DIOPHANTINE EQUATIONS INVOLVING GENERAL MEIXNER AND KRAWTCHOUK POLYNOMIALS

THOMAS STOLL AND ROBERT F. TICHY

Abstract. While counting lattice points in octahedra of different dimensions n and m, it is an interesting question to ask, how many octahedra exist containing equally many such points. This gives rise to the Diophantine equation $p_n(x) = p_m(y)$ in rational integers x, y, where $\{p_k(x)\}$ denote special Meixner polynomials $\{M_k^{(\beta,c)}(x)\}$ with $\beta = 1$, $c = -1$. In this paper we join the algorithmic criterion of Bilu and Tichy [4] with a famous result of Erdös and Selfridge [6] and prove that the Diophantine equation

$$M_n^{(\beta,c_1)}(x) = M_m^{(\beta,c_2)}(y)$$

with $m, n \geq 3$, $\beta \in \mathbb{Z} \setminus \{0, -1, -2, -\max(n,m)+1\}$ and $c_1, c_2 \in \mathbb{Q} \setminus \{0, 1\}$ only admits a finite number of integral solutions x, y. This generalizes a result given by Bilu and the authors [3]. As an immediate consequence of the investigation an analogous result for general Krawtchouk polynomials $\{K_k^{(p,N)}(x)\}$ is obtained.

1. Introduction

Sometimes polynomial Diophantine equations of type

$$P(x) = Q(y) \tag{1}$$

with $P(x), Q(x) \in \mathbb{Q}[x]$ come across by means of combinatorial counting problems [3, 8, 9, 10, 11, 12, 13]. For instance, consider the following question [3, 11]:

Given distinct positive integers n, m, how often can two octahedra of dimensions n and m respectively, contain equally many integral points?

Recall that an n–dimensional octahedron of size $r \in \mathbb{Z}_{>0}$ is the convex body in $\mathbb{R}^n$ defined by $|x_1| + \cdots + |x_n| \leq r$. Let $p_n(r)$ denote the number of such integral points $(x_1, \ldots, x_n)$ satisfying the inequality. Erhardt [5] proved that $p_n(r)$ is a polynomial in r of degree n indeed for any general lattice polytope described by

$$\frac{|x_1|}{a_1} + \frac{|x_2|}{a_2} + \cdots + \frac{|x_n|}{a_n} \leq r,$$

where $a_1, \ldots, a_n$ are positive integers. In the general case the Ehrhart polynomial is difficult to access and its coefficients involve Dedekind sums and their higher analogues [1]. However, in the special case of symmetric octahedra, Kirschenhofer, Pethö and Tichy [11] could show that $p_n(r)$ can be made explicit, namely

$$p_n(r) = \sum_{i=0}^{n} 2^i \binom{n}{i}\binom{r}{i} = {}_2F_1\left[\begin{matrix} -n, -r \\ 1 \end{matrix} ; 2\right], \tag{2}$$

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where

$$ {}_2F_1\left[\begin{array}{c} a, b \\ c \end{array} ; z\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} z^k \tag{3} $$

is the Gauss hypergeometric function with $(a)_0 = 1$ and $(a)_k = a(a+1)\dots(a+k-1)$ being the Pochhammer symbol. Thus, the original combinatorial counting problem can be restated by means of a polynomial Diophantine equation:

How many solutions in positive integers x, y can the equation $p_n(x) = p_m(y)$ have?

The general answer to the problem has been obtained in [3] by an approach suggested by Bilu and Tichy in [4].

Theorem 1 (Bilu-Stoll-Tichy, 2000)**.** *Let n and m be distinct integers satisfying $m, n \geq 2$. Then the equation*

$$ p_n(x) = p_m(y) $$

has only finitely many solutions in rational integers x, y.

In other words, sufficiently large octahedra of distinct dimensions n, m cannot have equally many lattice points.

In this work we extend the result of [3] by taking advantage of a well-known result of Erdös and Selfridge [6] which states that the product of consecutive integers can never be a perfect power. We obtain finiteness results for Diophantine equations involving general Meixner polynomials $M_k^{(\beta,c)}(x)$ which again depend on the additional parameters β and c. By putting $\beta = 1$ and $c = -1$ we get Theorem 1 for $m, n \geq 3$ as a special case. Results of the same type for other polynomials including several parameters have been obtained by Kirschenhofer and Pfeiffer in [12, 13] and by the authors in [15, 16, 17]. However, in the present paper a major difference has to be noted. The conditions on the parameters β and c are explicit and we don't use distinctive analytical properties (such as orthogonality) of the polynomials under consideration.

