Interaction Graphs: Additives

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Abstract
Geometry of Interaction (GoI) is a kind of semantics of linear logic proofs that aims at accounting for the dynamical aspects of cut elimination. We present here a parametrized construction of a Geometry of Interaction for Multiplicative Additive Linear Logic (MALL) in which proofs are represented by families of directed weighted graphs. Contrarily to former Geometries of Interaction dealing with additive connectives [Gir95a, Gir11a], the proofs of MALL are interpreted by finite objects in this model. Moreover, we show how one can obtain, for each value of the parameter, a denotational semantics for MALL from this GoI. Finally, we will show how this setting is related to Girard’s various constructions: two particulars choices of the parameter respectively give a combinatorial version of his latest GoI [Gir11a] or a refined version of older Geometries of Interaction [Gir89a, Gir88, Gir95a].

1 Introduction

The Geometry of Interaction program [Gir89b]. It was introduced by Girard a couple of years after his discovery of Linear Logic [Gir87a]. It aims at giving a semantics of linear logic proofs that would account for the dynamical aspects of cut-elimination, hence of computation through the proofs-as-program correspondence. Informally, a Geometry of Interaction (GoI) consists in:

- a set of mathematical objects — paraproofs — that will contain, among other things, the interpretations of proofs (or λ-terms);
- a notion of execution that will represent the dynamics of cut-elimination (or β-reduction).

Then, from these basic notions, one should be able to “reconstruct” the logic from the way the paraproofs interact:

- From the notion of execution, one defines a notion of orthogonality between the paraproofs that will allow to define formulas — types — as sets of paraproofs closed under bi-orthogonality (a usual construction in realizability). The notion of orthogonality should be thought of as a way of defining negation based on its computational effect.
• The connectives on formulas are defined from "low-level" operations on the paraproofs, following the idea that the rules governing the use of a connective should be defined by the way this connective acts at the level of proofs, i.e. by its computational effect.

Throughout the years, Girard defined several such semantics, mainly based on the interpretation of a proof as an operator on an infinite-dimensional Hilbert space. In particular, two such constructions offer a treatment of additive connectives of linear logic [Gir95a, Gir11a]. It is also worth noting that the first version of GoI [Gir89a] allowed Abadi, Gonthier, and Levy [AGL92] to explain the optimal reduction of $\lambda$-calculus defined by Lamping [Lam90].

The latest version of GoI [Gir11a], from which this work is greatly inspired, is related to quantum coherent spaces [Gir03], which suggest future applications to quantum computing. Moreover, the great generality and flexibility of the definition of exponentials also seem promising when it comes to the study of complexity.

This work. Departing from the realm of infinite-dimensional vector spaces and linear maps between them, we propose a graph-theoretical GoI where proofs are interpreted by finite objects. In this framework, it is possible to define the multiplicative and additive connectives of Linear Logic. This construction is moreover parametrized by a function from the interval $[0, 1]$ to $\mathbb{R}_{\geq 0} \cup \{\infty\}$, and therefore yields not just one but a whole family of models.

We then proceed to show how, from any of these models, one can obtain a $*$-autonomous category with $\neg \not\otimes$ and $1 \not\otimes \perp$, i.e. a non-degenerate denotational semantics for Multiplicative Linear Logic (MLL). However, as in all the versions of GoI dealing with additive connectives, our construction of additives does not define a categorical product. But, introducing a notion of observational equivalence within the model, we are able to define a categorical product from our additive connectives when considering classes of observationally equivalent objects, obtaining a denotational semantics for Multiplicative Additive Linear Logic (MALL).

Finally, we show how our framework is related to Girard’s versions of GoI. Indeed, a first choice of map gives us a model that can be embedded in Girard’s GoI5 framework [Gir11a]. It can be shown that it is a combinatorial version of (the multiplicative additive) fragment of GoI5, offering insights on its notion of orthogonality and constructions. On the other hand, a second choice of map defines a model where orthogonality is defined by nilpotency: our construction thus defines in this case a (refined) version of older Geometries of Interaction [Gir89a, Gir88, Gir95a].

2 Graphs and Cycles

We first recall some definitions and notations of our earlier paper [Sei12].

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1 A first result in this direction was obtained by Girard [Gir11c] who obtained a characterization of NL — the class of non-deterministic logspace algorithms.

2 Even though the graphs we consider can have an infinite set of edges, linear logic proofs are represented by finite graphs (disjoint unions of transpositions).
Definition 1. A directed weighted graph is a tuple $G$, where $V^G$ is the set of vertices, $E^G$ is the set of edges, $s^G$ and $t^G$ are two functions from $E^G$ to $V^G$, the source and target functions, and $\omega^G$ is a function $E^G \rightarrow [0, 1]$.

In this paper, we will work with directed weighted graphs where the set of vertices is finite, and the set of edges is finite or countably infinite.

We will write $E^G(v, w)$ the set of $e \in E^G$ satisfying $s^G(e) = v$ and $t^G(e) = w$. Moreover, we will sometimes forget the exponents when the context is clear.

Definition 2 (Plugging). Given two graphs $G$ and $H$, we define the graph $G \square H$ as the union graph of $G$ and $H$, together with a coloring function $\delta$ from $E^G \sqcup E^H$ to $\{0, 1\}$ such that

$$
\begin{cases}
\delta(x) = 0 & \text{if } x \in E^G \\
\delta(x) = 1 & \text{if } x \in E^H
\end{cases}
$$

We refer to $G \square H$ as the plugging of $G$ and $H$.

Definition 3 (Paths, cycles and $k$-cycles). A path in a graph $G$ is a finite sequence of edges $(e_i)_{0 \leq i \leq n}$ ($n \in N$) in $E^G$ such that $s(e_{i+1}) = t(e_i)$ for all $0 \leq i \leq n - 1$. We will call the vertices $s(\pi) = s(e_0)$ and $t(\pi) = t(e_n)$ the beginning and the end of the path.

