$$\begin{aligned} g_n(x) &= u_n^{[e]}(x) + u_n^{|b|}(x) - u_n(x) \\ &= u_n(x) - eu_{n-1}^{(1)}(x) + \frac{1}{4}(1 - b)u_{n-2}^{(2)}(x). \end{aligned}$$

The second part of the statement is then a direct calculation from (3.7). \square

From an other point of view, we may write $g_n(x)$ also in the so-called *combinatorial form*, from which we again can retrieve (3.8). Consider the general case of the r -term linear recurrence with constant coefficients,

$$V_{n+1} = a_0V_n + a_1V_{n-1} + \dots + a_{r-1}V_{n-r+1}, \quad n \geq r - 1, \quad (3.9)$$

with specified initial conditions V_0, \dots, V_{r-1} . It is well-known [16], that solutions of (3.9) are of the form

$$V_n = \varepsilon_0\rho(n, r) + \varepsilon_1\rho(n - 1, r) + \dots + \varepsilon_{r-1}\rho(n - r + 1, r),$$

where $\varepsilon_m = a_{r-1}V_m + \dots + a_mV_{r-1}$ for $0 \leq m \leq r - 1$ and

$$\rho(n, m) = \sum_{j_0+2j_1+\dots+rj_{r-1}=n-m} \frac{(j_0 + j_1 + \dots + j_{r-1})!}{j_0!j_1!\dots j_{r-1}!} a_0^{j_0} a_1^{j_1} \dots a_{r-1}^{j_{r-1}}.$$

Regarding (2.2), it is a direct calculation to verify $g_n = \varepsilon_0\rho(n, 2) + \varepsilon_1\rho(n - 1, 2)$ with

$$\varepsilon_0 = a_1g_0 + a_0g_1 = -\frac{1}{4}b + x(x - e), \quad \varepsilon_1 = a_1g_1 = -\frac{1}{4}(x - e).$$

Thus we get

$$g_n(x) = \left(x^2 - ex - \frac{b}{4}\right) \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{(n-2-j)!}{(n-2-2j)!j!} x^{n-2-2j} \left(-\frac{1}{4}\right)^j \\ - \left(\frac{x}{4} - \frac{e}{4}\right) \cdot \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \frac{(n-3-j)!}{(n-3-2j)!j!} x^{n-3-2j} \left(-\frac{1}{4}\right)^j,$$

which is equivalent to (3.8) by (3.7).

Finally, since in (2.2) coefficients are independent of n , we may express $g_n(x)$ also by a Binet type formula. For the sake of completeness, we give

Proposition 3.2. *We have*

$$g_n(x) = \frac{b + 2(x - bx - e)(x - \sqrt{x^2 - 1})}{\sqrt{x^2 - 1}} \left(\frac{1}{2}(x + \sqrt{x^2 - 1})\right)^{n+1} \\ - \frac{b + 2(x - bx - e)(x + \sqrt{x^2 - 1})}{\sqrt{x^2 - 1}} \left(\frac{1}{2}(x - \sqrt{x^2 - 1})\right)^{n+1}.$$

Proof. The characteristic polynomial $z^2 - zx + \frac{1}{4}$ has roots

$$\xi_{1,2} = \frac{1}{2} \left(x \pm \sqrt{x^2 - 1}\right).$$

Moreover, the ordinary generating function $G(z)$ of $g_n(x)$ is

$$G(z) = \frac{b + z(x - bx - e)}{1 - xz + \frac{1}{4}z^2},$$

which can be easily seen by writing $G(z) = g_0(x) + g_1(x)z + g_2(x)z^2 + \dots$, subtracting $xzG(z) - \frac{1}{4}z^2G(z)$ and using the recurrence relation (2.2). The result now directly follows by the Rational Expansion Theorem [12]. \square

In general, perturbed classical orthogonal polynomials satisfy differential equations of fourth order with polynomial coefficients of fixed degree (see [10], [21]). With the method of the proof of Theorem 1 in [10], we can directly calculate such a differential equation of fourth order for g_n . As for the special case of perturbed Chebyshev polynomials (2.2), there is a differential equation of second order with polynomial coefficients of degree ≤ 4 .