We also remark that the idea of the proof applies whenever the leading coefficients of the polynomials are built up by factorials. So, for instance, the approach also works fine in the case of the Pochhammer–Wilkinson polynomials defined by $w_k(x) = (1 + x)(1 + 2x)\dots(1 + kx)$.

2. General Meixner and Krawtchouk polynomials

There exist various definitions of *Meixner polynomials*. From a historical point of view the so-called *Meixner polynomials of the first* and *second kind* are of great interest, they form orthogonal Sheffer families [11]. The family $\{p_k(x)\}$, defined in the previous section, does not form an orthogonal family, However, orthogonality can be achieved by a substitution [11, Theorem 3.3]. In fact

$$ M_{2,k}(x) = i^k k! p_k(-1/2 - ix/2) \tag{4} $$

with $M_{2,k}(x)$ denoting the orthogonal *Meixner polynomials of the second kind*. The classical definition of *Meixner polynomials of the first kind* involves a hypergeometric function of type (3),

$$ M_{1,k}^{(\beta,c)}(x) = (\beta)_n \, {}_2F_1\left[\begin{array}{c} -k, -x \\ \beta \end{array} ; 1 - \frac{1}{c}\right]. $$

Above all, the modern Askey-scheme [14] suggests one single definition of Meixner polynomials which does not use the leading Pochhammer symbol, i.e.

$$M_k^{(\beta,c)}(x) = {}_2F_1\left[\begin{matrix}-k,-x\\ \beta\end{matrix};1-\frac{1}{c}\right]. \tag{5}$$

We use this definition while noting that by (2) and $\beta=1$, $c=-1$ we have

$$p_k(x) = M_k^{(1,-1)}(x). \tag{6}$$

It is well-known that the general family $\{M_k^{(\beta,c)}(x)\}$ defines a discrete orthogonal polynomial family if and only if $\beta>0$ and $0<c<1$. To avoid orthogonality arguments in the forthcoming investigation therefore also makes it possible to let the parameters vary more freely.

In [14] Koekoek and Swarttouw also give a two-parametric definition of the *Krawtchouk polynomials*:

$$K_n^{(p,N)}(x) = {}_2F_1\left[\begin{matrix}-n,-x\\ -N\end{matrix};\frac{1}{p}\right] \qquad n=0,1,2,\dots,N. \tag{7}$$

Comparing with (5), the following connection to Meixner polynomials is straightforward:

$$K_n^{(p,N)}(x) = M_n^{(-N,p/(p-1))}(x). \tag{8}$$

3. Main Theorem

Theorem 2 (General Meixner polynomials)**.** *Let n and m be distinct integers satisfying $m,n\geq 3$, further let $\beta\in\mathbb{Z}\setminus\{0,-1,-2,-\max(n,m)+1\}$ and $c_1,c_2\in\mathbb{Q}\setminus\{0,1\}$. Then the equation*

$$M_n^{(\beta,c_1)}(x) = M_m^{(\beta,c_2)}(y) \tag{9}$$

has only finitely many solutions in integers x,y.

In particular, the main part ($m,n\geq 3$) of Theorem 1 now follows easily from (6) by putting $\beta=1$ and $c_1=c_2=-1$. On the other hand, by (8) and by choosing $\beta=-N$, $c_1=p_1/(p_1-1)$ and $c_2=p_2/(p_2-1)$ we have

Corollary 3 (Krawtchouk polynomials)**.** *Let n and m be distinct integers satisfying $m,n\geq 3$, further let $N\geq\max(m,n)$ and $p_1,p_2\in\mathbb{Q}\setminus\{0,1\}$. Then the equation*

$$K_n^{(p_1,N)}(x) = K_m^{(p_2,N)}(y) \tag{10}$$

has only finitely many solutions in integers x,y.