We will also call a cycle a path $\pi = (e_i)_{0 \leq i \leq n}$ such that $s(e_0) = t(e_n)$. If $\pi$ is a cycle, and $k$ is the greatest integer such that there exists a cycle $\rho$ with $^3 \pi = \rho^k$, we will say that $\pi$ is a $k$-cycle.

Definition 4 (Alternating paths). Let $G$ and $H$ be two graphs. We define the alternating paths between $G$ and $H$ as the paths $(e_i)$ in $G \square H$ which satisfy

$$
\delta(e_i) \neq \delta(e_{i+1}) \quad (i = 0, \ldots, n - 1)
$$

We will call an alternating cycle in $G \square H$ a cycle $(e_i)_{0 \leq i \leq n}$ in $G \square H$ which is an alternating path and such that $\delta(e_n) \neq \delta(e_0)$.

The set of alternating paths in $G \square H$ will be denoted by $\text{Alt}(G, H)$, while $\text{Alt}(G, H)_V$ will mean the subset of alternating paths in $G \square H$ with source and target in a given set of vertices $V$.

Proposition 5. Let $\rho = (e_i)_{0 \leq i \leq n-1}$ be a cycle, and let $\sigma$ be the permutation taking $i$ to $i+1$ ($i = 0, \ldots, n-2$) and $n-1$ to $0$. We define the set

$$
\tilde{\rho} = \{(e_{\sigma^k(i)})_{0 \leq i \leq n-1} \mid 0 \leq k \leq n - 1\}
$$

Then $\rho$ is a $k$-cycle if and only if the cardinality of $\tilde{\rho}$ is equal to $n/k$. In the following, we will refer to such an equivalence class modulo $k$-circuit permutations as a $k$-circuit.

Proof. We use classical cyclic groups techniques here. We will abusively denote by $\sigma^k(\rho)$ the path $(e_{\sigma^k(i)})_{0 \leq i \leq n-1}$.

First, notice that if $\rho$ is a $k$-cycle, then $\sigma^{n/k}(\rho) = \rho$. Now, if $s$ is the smallest integer such that $\sigma^s(\rho) = \rho$, we have that $e_{i+s} = e_i$. Hence, writing $m = n/s$, we have $\rho = \pi^m$ where $\pi = (e_i)_{0 \leq i \leq s-1}$. This implies that $k = n/s$ from the

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Here, we denote by $\rho^k$ the concatenation of $k$ copies of $\rho$. 

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3
maximality of $k$. Hence $\rho$ is a $k$-cycle if and only if the smallest integer $s$ such that $\sigma^s(\rho) = \rho$ is equal to $n/k$.

Let $s$ be the smallest integer such that $\sigma^s(\rho) = \rho$. We have that for any integers $p, q$ such that $0 \leq q < s$, $\sigma^{p+q}(\rho) = \sigma^q(\rho)$. Indeed, it is a direct consequence of the fact that $\sigma^p(\rho) = \rho$ for any integer $p$. Moreover, since $\sigma^n(\rho) = \rho$, we have that $s$ divides $n$. Hence, we have that the cardinality of $\bar{\rho}$ is at most $s$. To show that the cardinality of $\bar{\rho}$ is exactly $s$, we only need to show that $\sigma^i(\rho) \neq \sigma^j(\rho)$ for $i < j$ between 0 and $s - 1$. But if it were the case, we would have, since $\sigma$ is a bijection, $\rho = \sigma^{j-i}(\rho)$, an equality contradicting the minimality of $s$.

Definition 6 (The set of 1-circuits). We will denote by $\mathcal{C}_1(G, H)$ the set of alternating 1-circuits in $G \square H$, i.e. the quotient of the set of alternating 1-cycles by cyclic permutations.

Definition 7. Let $F$ and $G$ be two graphs. We define the execution of $F$ and $G$ as the graph $\bar{F} : \bar{G}$ defined by:

\[
\begin{align*}
V^{\bar{F}} : \bar{G} &= V^{\bar{F}} \Delta V^{\bar{G}} = (V^F \cup V^G) - (V^F \cap V^G) \\
E^{\bar{F}} : \bar{G} &= \text{Alt}(F, G)_{V^F \Delta V^G} \\
s^{\bar{F}} : \bar{G} &= \pi \mapsto s(\pi) \\
i^{\bar{F}} : \bar{G} &= \pi \mapsto t(\pi) \\
\omega^{\bar{F}} : \bar{G} &= \pi = (e_i)_{i=0}^n \mapsto \prod_{i=0}^n \omega^{G \square H}(e_i)
\end{align*}
\]

When $V^F \cap V^G = \emptyset$, we will write $F \cup G$ instead of $F : G$.

Proposition 8 (Associativity). Let $G_0, G_1, G_2$ be three graphs with $V^{G_0} \cap V^{G_1} \cap V^{G_2} = \emptyset$. We have:

\[
G_0 : (G_1 : G_2) = (G_0 : G_1) : G_2
\]

Proof. Let us first define the 3-colored graph $G_0 \square G_1 \square G_2$ as the union graph $(\bigcup V^i, \bigcup E^i, \bigcup s^i, \bigcup t^i)$ together with the coloring function $\delta$ from $\bigcup E^i$ into $\{0, 1, 2\}$ which associates to each edge the number $i$ of the graph $G_i$ it comes from. We consider the 3-alternating paths between $G_0, G_1, G_2$, that is the paths $(e_i)$ in $G_0 \square G_1 \square G_2$ satisfying:

\[
\delta(e_i) \neq \delta(e_{i+1})
\]

Then, we can define the simultaneous reduction of $G_0, G_1, G_2$ as the graph $\vdash_i G_i = (V^0 \Delta V^1 \Delta V^2, F, s^F, t^F)$, where $F$ is the set of 3-alternating paths between $G_0, G_1, G_2$, $s^F(e)$ is the beginning of the path $e$ and $t^F(e)$ is its end.