Proposition 3.3. *The polynomials $y = g_n(x)$ satisfy the differential equation*

$$\sigma(x)y'' + \tau(x)y' - \lambda(x)y = 0, \tag{3.10}$$

where

$$\sigma(x) = A_4x^4 - eA_3x^3 - A_2x^2 + eA_1x + A_0, \\ \tau(x) = B_3x^3 - eB_2x^2 - B_1x - eB_0, \\ \lambda(x) = C_2x^2 - eC_1x - C_0$$

with

$$\begin{aligned}
A_4 &= 4n(b-1) = B_3, \\
A_3 &= 4n(b-2) - 2b = A_1 = B_0, \\
A_2 &= n(b^2 + 4b + 4e^2 - 4) - b(b-2), \\
A_0 &= n(b^2 + 4e^2) - b(b-2), \\
B_2 &= 8n(b-2) - 4b, \\
B_1 &= n(3b^2 - 8b + 12e^2 + 8) - 3b(b-2), \\
C_2 &= 4n^3(b-1), \\
C_1 &= 4n^3(b-2) - 6n^2b + 2n(b-2), \\
C_0 &= n^3(b^2 + 4e^2) - 3bn^2(b-2) + n(2b^2 - 8b - 4e^2 + 8).
\end{aligned}$$

Proof. First, we use Proposition 3.1 together with the three identities [20, Chapter 3.4]

$$\begin{aligned}
t'_n(x) &= nu_{n-1}(x), \\
(1-x^2)t'_n(x) &= n\left(\frac{1}{2}t_{n-1}(x) - xt_n(x)\right) \quad \text{and} \\
t_{n-2}(x) &= 4xt_{n-1}(x) - 4t_n(x)
\end{aligned}$$

to write

$$\begin{aligned}
(1-x^2)g_n(x) &= (1-x^2)\left((x-e)u_{n-1}(x) - \frac{b}{4}u_{n-2}(x)\right) \\
&= \left(-x^2 + ex + \frac{b}{2}\right)t_n(x) + \left(\left(\frac{1}{2} - \frac{b}{4}\right)x - \frac{e}{2}\right)t_{n-1}(x). \tag{3.11}
\end{aligned}$$

We differentiate (3.11) twice to get

$$\begin{aligned}
4(1-x^2)(-2xg_n + (1-x^2)g'_n) & \tag{3.12} \\
&= ((4n+8)x^3 - 4e(n+1)x^2 - 2(2n+2+b)x + 4en)t_n \\
&\quad + ((-4-nb+2b)x^2 + 2ex + nb + 2 - b)t_{n-1}
\end{aligned}$$

and

$$\begin{aligned}
4(1-x^2)^2(-2g_n - 4xg'_n + (1-x^2)g''_n) & \\
&= (-4(n+2)(n+1)x^4 + 4en(n+1)x^3 + (12+2b+20n+4n^2+2n^2b-4nb)x^2 \\
&\quad - 4e(n^2+n+1)x - 2n^2b - 8n + 4nb - 4b)t_n \\
&\quad + ((2n^2+4+3nb-n^2b-2b)x^3 - 2n^2ex^2 \\
&\quad + (3b-3nb+n^2b-2n^2-6)x + 2e(1+n^2))t_{n-1}. \tag{3.13}
\end{aligned}$$

Finally, we eliminate t_n , t_{n-1} from (3.11), (3.12) and (3.13) to get the statement. \square

We recall a result of Veselić [28] (see also [5, Theorem 2.4.1]) concerning the minimum number of real zeroes of polynomials defined by three-term recurrences (in what follows, $\text{sign}(0) = +1$).

Proposition 3.4 (Veselić [28]). *Let the polynomial family $\{p_n\}$ satisfy*

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_np_{n-1},$$

where $\beta_n, \gamma_n \in \mathbb{R}$. Form the sequence

$$\mathcal{L}_n = 1, \beta_0, \beta_0\beta_1, \dots, \beta_0 \cdots \beta_{n-2}.$$

Denote by k_{\pm} the number of positive and negative signs in \mathcal{L}_n and set $k = \min(k_+, k_-)$. If $n - 2k > 0$, then p_n has at least $n - 2k$ different real zeros. If, in addition, $n - 3k > 0$, then at least $n - 3k$ of these real zeroes are simple.

From this we get the following decomposition result.