4. Methods and tools

To begin with, we restate the Theorem of Bilu and Tichy of [4]. Let $\gamma,\delta\in\mathbb{Q}\setminus\{0\}$, $q,s,t\in\mathbb{Z}_{>0}$, $r\in\mathbb{Z}_{\geq 0}$ and $v(x)\in\mathbb{Q}[x]$ a non-zero polynomial (which may be constant). Further let $D_s(x,\gamma)$ denote the *Dickson* polynomials which can be defined via

$$D_s(x,\gamma) = \sum_{i=0}^{\lfloor s/2\rfloor} d_{s,i}x^{s-2i} \qquad \text{with} \qquad d_{s,i} = \frac{s}{s-i}\binom{s-i}{i}(-\gamma)^i.$$

The pair $(f(x),g(x))$ or viceversa $(g(x),f(x))$ is called a *standard pair over* $\mathbb{Q}$ if it can be represented by an explicit form listed below. In such a case we call (f,g) a standard pair of the *first*, *second*, *third*, *fourth*, *fifth kind*, respectively.

kind	explicit form of (f,g) resp. (g,f)	parameter restrictions
first	$(x^q, \gamma x^r v(x)^q)$	with $0 \le r < q,\ (r,q)=1,\ r+\deg v > 0$
second	$(x^2, (\gamma x^2+\delta)v(x)^2)$	–
third	$(D_s(x,\gamma^t), D_t(x,\gamma^s))$	with $(s,t)=1$
fourth	$\left(\gamma^{-s/2}D_s(x,\gamma), -\delta^{-t/2}D_t(x,\delta)\right)$	with $(s,t)=2$
fifth	$\left((\gamma x^2-1)^3, 3x^4-4x^3\right)$	–

Theorem 4 (Bilu-Tichy, 2000)**.** *Let $P(x), Q(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two assertions are equivalent:*

(a) *The equation $P(x) = Q(y)$ has infinitely many rational solutions with a bounded denominator.*
(b) *We can express $P \circ \kappa_1 = \phi \circ f$ and $Q \circ \kappa_2 = \phi \circ g$ where $\kappa_1, \kappa_2 \in \mathbb{Q}[x]$ are linear, $\phi(x) \in \mathbb{Q}[x]$, and (f,g) is a standard pair over $\mathbb{Q}$.*

If we are able to get contradictions for decomposition of P and Q as demanded in (b) of Theorem 4 then finiteness of number of integral solutions x, y of the original Diophantine equation $P(x) = Q(y)$ is guaranteed.

Secondly, we restate a well-known result obtained by Erdös and Selfridge in [6]:

Theorem 5 (Erdös-Selfridge, 1975)**.** *The equation*

$$x(x+1)\cdots(x+k-1) = y^l$$

has no solution in rational integers $x > 0,\ k > 1,\ l > 1,\ y > 1$.

Interestingly, simple comparison of the leading coefficients of the polynomials (appearing in Theorem 4, (b)) gives an equation very similar to that of Theorem 5. Therefore, there are no parameters that satisfy such a coefficient equation. In other words, we can easily derive a contradiction if we suppose a higher degree polynomial representation in Theorem 4.

The following crucial Corollary 6 is a direct consequence of Theorem 5:

Corollary 6. *Let $\beta \in \mathbb{Z}_{>0}$ be fixed. Then the only solutions to*

$$\sqrt[k]{\frac{(\beta)_{ks}}{(\beta)_{kt}}} \in \mathbb{Q} \tag{11}$$

in rational integer triples (k,s,t) with $k \ge 1,\ s \ge 1,\ t \ge 1$ are (k,s,s) and $(1,s,t)$. The same holds in the case $-\beta \in \mathbb{Z}_{>0}$ provided $-\beta \ge \max(ks, kt)$.

Proof. First suppose $\beta > 0$. The cases $(1,s,t)$ and (k,s,s) are obvious solutions of (11). Now, suppose $k \ge 2$; without loss of generality we further may assume that $s > t$ (otherwise just take the reciprocal). Then relation (11) reads

$$\sqrt[k]{\frac{(\beta)_{ks}}{(\beta)_{kt}}} = \sqrt[k]{(\beta+kt)(\beta+kt+1)\cdots(\beta+ks-1)} \in \mathbb{Q}.$$

Thus, the product $(\beta+kt)(\beta+kt+1)\cdots(\beta+ks-1)$ must be a perfect kth-power of a rational integer, namely

$$(\beta+kt)\left((\beta+kt)+1\right)\cdots\left((\beta+kt)+k(s-t)-1\right) = y^k,$$

for some $y \in \mathbb{Z}$. Observe that $y \neq 1$. As $\beta + kt > 0$, $k(s-t) > 1$, $k > 1$ and $y > 1$, the conditions of Theorem 5 are satisfied, the first statement follows. For the second one let $\beta \in \mathbb{Z}_{>0}$ with $\beta \geq \max(ks, kt)$. Then

$$\sqrt[k]{\frac{(-\beta)_{ks}}{(-\beta)_{kt}}} = (-1)^{s-t} \sqrt[k]{\frac{(\beta-(ks-1))_{ks}}{(\beta-(kt-1))_{kt}}} \in \mathbb{Q},$$