We then show that this induced graph $\vdash_i G_i$ is equal to $(G_0 : G_1) : G_2$ and $G_0 : (G_1 : G_2)$. This is a simple verification. Indeed, to prove for instance that $\vdash_i G_i$ is equal to $(G_0 : G_1) : G_2$, we just write the 3-alternating paths in $G_0, G_1, G_2$ as an alternating sequence of alternating paths in $G_0 \square G_1$ (with source and target in $V^0 \Delta V^1$, i.e. an edge of $G_0 : G_1$) and edges in $G_2$.

\footnote{This is where the hypothesis $V^0 \cap V^1 \cap V^2 = \emptyset$ is important. If this is not satisfied, one gets some 3-alternating paths of the form $\rho x$, where $x$ is an edge in $G_2$ and $\rho$ is an alternating path in $G_0 \square G_1$, but such that $\rho$ does not correspond to an edge in $G_0 : G_1$.}
Definition 9. For any function \( m : [0, 1] \to \mathbb{R}_{\geq 0} \cup \{\infty\} \), let us define a measure on graphs by \( J_{F,G} = \sum_{\pi \in \mathcal{C}_y(F,G)} m(\omega_{F \Box G}(\pi)) \).

Proposition 10 (Geometric cyclic property). Let \( F, G, H \) be three graphs such that \( V^F \cap V^G \cap V^H = \emptyset \). Then:

\[
\mathcal{C}_y(F, G) \cup \mathcal{C}_y(F : G, H) \cong \mathcal{C}_y(F : G, H) \cup \mathcal{C}_y(G : H, F) \\
\cong \mathcal{C}_y(H, F) \cup \mathcal{C}_y(H : F, G)
\]

Moreover, these bijections preserve weights.

Proof. We prove this following the idea of the proof of associativity.

Let \( A = F \Box G \Box H \) the three-colored graph obtained from \( F, G, H \) together with its 3-coloring function \( \rho \), and consider a 3-alternating 1-cycle in \( F \Box G \Box H \), i.e. a 1-cycle \((e_i)_{i=0}^n\) which is 3-alternating and that satisfies \( \rho(e_0) \neq \rho(e_n) \). Denote by \( \mathcal{C}_y^3(F, G, H) \) the set of 3-alternating 1-cycles in \( F \Box G \Box H \). Then \( \pi \in \mathcal{C}_y^3(F, G, H) \) is either is an alternating 1-cycle in \( F \Box G \) (if it does not contain any edge from \( H \)) or it is a 1-cycle composed of alternations between alternating paths (with source and target in \( V^F \Delta V^G \), see Footnote 4) in \( F \Box G \Box H \). Hence, \( \pi \) is either in \( \mathcal{C}_y(F, G) \) or in \( \mathcal{C}_y(F : G, H) \). Conversely, any alternating 1-cycle in \( F \Box G \Box H \) corresponds to a unique 3-alternating cycle in \( F \Box G \Box H \). Hence we have a bijection \( \mathcal{C}_y^3(F, G, H) \cong \mathcal{C}_y(F, G) \cup \mathcal{C}_y(F : G, H) \).

A similar argument shows that \( \mathcal{C}_y^3(F, G, H) \cong \mathcal{C}_y(G, H) \cup \mathcal{C}_y(G : H, F) \) and that \( \mathcal{C}_y^3(F, G, H) \cong \mathcal{C}_y(H, F) \cup \mathcal{C}_y(H : F, G) \).

Corollary 10.1 (Scalar cyclic property). Let \( F, G, H \) be such as in the preceding proposition. Then:

\[
[F, G] + [F : G, H] = [G, H] + [G : H, F] = [H, F] + [H : F, G]
\]

Corollary 10.2 (Geometric three-terms adjunction). Let \( F, G, \) and \( H \) be weighted graphs such that \( V^G \cap V^H = \emptyset \). We have

\[
\mathcal{C}_y(F, G \cup H) \cong \mathcal{C}_y(F, G) \cup \mathcal{C}_y(F : G, H)
\]

Corollary 10.3 (Scalar three-terms adjunction). With the hypotheses of the last corollary:

\[
[F, G \cup H] = [F, G] + [F : G, H]
\]

3 The Additive Construction

The additive construction consists in considering finite weighted families of graphs \( G_i \) on the same set of vertices. This can be related to the way additives are dealt with in proof nets [Gir95b, HG03]. The main interest of this approach is that it allows us to juxtapose two graphs while making sure they cannot interact: such an operation should be the interpretation of the \& connective.
Definition 11. A sliced graph \( F \) of carrier \( V^F \) is a family \( F = \sum_{i \in I^F} \alpha_i^F F_i \) where \( I^F \) is a finite set, and, for all \( i \in I^F \), \( \alpha_i^F \) is a real number and \( F_i \) is a graph on the set of vertices \( V^F \).

Given a sliced graph \( F \), we define \( 1_F = \sum_{i \in I^F} \alpha_i^F \).

We can now easily extend execution and measurement on sliced graphs.

Definition 12. Let \( F \) and \( G \) be two sliced graphs. We define their execution as:

\[
\left( \sum_{i \in I^F} \alpha_i^F F_i \right) :: \left( \sum_{i \in I^G} \alpha_i^G G_i \right) = \sum_{(i,j) \in I^F \times I^G} \alpha_i^F \alpha_j^G F_i :: G_j
\]

When \( V^F \cap V^G = \emptyset \), we will denote the execution by \( F \cup G \).

Definition 13. Let \( F \) and \( G \) be two sliced graphs. We define the measurement:

\[
\left[ \sum_{i \in I^F} \alpha_i^F F_i, \sum_{i \in I^G} \alpha_i^G G_i \right] = \sum_{(i,j) \in I^F \times I^G} \alpha_i^F \alpha_j^G [F_i, G_j]
\]

In the case where some of the \([F_i, G_j]\) are equal to \( \infty \), we set \([F, G]\) to be equal to \( \infty \).

And a simple computation gives us the cyclic equality for sliced graphs as a consequence of the cyclic equality for graphs.