Corollary 3.5. *Let $g_n = r \circ q$ with $r, q \in \mathbb{R}[x]$ and $\min(\deg r, \deg q) \geq 2$. If $(b, e) \neq (2, 0)$ then*

$$\deg q \leq 7.$$

Proof. First, note that for all $b \in \mathbb{R}$ and $n \geq 0$ we have $\lambda(x)^2 \neq 0$. Moreover, the polynomial $\omega(x) = (2\tau(x) - \sigma'(x))\lambda(x) + \sigma(x)\lambda'(x)$ is non-zero for $(b, e) \neq (2, 0)$ and has degree at most 4. Define the Sonin-type function

$$h(x) = g_n(x)^2 - \frac{\sigma(x)}{\lambda(x)} g_n'(x)^2, \quad (3.14)$$

which has first derivative $h'(x) = \omega(x)(g_n'(x))^2$. The function $\omega(x)$ changes at most four times its sign for $x \in \mathbb{R}$, thus splitting the real line into five intervals. Since g_n has at least $n - 2$ different real zeros by Proposition 3.4 and there are at most two additional complex roots of g_n , we conclude

$$\deg \gcd(g_n - \zeta, g_n') \leq 7,$$

uniformly in $\zeta \in \mathbb{C}$. Now, suppose a non-trivial decomposition $g_n = r \circ q$ and denote by ζ_0 a root of r' , which exists by $\deg r \geq 2$. Then both $g_n(x) - r(\zeta_0)$ and $g_n'(x)$ are divisible by $q(x) - \zeta_0$. Then,

$$\deg q \leq \deg \gcd(g_n - r(\zeta_0), g_n') \leq 7,$$

which finishes the proof. \square

4 The decomposition algorithm for right components of fixed degree

As the main tool, we provide a decomposition algorithm, which follows an approach of Binder [2].

Proposition 4.1. *Let $g \in \mathbb{R}[x]$ with $\deg g = mk$ be a monic polynomial. Further let $q \in \mathbb{R}[x]$ with $q(0) = 0$ and $\deg q = m \geq 1$ and suppose*

$$\deg(g - q^k) \leq mk - j$$

for some $1 \leq j < m$. Then for $\alpha_j = \frac{1}{k} \text{coeff}(g - q^k, [x^{m-j}])$ we have

$$\deg(g - (q + \alpha_k x^{m-j})^k) \leq mk - j - 1. \quad (4.1)$$

Proof. The result immediately follows from

$$\deg(g - (q + \alpha_k x^{m-j})^k) = \deg(g - q^k - kq^{k-1}\alpha_k x^{m-j} - \dots)$$

and the fact that the omitted terms have degree $\leq mk - j - 1$. \square

We apply Proposition 4.1 subsequently for $1 \leq j \leq m-1$ with $q \mapsto q + \alpha_j x^{m-j}$ to define a polynomial

$$\hat{q}(x) := x^m + \alpha_1 x^{m-1} + \cdots + \alpha_{m-1} x. \quad (4.2)$$

Note that $\hat{q}(x)$ can also be calculated if the coefficients of $g(x)$ depend on several parameters (k is also considered as a parameter). We put on record the procedure as the following algorithm.

ALGORITHM 1:

Input: $m \geq 2, g \in \mathbb{Q}[x]$.

Output: $\hat{q} \in \mathbb{Q}[x]$.

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 $q := x^m;$ 
for  $1 \leq j \leq m-1$  do
   $\alpha_j = \frac{1}{k} \text{coeff}(g - q^k, [x^{m-j}]);$ 
   $q := q + \alpha_j x^{m-j};$ 
od;
 $\hat{q} := q;$ 

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With the help of (4.2) we have an indecomposability criterion for binary decompositions with right components of fixed degree.

Proposition 4.2. *Let g be monic and $m \geq 2$ a positive integer. Furthermore, let*

$$g(x) = \hat{q}(x)^k + \beta_1 \hat{q}(x)^{k-1} + \cdots + \beta_l \hat{q}(x)^{k-l} + \mathcal{R}(x), \quad (4.3)$$

for some constants $\beta_j \in \mathbb{R}$, $0 \leq l < k$ with $\deg \mathcal{R} \leq mk - m$ and $m \nmid \deg \mathcal{R}$. Then g is indecomposable with right components of degree m .