which in the same way as above gives the result. □

5. Proof of Main Theorem

5.1. **Preliminaries.** To start with, we need explicit knowledge of the coefficients of the polynomial

$$M_n^{(\beta,c)}(Ax+B) = k_n^{(n)}x^n + k_{n-1}^{(n)}x^{n-1} + \cdots + k_0^{(n)}$$
$$= \sum_{k=0}^{n} \frac{\left(1-\frac{1}{c}\right)^k}{(\beta)_k}\binom{n}{k}(Ax+B)(Ax+B-1)\cdots(Ax+B-(k-1))$$

in order to compare them as Theorem 4 suggests. Recall that $A \neq 0$, $\beta \in \mathbb{Z} \setminus \{0,-1,\ldots,-n+1\}$ and $c \in \mathbb{Q} \setminus \{0,1\}$. We have

$$k_{n-j}^{(n)} = A^{n-j}\sum_{l=0}^{j} \eta_{n-l}\xi_{j-l,n-l}, \qquad j = 0,1,\ldots,n$$

with

$$\eta_k = \frac{\left(1-\frac{1}{c}\right)^k}{(\beta)_k}\binom{n}{k} \qquad \text{and}$$

$$\xi_{l,k} = \sum_{0\leq i_1<i_2<\cdots<i_l\leq k-1} \prod_{\nu=1}^{l}(B-i_\nu),$$

with $0 \leq k,l \leq n$. Moreover, define

$$\tilde{k}_{n-j}^{(n)} = \frac{(n-j)!(\beta)_n c^j}{n!A^{n-j}\left(1-\frac{1}{c}\right)^{n-j}}\, k_{n-j}^{(n)}.$$

Note that $k_{n-j}^{(n)} = 0$ is equivalent to $\tilde{k}_{n-j}^{(n)} = 0$ provided $n \neq 0,1,\ldots,j-1$. With help of MAPLE we explicitly calculate

$$\tilde{k}_{n-1}^{(n)} = \frac{1}{2}(c+1)n + Bc - B + \beta c - \frac{1}{2}c - \frac{1}{2}.$$

The equation $\tilde{k}_{n-1}^{(n)} = 0$ yields

$$B = \frac{c+1-cn-n-2c\beta}{2(c-1)}. \tag{12}$$

By this choice of B we write down some more coefficients

$$\tilde{k}_{n-2}^{(n)} = \frac{1}{24}\{(-c^2-10c-1)n - c^2 + 14c - 12\beta c - 1\},$$
$$\tilde{k}_{n-3}^{(n)} = \frac{1}{6}\,c(c+1)(\beta+n-1),$$
$$\tilde{k}_{n-4}^{(n)} = \frac{1}{5760}\{5\left(c^2+10c+1\right)^2 n^2 + \left(120c^3\beta - 288c^3 - 2328c^2 + 1200c^2\beta + 12c^4 - 288c + 120\beta c\right.$$
$$\left.+12\right)n + 7c^4 + 92c - 120\beta c + 92c^3 - 120c^3\beta + 1962c^2 + 7 + 720\beta^2c^2 - 2640c^2\beta\}.$$

Now, assume

$$M_n^{(\beta,c_1)}(Ax+B)=\varphi(f(x)) \qquad \text{and} \qquad M_m^{(\beta,c_2)}(\widetilde{A}x+\widetilde{B})=\varphi(g(x)), \tag{13}$$

where (f,g) is a standard pair, $A, B, \widetilde{A}, \widetilde{B}$ are rational numbers with $A\widetilde{A}\neq 0$ and

$$\varphi(x)=e_kx^k+e_{k-1}x^{k-1}+\cdots+e_0 \qquad \text{with } e_k\neq 0 \qquad (k\geq 1).$$

By comparing the leading coefficients in (13) we have

$$\frac{\left(1-\frac{1}{c_1}\right)^n A^n}{(\beta)_n}=e_kf_s^k \qquad \text{and} \qquad \frac{\left(1-\frac{1}{c_2}\right)^m \widetilde{A}^m}{(\beta)_m}=e_kg_t^k,$$

where $f(x)=f_sx^s+\cdots+f_0$ and $g(x)=g_tx^t+\cdots+g_0$ with $f_sg_t\neq 0$. Obviously, $n=ks$ and $m=kt$ by comparison of degrees. By taking the quotient of the previous two equations we get

$$\frac{(\beta)_{ks}}{(\beta)_{kt}}=\left(\frac{\left(1-\frac{1}{c_1}\right)^s A^s g_t}{\left(1-\frac{1}{c_2}\right)^t \widetilde{A}^t f_s}\right)^k. \tag{14}$$