Proposition 14. Let \( F, G, H \) be sliced graphs such that \( V^F \cap V^G \cap V^H = \emptyset \). Then

\[
1_H [F, G] + [F :: G, H] = 1_F [G, H] + [G :: H, F] = 1_G [H, F] + [H :: F, G]
\]  \( \text{(1)} \)

Corollary 14.1. Let \( F, G, \) and \( H \) be sliced graphs such that \( V^G \cap V^H = \emptyset \). Then:

\[
[F, G \cup H] = 1_H [F, G] + [F :: G, H]
\]

In order to get a usual adjunction, relating only the terms \([F, G \cup H]\) and \([F :: G, H]\), we need to get rid of the additional term \(1_H [F, G] \). A good way of doing so is it capture this term in a real number that will be associated to our graphs, the wager. Hence, the objects we will be working with will be couples of a wager and a sliced graph.

Definition 15. A project is a couple \( a = (a, A) \) where \( A \) is a sliced graph and \( a \in \mathbb{R} \cup \{\infty\} \). We will call \( V^A \) the carrier of \( a \) and denote it by \( \text{car}_a \).

Since we will work in \( \mathbb{R} \cup \{\infty\} \), we need to explain how sums and products are defined on \( \infty \). We will follow a very simple rule: any sum and any product containing \( \infty \) will be equal to \( \infty \).

Definition 16. Let \( a, b \) be projects. We define the measurement:

\[
\ll a, b \gg = a1_B + 1_A b + [A, B]
\]

Definition 17. Let \( f \) and \( g \) be projects. We define the cut between \( f \) and \( g \) by:

\[
f :: g = (\ll f, g \gg, F :: G)
\]
Definition 18. Let \(a, b\) be projects of disjoint carriers. We define the tensor product as:
\[ a \otimes b = (\langle a, b \rangle, A \cup B) \]

The following theorem is a straightforward application of Proposition 14.

Theorem 19 (Cyclic Property). Let \(f, g, h\) be projects such that \(V^F \cap V^G \cap V^H = \emptyset\). Then:
\[ \langle f :: g, h \rangle = \langle f :: h, g \rangle = \langle g :: h, f \rangle \]

Corollary 19.1 (Adjunction). Let \(f, a, b\) be projects such that \(V^A \cap V^B = \emptyset\). Then:
\[ \langle f, a \otimes b \rangle = \langle f :: a, b \rangle \]

4 Localized Connectives

4.1 Multiplicatives

Definition 20 (Orthogonality). Two projects \(a, b\) on the same carrier are orthogonal, denoted by \(a \perp b\), if \(\langle a, b \rangle \neq 0, \infty\).

If \(A\) is a set of projects, the orthogonal \(A \perp\) of \(A\) is defined as the set \(\{b \mid \forall a \in A, a \perp b\}\).

Definition 21 (Conducts). A conduct is a set \(A\) of projects which is equal to its bi-orthogonal together with a set \(V_A\) such that \(a \in A \Rightarrow \text{car}_a = V_A\). The set \(V_A\) will be called the carrier of the conduct, and denoted by \(\text{car}_A\).

Proposition 22. Let \(A\) be a conduct, and \(a = (\infty, A) \in A\) be a project with an infinite wager. Then \(A \perp = \emptyset\). We will denote by \(0_{V^A} = A^\perp\) the empty conduct of carrier \(V^A\) and by \(T_{V^A} = A\) the conduct that contains all projects of carrier \(V^A\).

Remark. If there exists only one conduct (up to the choice of a carrier) containing projects with infinite wager, then why introduce them? In fact, the introduction of infinite wagers insures that the application \(f :: a\) is always defined. Technically, this allows us to have the equality between \(0_V \rightarrow A\) and \(A^\perp \rightarrow T_V\), which wouldn’t be the case if application were not always defined. Indeed, by definition of the linear implication (Definition 26 below), the conduct \(0_V \rightarrow A\) would be equal to \(T_{V \cup V^A}\), while the conduct \(A^\perp \rightarrow T_V\) would contain only the projects \(f\) with \(\text{car}_f = V \cup V^A\) such that \(f :: a\) is defined for all \(a \in A^\perp\).

Definition 23 (Tensor on Conducts). Let \(A, B\) be conducts of disjoint carrier. We can form the conduct \(A \otimes B\)
\[ A \otimes B = \{a \otimes b \mid a \in A, b \in B\}^{\perp\perp} \]

Definition 24. We will write \(0_V\) the project \((0, (V, \emptyset))\) where \((V, \emptyset)\) is the empty graph on the set of vertices \(V\) considered as a one-sliced graph. Then we will denote by \(1_V = \{0_V\}^{\perp\perp}\).

Proposition 25 (Properties of the Tensor). The tensor product is commutative and associative. Moreover it has a neutral element, namely \(1_\emptyset\).
Definition 26 (Linear Implication). Let $A, B$ be conducts of disjoint carriers $V^A$ and $V^B$.

$$A \rightarrow B = \{ f \mid \text{car}_f = V^A \cup V^B \land \forall a \in A, f:: a \in B \}$$

The fact that this defines a conduct is justified by the following proposition, which is a simple corollary of the adjunction (Corollary 19.1).

Proposition 27 (Duality). We have the following

$$A \rightarrow B = (A \otimes B^\perp)^\perp$$

Definition 28 (Delocations). Let $a$ be a project of carrier $V^A$, and $\phi : V^A \rightarrow V^B$ a bijection. We define the delocation of a graph $G$ as the graph $\phi(G) = (V^B, E^\phi, \phi \circ t^A, \phi \circ t^A, \omega^A)$. This extends to projects: the delocation of $a = (a, \sum_{i \in I_A} \alpha_i^A A_i)$ is defined as $\phi(a) = (a, \sum_{i \in I_A} \alpha_i^A \phi(A_i))$.

Similarly, the delocation of a conduct $A$ of carrier $V^A$ is defined as the conduct $\phi(A) = \{ \phi(a) \mid a \in A \}$ of carrier $V^B$.