Proof. Put

$$\begin{aligned} \mathcal{S}(x) &:= \hat{q}(x)^k + \beta_1 \hat{q}(x)^{k-1} + \cdots + \beta_l \hat{q}(x)^{k-l} \\ &= (x^k + \beta_1 x^{k-1} + \cdots + \beta_l x^{k-l}) \circ \hat{q}(x) =: s \circ \hat{q}. \end{aligned}$$

We have $\deg(\mathcal{S} - \hat{q}^k) \leq mk - m$. As $\deg \mathcal{R} \leq mk - m$ by assumption, this yields

$$\deg((\mathcal{S} + \mathcal{R}) - \hat{q}^k) = \deg((\mathcal{S} - \hat{q}^k) + \mathcal{R}) \leq mk - m.$$

If there is a decomposition of $\mathcal{S} + \mathcal{R}$ with a right component q of degree m then it is necessarily \hat{q} (up to equivalence). Suppose $\mathcal{S} + \mathcal{R} = r \circ \hat{q}$. Since $\mathcal{S} = s \circ \hat{q}$, we get $\mathcal{R} = (r - s) \circ \hat{q}$ which is a contradiction due to $m \nmid \deg \mathcal{R}$. Thus, $g = \mathcal{S} + \mathcal{R}$ is indecomposable with right components of degree m . \square

Denote by N_{par} the number of parameters which the coefficients of $g(x)$ depend on. Then, while expanding g in terms of \hat{q} , the numbers β_j are rational functions of these parameters, such that it is not straightforward to check the condition $m \mid \deg \mathcal{R}$. The answer depends on whether the current β_j vanishes or not. The following algorithm collects several of these coefficients into a system of N_{eqs} polynomial equations $\{\text{eq}_l = 0, 1 \leq l \leq N_{eqs}\}$, which can be solved by a calculation of an

associated Gröbner basis. From a practical point of view, the quantity N_{eqs} is some sort of a “precision parameter”, used to control the running time of the Gröbner calculations. In our case of perturbed Chebyshev polynomials (with the exception of the instances $\deg q = 4, 6$), it will be sufficient to let $N_{eqs} = N_{par} = 3$, corresponding to a system of three equations in the parameters k, b, e . We always assume $k \in \mathbb{Z}$ and $k \geq 2$.

ALGORITHM 2:

Input: $m \geq 2$, $g \in \mathbb{Q}[x]$ with $\deg g = mk$, $N_{eqs} > N_{par}$.

Output: Finds a decomposition of g with $\deg q = m$, or proves that there is no such decomposition, or stops after N_{eqs} coefficient equations (“precision”).

```

 $l := 1;$    $j := 0;$    $h := g;$    $S := \text{true};$ 
while  $l < N_{eqs}$  do
   $\beta_j = \text{lcoeff}(h);$    $d_j = \deg h;$   # the numbers  $\beta_j, d_j$  refer to the generic polynomial  $h$ 
  if  $m \mid d_j$  then  # we have to expand one more term
     $h := h - \beta_j \hat{q}^{m(k-j)};$ 
    if  $h \equiv 0$  then
       $\text{return}((\beta_0 x^k + \dots + \beta_j x^{k-j}) \circ \hat{q});$   # decomposition found for  $k \geq j$ 
      # we check the cases  $2 \leq k < j$  separately by ALGORITHM 2
    fi;
     $j := j + 1;$    $l_e := 1;$ 
  else
     $\text{eq}_l := \beta_j;$    $S := S \cap \text{solve}(\text{eq}_l = 0);$ 
    if  $S = \{\}$  then
       $\text{return}(\text{"no decomposition with } \deg q = m \text{ and } k \geq j + 1 \text{ possible"});$ 
      # we check the cases  $2 \leq k \leq j$  separately by ALGORITHM 2
    else
       $h := h - \text{eq}_l x^{m(k-j)-l_e};$ 
       $l_e := l_e + 1;$    $l := l + 1;$ 
      #  $l_e$  resp.  $l$  refer to the next exponent resp. the number of coefficient equations
    fi;
  fi;
od;

```

5 Application to the polynomials g_n

In the sequel, we show how ALGORITHM 1 and 2 can be used to find or/and to disprove decompositions with $\deg q = 2, 3, 4$. From the investigation given below, we get the cases (ii)–(vi) of Theorem 2.1 as well as the case (i) for the particular $n = mk$. As the calculations with the aid of Maple 10 get more and more involved and expressions quite large, we do not give the details for the three cases $\deg q = 5, 6, 7$ here. We refer to [26], where the complete data can be found.