Of course, a necessary condition for (14) is given by

$$\sqrt[k]{\frac{(\beta)_{ks}}{(\beta)_{kt}}}\in\mathbb{Q},$$

which by $m\neq n$ and Corollary 6 implies $k=1$. Thus, we just have to deal with the two representations

$$M_n^{(\beta,c_1)}(Ax+B)=e_1f(x)+e_0 \qquad \text{and}$$
$$M_m^{(\beta,c_2)}(\widetilde{A}x+\widetilde{B})=e_1g(x)+e_0.$$

Now, we take a closer look at the various standard pairs. To begin with, the standard pair (f,g) cannot be of the *second kind* due to $m,n\geq 3$. The rest of the investigation splits into four cases, corresponding to the four remaining kinds of standard pairs.

5.2. **Standard pair of the first kind** $(x^q, \alpha x^r v(x)^q)$ **or switched.** First, let $m,n\geq 4$. Without loss of generality suppose $M_n^{(\beta,c)}(Ax+B)=e_1x^q+e_0$ where $q=n\geq 4$. Hence $k_{n-1}^{(n)}=k_{n-2}^{(n)}=0$, or equivalently, $\tilde{k}_{n-1}^{(n)}=\tilde{k}_{n-2}^{(n)}=0$. This yields

$$n=-\frac{c^2-14c+12c\beta+1}{c^2+10c+1}. \tag{15}$$

Note that the denominator $c^2+10c+1$ is always non-zero as it has non-rational roots $c=-5\pm 2\sqrt{6}$. Finally $k_{n-3}^{(n)}=0$, or equivalently, $\tilde{k}_{n-3}^{(n)}=0$,

$$c(c-1)^2(c+1)(\beta-2)=0. \tag{16}$$

If $\beta=2$, then (15) gives $n=-1$, a contradiction. On the other hand, for $c=-1$ relation (15) yields $n=2-\frac{3}{2}\beta$, which is again a contradiction for $\beta\geq 1$ as well as for $-\beta\geq\max(n,m)\geq n$. Let now $\min(n,m)=3$. Without loss of generality we may assume $M_3^{(\beta,c)}(Ax+B)=e_1x^3+e_0$. In this case equations (12) and (15) imply $c^2+(3\beta+4)c+1=0$. This quadratic equation has solutions

$$c=-2-\frac{3}{2}\beta\pm\frac{1}{2}\sqrt{(3\beta+4)^2-4}.$$

There do not exist any two perfect integer squares having difference 4, hence $c \notin \mathbb{Q}$, again a contradiction.

5.3. **Standard pair of the third kind** $(D_s(x,\alpha^t), D_t(x,\alpha^s))$**.** By symmetry we may suppose without loss of generality that $s = n \geq 4$. Then,

$$M_n^{(\beta,c)}(Ax+B) = e_1 D_n(x,\delta) + e_0,$$

where $\delta = \alpha^t$ and

$$D_n(x,\delta) = x^n - n\delta x^{n-2} + \frac{n(n-3)}{2}\delta^2 x^{n-4} + \dots.$$

The condition $\tilde{k}_{n-1}^{(n)} = \tilde{k}_{n-3}^{(n)} = 0$ gives

$$c(c+1)(\beta+n-1) = 0, \tag{17}$$

which implies $c = -1$ since the other case $n = 1-\beta$ is not possible for $\beta \geq 1$ and $-\beta \geq \max(m,n) \geq n$. By comparing coefficients of the power x^{n-2} we have $-k_n^{(n)} n\delta = k_{n-2}^{(n)}$ which yields

$$\delta = -\frac{1}{24A^2}(n-1)(2n+3\beta-4).$$

First, suppose $n > 4$. Then $k_{n-4}^{(n)} = k_n^{(n)} n(n-3)\delta^2/2$ leads to an equation just in terms of n and β; its solutions are $n = \frac{7}{4} - \frac{15}{16}\beta \pm \frac{1}{16}\sqrt{-15\beta^2+120\beta+16}$. The latter solution can never be a rational integer as we have $-15\beta^2+120\beta+16 < 0$ for $\beta \leq -1$ and $\beta \geq 9$. The cases $\beta = 1,\dots,8$ can easily be seen to violate $n \in \mathbb{Z}_{>0}$. We end up with a contradiction for $n > 4$. Finally, the case $(n,m) = (4,3)$ or reversed $(3,4)$ has to be dealt with. Suppose

$$M_3^{(\beta,c_1)}(Ax+B) = e_1 D_3(x,\alpha^4) + e_0 = e_1(x^3 - 3\alpha^4 x) + e_0, \tag{18}$$

$$M_4^{(\beta,c_2)}(\widetilde{A}x+\widetilde{B}) = e_1 D_4(x,\alpha^3) + e_0 = e_1(x^4 - 4\alpha^3 x^2 + 2\alpha^6) + e_0. \tag{19}$$