Proposition 29. Keeping the notations of Definition 28 and supposing $V^A \cap V^B = \emptyset$, we define the project $\bar{\text{ar}}_\phi = (0, \{ \Phi \})$ whose slice has weight 1 and where:

$$E^\Phi = \{(a, \phi(a)) \mid a \in V^A\} \cup \{(\phi(a), a) \mid a \in V^A\}$$

$$\Phi = (V^A \cup V^B, E^\phi, \omega^\Phi(e) = 1)$$

Then $\bar{\text{ar}}_\phi \in A \rightarrow \phi(A)$.

4.2 Additives

4.2.1 Definitions

Definition 30. We extend the sum and product by a scalar to projects as follows:

$$a + \lambda b = (a + \lambda b, A + \lambda B)$$

where $A + \lambda B = \sum_{i \in I_A} \alpha_i^A A_i + \sum_{i \in I_B} \lambda \alpha_i^B B_i$.

Proposition 31. Let $a, b$ be projects, and $\lambda \in \mathbb{R}^\ast$. Then, for any project $c$, we have:

$$\ll a + \lambda b, c \gg = \ll a, c \gg + \lambda \ll b, c \gg$$

Corollary 31.1 (Homothety). Conducts are closed under homothety: for all $a \in A$ and all $\lambda \in \mathbb{R}$ with $\lambda \neq 0$, $\lambda a \in A$.

In order to define the $\oplus$ connective, we need to be able to associate to a project $a$ in a conduct $A$ a project $a_{\uparrow V}$ in the conduct $A \oplus B$. There is only one natural way of defining such a project $a_{\uparrow V}$.

Definition 32. Let $a = (a, A)$ be a project of carrier $V^A$, and $V$ a finite set such that $V \cap V^A = \emptyset$. We will write $a_{\uparrow V}$ the project $a \otimes \phi_V$.

If $A$ is a conduct of support $V^A$, then $A_{\uparrow V}$ will denote the set $\{ a_{\uparrow V} \mid a \in A \}$. 
Let $A$ be a conduct of carrier $V^A$ and let $V$ be a finite set such that $V \cap V^A = \emptyset$, then we want to define from $A$ a conduct of carrier $V^A \cup V$. This can be done in two different ways: we can consider the conduct $(A^\uparrow V)_{V^A}$ or we can consider $((A^\uparrow V)_{V^A})^\perp$. These two different ways will be the core of the definition of additives.

**Definition 33.** Let $A$ and $B$ be non-empty conducts of disjoint carriers, we define the set

$$A + B = \{a^\uparrow V_a + b^\uparrow V_b \mid a \in A, b \in B\}$$

**Definition 34** (Additive Connectives). Let $A, B$ be conducts. We define the conducts $A \& B$ and $A \oplus B$ as follows:

$$A \oplus B = ((A^\uparrow V_{A^B})_{V^A} \uplus (B^\uparrow V_{A^B})_{V^A})^\perp$$

$$A \& B = ((A^\perp_{V^A})_{V^A} \cap (B^\perp_{V^A})_{V^A})^\perp$$

**Proposition 35.**

$$(A \& B)^\perp = A^\perp \oplus B^\perp$$

**Proposition 36.** The $\&$ connective is commutative, associative, and has a neutral element: the full conduct on the empty carrier $T^\emptyset$.

However, this construction leaves us more or less empty-handed: how could one obtain a project in $A \& B$ from two projects $a, b$ respectively in $A, B$? We will need a more explicit construction of the $\&$ construction at the level of projects. In order to do this, one has to restrict to a particular class of conducts which we will call behaviours.

### 4.2.2 Behaviours

**Definition 37.** A behaviour $A$ of carrier $V$ is a conduct $A$ such that for all $\lambda \in \mathbb{R}$:

1. if $a = (a, A) \in A$, then $a + \lambda \cdot V \in A$;
2. if $a = (a, A) \in A^\perp$, then $a + \lambda \cdot V \in A^\perp$.

**Remark.** The orthogonal of a behaviour is a behaviour.

This definition of hat is a behaviour is however quite cumbersome and difficult to work with. The following characterization (Proposition 40) allows to simplify greatly the proofs of the results of this section.

**Proposition 38.** If $A$ is a non-empty set of projects of carrier $V$ such that $a \in A \Rightarrow a + \lambda \cdot V \in A$, then any project in $A^\perp$ is wager-free, i.e. if $(a, A) \in A^\perp$ then $a = 0$.

**Proof.** Choose $a = (a, A) \in A$, which is possible since $A$ is supposed to be non-empty. Then for any $b = (b, B) \in A^\perp$, $\langle a, b \rangle \neq 0$. But $a + \lambda \cdot V \in A$ for any $\lambda \in \mathbb{R}$. Then $\langle a + \lambda \cdot V, b \rangle = \langle a, b \rangle + b\lambda$ must be non-zero, hence $b$ must be equal to $0$. \(\square\)

5The behaviours are the Interaction Graphs’ counterpart of Girard’s dichologies in his latest paper [Gir11b]
Proposition 39. If $A$ is a non-empty set of projects of same carrier $V^A$ such that $(a, A) \in A$ implies $a = 0$, then $b \in A^\perp$ implies $b + \lambda \circ_V A \in A^\perp$ for all $\lambda \in \mathbb{R}$.

Proof. Chose $a = (a, A) \in A$. For any project $b$, we have $\langle a, b + \lambda \circ V \rangle = \langle a, b \rangle + \lambda a$. Since $a = 0$, we get that $\langle a, b + \lambda \circ V \rangle = \langle a, b \rangle$. Hence, if $b \in A^\perp$, $b + \lambda \circ V \in A^\perp$. $\square$

Corollary 39.1. If a conduct $A$ of carrier $V$ is such that
1. if $a \in A$, then $a + \lambda \circ_V \in A$;
2. if $a = (a, A) \in A$ then $a = 0$;
3. $A$ is non-empty.

Then $A$ is a behaviour, and its orthogonal satisfies all the conditions above. We call such a behaviour proper.