The case $\deg q = 2$:

ALGORITHM 1 yields

$$\hat{q} = x^2 - \frac{ex}{k},$$

where we used the expression for the second coefficient of g from Proposition 3.1. According to ALGORITHM 2 we have

$$g - \text{eq}_1 x^{2k-3} - \text{eq}_2 x^{2k-5} - \text{eq}_3 x^{2k-7} = \hat{q}^k + \beta_1 \hat{q}^{k-1} + \beta_2 \hat{q}^{k-2} + \beta_3 \hat{q}^{k-3} + \mathcal{R}(x), \quad (5.1)$$

where $\mathcal{R}(x)$ has degree at most $2k - 8$. Note that the expansion (5.1) is possible, provided that $k \geq 4$. Now, it is a straightforward algorithmic calculation from Proposition 3.1 to make parameters in (5.1) explicit. We used an implementation of ALGORITHM 2 in Maple 10, to get the following output,

$$\begin{aligned} \beta_1 &= -\frac{1}{4k}(2k^2 + 2e^2k - 2k + bk - 2e^2), \\ \beta_2 &= \frac{1}{48k^3}(6k^5 + 12k^4e^2 + 6k^4b - 21k^4 + 10e^4k^3 + 6e^2bk^3 - 48e^2k^3 - 9bk^3 + 18k^3 \\ &\quad - 36e^4k^2 + 60e^2k^2 - 18e^2k^2b + 38e^4k + 12e^2kb - 24e^2k - 12e^4), \\ \beta_3 &= -\frac{1}{2280k^5}(k-2)(600e^6 + 180e^2bk^5 + 450k^5 + 60k^7 - 330k^6 + 150e^4k^4b - 840e^4k^3b \\ &\quad + 810e^2k^3b + 1410e^4k^2b - 720e^4kb - 810e^2k^4b + 2430e^2k^4 + 300e^4k^5 - 1170e^2k^5 \\ &\quad - 225bk^5 + 180k^6e^2 + 90k^6b - 1620e^2k^3 - 4260e^4k^2 + 1440e^4k + 244e^6k^4 \\ &\quad - 1492e^6k^3 + 4500e^4k^3 + 2996e^6k^2 - 2348e^6k - 1980e^4k^4), \end{aligned}$$

and

$$\begin{aligned} \text{eq}_1 &= -\frac{e}{12}(k-1)(-6k + 4ke^2 + 3kb - 2e^2), \\ \text{eq}_2 &= \frac{e}{240}(2k-3)(k-2)(20k^3e^2 + 15k^3b - 30k^3 + 16e^4k^2 - 40e^2k^2 + 10e^2k^2b - 24e^4k \\ &\quad + 20e^2k - 10e^2kb + 8e^4), \\ \text{eq}_3 &= -\frac{e}{20160}(2k-5)(k-2)(k-3)(315k^5b + 420k^5e^2 - 630k^5 + 420e^2k^4b + 672e^4k^4 \\ &\quad - 1470e^2k^4 - 630e^2k^3b + 336e^4k^3b + 544e^6k^3 - 2352e^4k^3 + 1260e^2k^3 + 2688e^4k^2 \\ &\quad - 1632e^6k^2 - 840e^4k^2b - 1008e^4k + 504e^4kb + 1496e^6k - 408e^6). \end{aligned}$$

A reduced Gröbner basis over $\mathbb{Q}[k, b, e]$ for the ideal generated by $\{\text{eq}_i\}_{1 \leq i \leq 3}$ is given by

$$\begin{aligned} &[4ke^3 - 6ek + 3ekb - 2e^3, \\ &8k^8e^3 - 58k^7e^3 + 133k^6e^3 - 83k^5e^3 - 30e^3k^4, \\ &\quad - 24k^7e^3 + 166k^6e^3 + 4e^5k - 341k^5e^3 - 8e^5 + 120e^3k^4 + 150k^3e^3 + 60e^3k^2, \\ &48e^5b + 48e^7 + 360k^7e^3 - 96e^5 - 2594k^6e^3 + 5833k^5e^3 - 3270e^3k^4 - 1800k^3e^3], \end{aligned}$$

from which we deduce the solution set of $\{\text{eq}_i = 0\}_{1 \leq i \leq 3}$,

$$\begin{aligned} & \{k = k, b = b, e = 0\}, \{k = 2, b = 2 - e^2, e = e\}, \{k = -1/4, b = 1, e = -1/2\}, \\ & \{k = 5/2, b = -19/3, e = \pm 5\sqrt{5}/4\}, \{k = 3, b = -11/2, e = \pm 3\sqrt{3}/2\}, \\ & \{k = -1/4, b = 1, e = 1/2\}. \end{aligned}$$