From (19), (17) and (12) we obtain $c_2 = -1$ and $\widetilde{B} = -\frac{1}{2}\beta$. Comparison of coefficients in (19) gives

$$e_1 = \frac{16A^4}{(\beta)_4}, \qquad -4\alpha^3 e_1 = \frac{8A^2(3\beta+4)}{(\beta)_4},$$

$$2\alpha^6 e_1 + e_0 = \frac{3}{(\beta+1)(\beta+3)}.$$

Thus,

$$e_0 = -\frac{3\beta^2+12\beta+16}{2(\beta)_4}. \tag{20}$$

On the other hand, (18) and (12) imply $B = -\frac{c_1+1+c_1\beta}{c_1-1}$ and

$$e_0 = k_3^{(3)} = \frac{c_1+1}{c_1^2\beta(\beta+1)}. \tag{21}$$

Finally, from (20) and (21) follows a quadratic equation for the variable c_1 whose solutions are

$$c_1 = -\frac{\beta^2+5\beta+6 \pm \sqrt{-(\beta+2)(\beta+3)(5\beta^2+19\beta+26)}}{3\beta^2+12\beta+16}.$$

Observe that $3\beta^2+12\beta+16\neq 0$. The term under the square root is non-negative if and only if $-3\leq\beta\leq -2$, a contradiction arises.

5.4. **Standard pair of the fourth kind** $\left(\alpha_1^{-s/2}D_s(x,\alpha_1), -\alpha_2^{-t/2}D_t(x,\alpha_2)\right)$**.** We may assume $s>4$ since in case of $(n,m)=(4,3)$ (or reversed) the condition $(s,t)=2$ would be violated. Now,

$$M_n^{(\beta,c)}(Ax+B)=\widetilde{e_1}\alpha_1^{-n/2}D_n(x,\alpha_1)+e_0=e_1D_n(x,\alpha_1)+e_0.$$

The argument now follows exactly the previous lines, with δ replaced by α_1. Again, we end up with a contradiction.

5.5. **Standard pair of the fifth kind** $\left((\alpha x^2-1)^3, 3x^4-4x^3\right)$ **or switched.** Without loss of generality we have

$$\begin{aligned}M_6^{(\beta,c)}(Ax+B)&=e_1(\alpha x^2-1)^3+e_0\\&=e_1(\alpha^3x^6-3\alpha^2x^4+3\alpha x^2-1)+e_0.\end{aligned}$$

By $k_5^{(6)}=k_3^{(6)}=0$, (17) and (12) we get $c=-1$ and $B=-\frac{1}{2}\beta$. The remaining non-trivial coefficient equations for $M_6^{(\beta,c)}(Ax+B)$ are

$$\alpha^3e_1=\frac{64A^6}{(\beta)_6}, \tag{22}$$

$$-3\alpha^2e_1=\frac{80A^4(3\beta+8)}{(\beta)_6}, \tag{23}$$

$$3\alpha e_1=\frac{4A^2(45\beta^2+210\beta+184)}{(\beta)_6}. \tag{24}$$

We divide the square of (23) by the product of (22) and (24) in order to obtain a single equation only in terms of β,

$$3=25\cdot\frac{(3\beta+8)^2}{45\beta^2+210\beta+184}.$$

Note that $45\beta^2+210\beta+184\neq 0$ for $\beta\in\mathbb{Z}$. This implies $\beta=-\frac{19}{6}\pm \mathrm{i}\frac{1}{30}\sqrt{1455}$, a contradiction. This completes the proof of Theorem 2.

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Thomas Stoll, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8–10/104, A-1040 Wien, Austria;
E-mail address: stoll@dmg.tuwien.ac.at

Robert F. Tichy, Institut für Mathematik (A), Technische Universität Graz, Steyrergasse 30, A-8010 Graz, Austria;
E-mail address: tichy@tugraz.at