Proof. From the preceding proposition and the second and third conditions, we get that $A^\perp$ satisfies that $b \in A^\perp \Rightarrow b + \lambda \circ \in A^\perp$. Hence $A$ is a behaviour. Moreover, if $A^\perp$ were empty, any project $a$ of same carrier as $A$ would be in the orthogonal of $A^\perp$. Hence $A$ wouldn’t be a conduct, since it contains only wager-free projects. Finally, all projects in $A^\perp$ are wager-free from proposition 38. $\square$

Proposition 40. A behaviour is either proper, equal to $0_V = \emptyset$ or equal to $T_V = 0^\perp$.

Proof. Let $A$ be a behaviour. If it is empty, then $A = 0$. If $A^\perp$ is empty, then $A = T$. In the other cases, since $A^\perp$ is non-empty we get that $A$ contains only wager-free projects from proposition 38. Hence $A$ is proper since it satisfies all the needed conditions. $\square$

As a consequence of this proposition, it is easy to show that $1$ and $\perp$ are not behaviours. However, the class of behaviour, which is closed under taking the orthogonal, is closed under the multiplicative and additive connectives, as the following propositions state.

Proposition 41. If $A$ and $B$ are behaviours of disjoint carriers, then $A \to B$ and $A \otimes B$ are behaviours.

Proof. First suppose $A, B$ are proper. Let $f \in A \to B$, $a \in A$ and $b \in B^\perp$ be projects. Then, from the adjunction we have that $f \perp a \otimes b$ if and only if $f : a \perp b$. Since $A \otimes B^\perp$ is non-empty and contains only wager-free projects, $f \in A \rightarrow B$ implies $f + \lambda \in A \rightarrow B$. Moreover, for all $\lambda \in \mathbb{R}$, $f : a + \lambda \circ B$, hence $f \lambda A + f\lambda + \langle F, A \rangle = 0$. We therefore get that $f = 0$. Hence $A \rightarrow B$ contains only wager-free projects. Finally, either $A \rightarrow B$ is empty, and therefore equal to $0$, either it is non-empty and therefore satisfies the conditions needed to be a proper behaviour.

Now, if either $A = 0$ or $B = T$, it is easy to see that $A \rightarrow B = T$.

The last case is when $A = T$ and $B$ is proper or $B = 0$ while $A$ is proper. Then by definition of the linear implication we get that $A \rightarrow B = 0$ in the second case. To prove that $T \rightarrow A = 0$ when $A$ is proper, notice that if $f$ is of
carrier \( V \cup V^A \), then applying \( \mathfrak{f} \) to a project with infinite wager yields a project with infinite wager. Hence, if \( T \rightarrow A \) were to be non-empty, \( A \) should contain a project with an infinite wager, i.e. \( A \) would not be proper.

From the duality of multiplicative connectives, \( A \otimes B = (A \rightarrow B^\perp)^\perp \). Since \( A, B^\perp \) are behaviours, \( A \rightarrow B^\perp \) is a behaviour. Therefore \( A \otimes B \) is the orthogonal of a behaviour, hence a behaviour.

**Proposition 42.** Let \( A, B \) be behaviours. Then \( A \& B \) and \( A \oplus B \) are behaviours.

**Proof.** Let \( A \) be a proper behaviour and let \( V^B \) be such that \( V^B \cap V^A = \emptyset \). Then if \( a \otimes o \in A_{\uparrow_{V^B}}, a \otimes o + \lambda o \in A_{\uparrow_{V^B}} \) since \( (a + \lambda o) \otimes o = a \otimes o + \lambda o \).

Moreover, \( A \) contains only wager-free projects, hence \( A_{\uparrow_{V^B}} \) contains only wager-free projects. Since \( A_{\uparrow_{V^B}} \) is closed under expansion, non-empty and contains only wager-free projects, it generates a behaviour \( (A_{\uparrow_{V^B}})^\perp \). Thus if \( A \) is a proper behaviour, \( A^\perp \) is a proper behaviour, which implies as just showed that \((A^\perp)_{\uparrow_{V^B}} \) is a behaviour, and finally \((A^\perp)_{\uparrow_{V^B}})^\perp \) is a behaviour.

If \( A = T_{V^A} \), we have \( T_{V^A}^\perp = 0_{V^A} \), hence \( (T_{V^A}^\perp)_{\uparrow_{V^B}} = T_{V^A \cup V^B} \). Thus \((A^\perp)_{\uparrow_{V^B}} \) is a behaviour.

If \( A = 0_{V^A} \), then \( A^\perp = T_{V^A} \), hence \((A^\perp)_{\uparrow_{V^B}} = \{a \otimes o \in V^B \mid a \in T_{V^A} \} = (T_{V^A} \otimes 1_{V^B})^\perp \). But, since there exists infinite wager projects in \( T \), any \( f \in T \rightarrow \downarrow_{V^B} \) would yield a project with an infinite wager. Hence \( \downarrow_{V^B} \) should contain a project with an infinite wager. But this is not possible, since \( \downarrow_{V^B} \neq T_{V^B} \). Hence \( T \rightarrow \downarrow_{V^B} \) is necessarily empty, i.e. \((0_{V^A}^\perp)_{\uparrow_{V^B}} = 0_{V^A \cup V^B} \) is a behaviour.

This implies that if \( A, B \) are behaviours, the conduct \( A \& B \) is a behaviour, as the intersection of \((A^\perp)_{\uparrow_{V^B}} \) and \((B^\perp)_{\uparrow_{V^B}} \) which are behaviours.

Since the orthogonal of a behaviour is a behaviour, we get that if \( A, B \) are behaviours, then \( A^\perp, B^\perp \) are behaviours, hence \( A^\perp \& B^\perp \) is a behaviour, and eventually \( A \oplus B = (A^\perp \& B^\perp)^\perp \) is a behaviour.

Now, there is something more. We justified the restriction to behaviours by the fact that additive connectives on conducts were not obtained through an operation at the level of projects, and therefore did not allow us to interpret the additive rules of sequent calculus. As we show in the next results, this restriction bore its fruits since we are now able to define the interpretation of the & rule at the level of projects.