In the first instance, we obtain the (trivial) case (ii) of Theorem 2.1. For $k = 2, 3$ we get the solutions given in the solution set. Note that in these cases ALGORITHM 2 also delivers the explicit $\beta_j \in \mathbb{Q}$, from which we deduce the cases (vi) and (iv). The above Gröbner calculations can be performed with Maple 10 with aid of the following commands.

```
> with(Groebner):
> infolevel[GroebnerBasis] := 5;
> eq_1 := -e/12*(k-1)*(-6*k+4*k*e^2+3*k*b-2*e^2);
> eq_2 := e/240*(2*k-3)*(k-2)*(20*k^3*e^2+15*k^3*b-30*k^3+16*e^4*k^2-40*e^2*k^2
> +10*e^2*k^2*b-24*e^4*k+20*e^2*k-10*e^2*k*b+8*e^4);
> eq_3 := -e/20160*(2*k-5)*(k-2)*(k-3)*(315*k^5*b+420*k^5*e^2-630*k^5+420*e^2*k^4*b
> +672*e^4*k^4-1470*e^2*k^4-630*e^2*k^3*b+336*e^4*k^3*b+544*e^6*k^3-2352*e^4*k^3
> +1260*e^2*k^3+2688*e^4*k^2-1632*e^6*k^2-840*e^4*k^2*b-1008*e^4*k+504*e^4*k*b
> +1496*e^6*k-408*e^6);
> G := [eq_1, eq_2, eq_3];
> Basis(G, plex(b, e, k));
> Solve(%, [b, e, k]):
> map(L -> solve(convert(L[1], set), {b, e, k}), %);
```

First, Maple calculates a reduced Gröbner basis for G with respect to an appropriate term ordering and then converts it by the Gröbner walk strategy to a lexicographic Gröbner basis. By setting `infolevel`, we stay informed about the current status of the computation. We point out, that we may also succeed here with the command

```
> solve({eq_1, eq_2, eq_3}, [b, e, k]);
```

without resorting to the calculation of a Gröbner basis. Indeed, the `solve`-command also successfully solves the subsequent systems for $\deg q = 3, 4$. However, as for $\deg q = 5$, we waited hours while Maple was busy with computing a subresultant determinant of dimension 23 and length 10918. Similarly, for $\deg q = 6$ and $\deg q = 7$, we were not able to solve the coefficient systems, as the high-dimensional determinant computation did not stop.

The case $\deg q = 3$:

Here we calculate

$$\hat{q} = x^3 - \frac{ex^2}{k} - \frac{x}{4k^2}(3k^2 + 2ke^2 - 2k + kb - 2e^2)$$

and use the expansion ($\deg \mathcal{R} \leq 3k - 8$),

$$g - \text{eq}_1 x^{3k-4} - \text{eq}_2 x^{3k-5} - \text{eq}_3 x^{3k-7} = \hat{q}^k + \beta_1 \hat{q}^{k-1} + \beta_2 \hat{q}^{k-2} + \mathcal{R}(x),$$

with

$$\begin{aligned}\beta_1 &= -\frac{e}{12k^2}(3k^2b + 4k^2e^2 - 9k^2 - 3kb - 6ke^2 + 6k + 2e^2), \\ \beta_2 &= \frac{1}{5760k^4}(120be^4 + 128e^6 - 5580e^4k^4 + 1710bk^4 - 495b^2k^4 - 1560k^4 + 1440k^3 - 240k^2 \\ &\quad + 450k^5 + 840be^4k^4 + 360b^2e^2k^4 - 5040be^2k^4 + 1440e^2kb - 1260e^4kb - 360e^2kb^2 \\ &\quad + 3360k^2be^4 + 1260k^2b^2e^2 - 5760k^2be^2 - 3060k^3be^4 - 1260k^3b^2e^2 + 8280k^3be^2 \\ &\quad + 30k^2b^3 + 360k^2b - 1440e^2k + 10440k^3e^4 - 1620k^3b + 2240k^2e^6 - 180k^2b^2 + 6480k^2e^2 \\ &\quad - 8580k^2e^4 + 7020e^2k^4 + 15b^3k^4 + 135k^5b^2 - 1620k^5e^2 + 1080k^5e^4 - 540k^5b - 1920k^3e^6 \\ &\quad + 540k^3b^2 - 10440k^3e^2 - 45k^3b^3 - 960e^6k + 2880e^4k + 512e^6k^4 - 240e^4 + 1080k^5be^2),\end{aligned}$$