**Definition 43** (Equivalence). We define, given a conduct \( A \), an equivalence of the projects in \( A \) as follows:

\[
a \cong_A b \iff \forall \epsilon \in A^\perp, \ll a, c \gg \Rightarrow \ll b, c \gg
\]

We will denote the equivalence class of \( f \) by \([f]_A\), forgetting the subscript when it is clear from the context.

**Proposition 44.** Let \( A, B \) be non empty behaviours, and \( f \in A \& B \). Then there exists \( g \in A \) and \( h \in B \) such that

\[
f \cong_A \& B g + h
\]

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Lemma 45. Let $E$ be a set of projects on a carrier $V$, and $a, b \in E^\perp$. Then $a \cong_{E^\perp} b$ if and only if $\forall \varepsilon \in E, \langle a, c \rangle = \langle b, c \rangle$.

Proof. By definition, if $a \cong_{E^\perp} b$, we have $\forall f \in E^\perp, \langle a, f \rangle = \langle b, f \rangle$, which is equivalent to $\forall \lambda \in \mathbb{R}, \langle \lambda a - \lambda b, f \rangle = 0$. Thus the equivalence of $a$ and $b$ can be restated as $\forall \lambda \in \mathbb{R}, \forall c \in E^\perp, \langle c + \lambda a - \lambda b, c \rangle \neq 0$, i.e. $\langle \lambda a - \lambda b, c \rangle \neq 0$. Finally, we have shown that $a \cong_{E^\perp} b$ if and only if $\forall \varepsilon \in E, \langle a, c \rangle = \langle b, c \rangle$.

Proposition 46. Let $A, B$ be non empty behaviours, and $f \in A \oplus B$. Then there exists $h \in A_{V, B} \cup B_{V, A}$ such that

$$f \cong_{A \oplus B} h$$

Proof. Let $f \in A \oplus B$. Then, since $A \oplus B \subset (A^\perp + B^\perp)^\perp$, we have that $f$ is orthogonal to $a^\prime + b^\prime$ for any $a^\prime \in A^\perp$ and $b^\prime \in B^\perp$. If $A^\perp, B^\perp$ are non-empty, we can take $a_0, b_0$ in them. Then, for any $\lambda, \mu$, the projects $\lambda a_0, \mu b_0$ are in $A^\perp, B^\perp$ respectively, hence $\langle f, \lambda a_0 + \mu b_0 \rangle = \lambda \langle f, a_0 \rangle + \mu \langle f, b_0 \rangle \neq 0, \infty$. Since this must be true for any $\lambda, \mu \neq 0$, we have that either $\langle f, a_0 \rangle = 0$ and $\langle f, b_0 \rangle \neq 0, \infty$, or $\langle f, a_0 \rangle = 0$ and $\langle f, b_0 \rangle = 0, \infty$, or $\langle f, a_0 \rangle = 0, \infty$ and $\langle f, b_0 \rangle = 0, \infty$. Without loss of generality, we can suppose we are in the first case, i.e. $\langle f, a_0 \rangle = 0$ and $\langle f, b_0 \rangle \neq 0, \infty$. Then, for any $a^\prime \in A^\perp, \langle f, a^\prime \rangle = 0$ since $\forall \lambda, \mu, \langle f, \lambda a^\prime + \mu b_0 \rangle \neq 0, \infty$. Then, we have that for any $b^\prime \in B^\perp, \langle f, b^\prime \rangle = 0, \infty$. This gives us that $\langle f, b^\prime \rangle \cong_{A_{V, B} \cup B_{V, A}} b^\prime$.}

Now, we want to show that $f \cong_{A \oplus B} f \cong_{A_{V, B} \cup B_{V, A}} f \cong_{A \oplus B} f \cong_{A \oplus B} I_{A_{V, B} \cup B_{V, A}} I_{A \oplus B}$. Taking an element $g \in A^\perp$ and $B^\perp$, we therefore want to show that $\langle f, g \rangle = \langle f \cong_{A \oplus B} \cong_{A \oplus B} \rangle$. But, using the preceding proposition, there are two projects $g_1$ and $g_2$ in $A^\perp, B^\perp$ respectively, such that $\langle g, c \rangle = \langle g_1 + g_2, c \rangle$ for all $c \in A \oplus B$. Hence, $\langle f, g \rangle = \langle f, g_1 + g_2 \rangle$ and $\langle f \cong_{A \oplus B} \cong_{A \oplus B} \rangle$. But $\langle f, g_1 + g_2 \rangle = \langle f, g_1 \rangle + \langle f, g_2 \rangle$ since $\langle f, a^\prime \rangle = 0$ for all
a' ∈ A\perp. On the other hand, \( \langle(f ◦ o) ◦ o, g_{2} + g_{2} \rangle = \langle(f ◦ o) ◦ o, g_{2} \rangle + \langle(f ◦ o) ◦ o, g_{2} \rangle \). Since \( \langle(f ◦ o) ◦ o, g_{2} \rangle = 0 \), we have:

\[
\langle(f ◦ o) ◦ o, g_{2} + g_{2} \rangle = \langle(f ◦ o) ◦ o, g_{2} \rangle
\]

Eventually, we obtained that for all \( g ∈ A\perp & B\perp \), the equality \( \langle f, g \rangle = \langle(f ◦ o) ◦ o, g \rangle \) is satisfied.

In order to prove distributivity, the following proposition and corollaries are useful.

**Proposition 47.** We denote by \( A \oplus B \) the set \( \{ a ⊕ b | a ∈ A, b ∈ B \} \).

Let \( E, F \) be non-empty sets of projects of respective carriers \( V, W \) with \( V ∩ W = \emptyset \). Then

\[
(E \odot F)^{\perp \perp} = (E^{\perp \perp} \odot F^{\perp \perp})^{\perp \perp}
\]

**Proof.** Obviously, we have \( E ⊆ E^{\perp \perp} \) and \( F ⊆ F^{\perp \perp} \), hence \( E \odot F ⊆ E^{\perp \perp} \odot F^{\perp \perp} \) and finally we get a first inclusion \( (E \odot F) ⊆ (E^{\perp \perp} \odot F^{\perp \perp})^{\perp \perp} \).