and

$$\begin{aligned}\text{eq}_1 &= -\frac{1}{96k^3}(k-1)(24k^2be^2 + 24k^2e^4 - 12k^2b + 12k^2 + 3k^2b^2 - 36k^2e^2 - 12e^2kb - 20e^4k \\ &\quad + 24e^2k + 4e^4), \\ \text{eq}_2 &= \frac{e}{480k^4}(k-1)(-15k^3b^2 + 60k^3 - 60k^3e^2 + 24k^3e^4 - 80e^2k^2b - 116e^4k^2 + 160e^2k^2 \\ &\quad - 80e^2k + 84e^4k + 40e^2kb - 16e^4), \\ \text{eq}_3 &= \frac{e}{13440k^6}(k-2)(160e^6 - 1260k^6b + 2016k^6e^4 - 2520k^6e^2 - 8400k^5be^2 + 840k^5b^2e^2 \\ &\quad - 3556k^4be^4 + 10360k^4be^2 - 1960k^4b^2e^2 + 2520k^6be^2 - 5600k^3be^2 + 1400k^3b^2e^2 \\ &\quad - 280e^2k^2b^2 + 196e^4k^2b + 1120e^2k^2b - 280e^4kb + 1512k^5be^4 - 560k^3 + 3080k^4 - 2100k^5 \\ &\quad - 175k^4b^3 - 3780k^4b - 4368k^3e^4 + 160k^3e^6 + 70k^3b^3 + 5600k^3e^2 + 840k^3b - 392e^4k^2 \\ &\quad + 1440e^6k^2 - 1120e^2k^2 + 560e^4k - 420k^3b^2 - 928e^6k + 2128k^3be^4 + 630k^6b^2 + 768k^5e^6 \\ &\quad - 1680k^5b^2 + 105k^5b^3 + 3990k^5b - 7392k^5e^4 + 8400k^5e^2 - 10360k^4e^2 + 9576k^4e^4 \\ &\quad - 1600k^4e^6 + 1470k^4b^2).\end{aligned}$$

The system has the only admissible solutions $(k, b, e) \in \{(k, 2, 0), (2, -\frac{10}{3}, \pm\frac{2\sqrt{3}}{3})\}$, which correspond to the cases (i) and (v) of Theorem 2.1.

The case $\deg q = 4$:

This case shows a new feature of ALGORITHM 2. First,

$$\begin{aligned}\hat{q} &= x^4 - \frac{ex^3}{k} - \frac{x^2}{4k^2}(4k^2 + 2ke^2 + kb - 2k - 2e^2) \\ &\quad - \frac{ex}{12k^3}(3k^2b + 4k^2e^2 - 12k^2 - 6ke^2 - 3kb + 6k + 2e^2)\end{aligned}$$

and ($\deg \mathcal{R} \leq 4k - 8$),

$$g - \text{eq}_1x^{4k-5} - \text{eq}_2x^{4k-6} - \text{eq}_3x^{4k-7} = \hat{q}^k + \beta_1\hat{q}^{k-1} + \mathcal{R}(x),$$

with

$$\begin{aligned}
\beta_1 &= -\frac{1}{96k^3}(-12k^4 - 48e^2k^3 + 24e^4k^3 + 24e^2bk^3 + 24k^3 - 18bk^3 + 3b^2k^3 - 44k^2e^4 + 72k^2e^2 \\
&\quad - 36k^2e^2b - 3k^2b^2 - 12k^2 + 12k^2b + 24e^4k - 24e^2k + 12e^2kb - 4e^4), \\
\text{eq}_1 &= -\frac{e}{480k^4}(k-1)(-90k^3b + 60k^3 + 96k^3e^4 + 120k^3e^2b + 30k^3b^2 - 160k^3e^2 - 104k^2e^4 \\
&\quad + 160k^2e^2 + 60k^2b - 60k^2 - 15k^2b^2 - 100k^2e^2b - 40e^2k + 36e^4k + 20e^2kb - 4e^4), \\
\text{eq}_2 &= \frac{1}{5760k^5}(-180k^4e^2b^2 + 540k^3e^2b - 1080k^3e^4b + 900e^4k^2b + 90e^2k^2b^2 - 360e^2k^2b \\
&\quad - 180e^4kb - 120k^4 + 360e^4k + 360e^2k^2 + 15k^3b^3 - 480k^4e^4 + 360k^4e^2 - 90k^3e^2b^2 \\
&\quad - 120k^3 - 1560e^4k^2 + 1112e^6k^2 + 180k^3b - 90k^3b^2 + 40e^6 - 30k^4b^3 + 90k^4b^2 + 192k^4e^6 \\
&\quad - 1168k^3e^6 + 1920k^3e^4 - 720k^3e^2 - 368e^6k), \\
\text{eq}_3 &= \frac{e}{40320k^6}(k-1)(120e^6 - 420e^4kb + 2520k^3e^2b^2 + 2436k^3e^4b - 10500k^3e^2b - 210e^2k^2b^2 \\
&\quad + 1176e^4k^2b + 840e^2k^2b + 27300k^4e^2b + 1260k^5e^2b^2 - 4830k^4e^2b^2 - 7224k^4e^4b \\
&\quad + 10080k^6e^2b - 26880k^5e^2b + 2016k^5e^4b - 440k^3e^6 - 840e^2k^2 - 7728k^3e^4 + 840e^4k \\
&\quad - 11760k^5 + 10920k^4 - 2520k^3 + 10920k^3e^2 + 2360e^6k^2 - 1512e^4k^2 - 1890k^3b^2 \\
&\quad - 1000e^6k - 735k^4b^3 - 31080k^4e^2 - 13860k^4b - 22848k^5e^4 - 6300k^5b^2 + 8064k^6e^4 \\
&\quad + 34440k^5e^2 + 3780k^3b + 210k^5b^3 + 17640k^5b + 960k^5e^6 + 5670k^4b^2 - 2960k^4e^6 \\
&\quad + 315k^3b^3 + 24192k^4e^4 + 2520k^6b^2 - 13440k^6e^2 - 7560k^6b + 5040k^6).
\end{aligned}$$

From this we retrieve case (i) of Theorem 2.1 as well as the solution triple

$$(k = k, \quad b = -\frac{2(k+1)}{2k-1}, \quad e = 0).$$

Since there is no final decision about a polynomial decomposition in this case, we set $N_{eqs} = 4$ and put

$$g - \text{eq}_4 x^{4k-9} = \hat{q}^k + \beta_1 \hat{q}^{k-1} + \beta_2 \hat{q}^{k-2} + \mathcal{R}(x), \quad \deg \mathcal{R} \leq 4k - 10,$$

and

$$\beta_1 = \frac{k(4k^2 - 19k + 13)}{8(2k-1)^2}, \quad \beta_2 = \frac{k(4k-5)(8k^4 - 78k^3 + 204k^2 - 208k + 69)}{256(2k-1)^4}.$$

This yields $\text{eq}_4 \equiv 0$. We increase the precision again by one. Let $N_{eqs} = 5$ and consider

$$g - \text{eq}_5 x^{4k-10} = \hat{q}^k + \beta_1 \hat{q}^{k-1} + \beta_2 \hat{q}^{k-2} + \mathcal{R}(x),$$

with $\deg \mathcal{R} \leq 4k - 11$ and $k \geq 3$. Then finally

$$\text{eq}_5 = \frac{9k(k-2)(2k-3)(4k+1)}{1280(2k-1)^4} \neq 0.$$

Therefore, the only possibility is $k = 2$ and thus $(k, b, e) = (4, -2, 0)$, which is the case (iii) in Theorem 2.1. We point out that the calculations for $\deg q \leq 4$ performed on an INTEL-P4 (CPU 2.20GHz, 512 MB RAM) all lasted shorter than one minute. The situation dramatically changes for the remaining three cases. For the exact data for the three systems we refer to [26].

Right component	Reduced Gröbner basis	Solving the system
$\deg q = 5$	4'55''	4''
$\deg q = 6$	9'10''	21''
$\deg q = 7$	5 h 3'29''	2'25''

The case $\deg q = 6$ with three coefficient equations yields $e = 0$, which is then similarly treated as the case $\deg q = 4$. In the other two cases, we directly conclude with three coefficient equations. Summing up the results, we get no new sporadic decompositions. This completes the proof of Theorem 2.1.

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