For the other inclusion, we prove that \( (E \odot F)^{\perp \perp} ⊆ (E^{\perp \perp} \odot F^{\perp \perp})^{\perp \perp} \). Let \( a \) be a project in \((E \odot F)^{\perp \perp}\). Then for all \( e ∈ E \) and \( f ∈ F \) we have \( \langle a, e ◦ f \rangle \neq 0, ∞ \). By the adjunction this means that \( \langle a : e, f' \rangle \neq 0, ∞ \), i.e. \( a : e ∈ F^{\perp \perp} \). Thus \( \langle a : e, f' \rangle \neq 0, ∞ \) for all \( e ∈ E \) and \( f' ∈ F^{\perp \perp} \). Since \( \langle a : e, f' \rangle = \langle a : e', f' \rangle \), we deduce that \( a : e' ∈ E^{\perp \perp} \), which means that \( a : e' \perp e' \) for all \( e' ∈ F^{\perp \perp} \) and \( e' ∈ E^{\perp \perp} \). To conclude, just notice that this is equivalent to the fact that for all \( e' ∈ E^{\perp \perp} \) and for all \( f' ∈ F^{\perp \perp} \), \( \langle a, e' ◦ f' \rangle \neq 0, ∞ \). This implies that \( a ∈ (E^{\perp \perp} \odot F^{\perp \perp})^{\perp \perp} \) which gives us the second inclusion.

**Corollary 47.1.** Let \( A \) be a conduct. Then \( (A \odot)_V^{\perp \perp} = A \odot 1_V \).

**Proposition 48 (Distributivity).** For any behaviours \( A, B, C \), and delocations \( φ, ψ, θ, ρ \) of \( A, A, B, C \) respectively, there is a project \( \text{distr} \) in the behaviour \( ((φ(A) ⊸ θ(B)) & (ψ(A) ⊸ ρ(C))) ⊸ (A ⊸ (B & C)) \).

**Proof.** Let \( g \) be a project in \( (φ(A) ⊸ θ(B)) & (ψ(A) ⊸ ρ(C)) \). Using the definition of \& and propositions 27 and 47.1, we get:

\[
(φ(A) ⊸ θ(B)) & (ψ(A) ⊸ ρ(C)) = (((φ(A) ⊸ θ(B)))_V^{\perp \perp})_V^{\perp \perp} \cap (((ψ(A) ⊸ ρ(C)))_V^{\perp \perp})_V^{\perp \perp}
\]

Now, define the projects:

\[
\text{distr} = f_1 + f_2
\]
We have $\text{Distr} \vdash g = f_1 \vdash g + f_2 \vdash g$. Let us compute $(f_1 \vdash g) \vdash a$ for $a \in A$:

$$(f_1 \vdash g) \vdash a = ((\overline{c} a_1 \otimes \overline{c} a_2 \otimes o_{(V\lambda)}) \otimes o_{Vc}) : g) \vdash a$$
$$(f_1 \vdash g) \vdash a = (\overline{c} a_1 \otimes o_{Vc}) \vdash g = \overline{c} a_1 \otimes o_{Vc} \vdash g : \overline{c} a_1 \otimes o_{Vc} \vdash (g \vdash \phi(a))$$

Since, as we have shown earlier, $g \in (\phi(A) \rightarrow (1_{\phi(V\lambda)}) \rightarrow \theta(B)))$, the project $((g \vdash \phi(a)) \vdash o_{(V\lambda)})$ is in $\theta(B)$, hence it is equal to $\theta(b)$ with $b \in B$. This yields:

$$(f_1 \vdash g) \vdash a = (\overline{c} a_1 \otimes o_{Vc}) \vdash \theta(b) = b \otimes o_{Vc}$$

Similarly, one gets that $(f_2 \vdash g) \vdash a = c \otimes o_{Vc}$ for $c \in C$. Hence $\text{Distr} \vdash g = a \in B + C \subseteq B \& C$.

Eventually, $\text{Distr} \vdash g \in A \rightarrow B \to C$, which implies that the project $\text{Distr}$ we defined is the one we were looking for.

Distributivity is quite easy to grasp without going into details: the project $\text{Distr}$ just superimpose elements of $((\phi(A) \rightarrow \theta(B))$ and $(\psi(A) \rightarrow \rho(C))$ over the carrier of $A$.

**Proposition 49.** The mix rule is never satisfied for proper behaviours.

**Proof.** Let $A, B$ be behaviours, and let $a, a', b, b'$ be projects in respectively $A, A^-, B, B^+$. Then, noticing that

$$\ll A \cup B, A' \cup B' \gg = \sum_{i,j,k,l} a_i^A a_j^B a_k^A' a_l^B' \ll A_i \cup B_j, A_k' \cup B_l' \gg$$

we can compute $\ll a \otimes b, a' \otimes b' \gg$ as follows:

$$\ll a \otimes b, a' \otimes b' \gg = 1_A \ll B, (1_A b + 1_B a) + 1_A \ll B, (1_A b' + 1_B a') + \ll A \cup B, A' \cup B' \gg$$

$$= 1_A \ll B, b + b' + \ll B, B' \gg + 1_A \ll A, (A' + \ll A, A' \gg)$$

Since $\ll a, a' \gg$ and $\ll b, b' \gg$ are different from 0 and $\infty$, it is possible to make the last expression be equal to 0 by changing the value of $1_A$ for instance, using the fact that for $c = a + \infty$, we have $\ll c, a' \gg = \ll a, a' \gg$ and that $1_C = 1_A + \lambda$.

**Proposition 50.** Weakening does not hold for non-empty behaviours.
Proof. Let $A, B$ be conducts, let $C$ be a behaviour, and let $f : A \to B$. Then $f \otimes \circ_C$ is not an element of $A \otimes C \rightarrowtail B$. Indeed, chose $a \otimes c$ in $A \otimes C$. Then $\langle f \otimes a, a \otimes c \rangle = 1_C \langle f, c \rangle$. Moreover, $(F \otimes 0) : (A \otimes C) = 1_C F : A$. This yields that $(f \otimes a) : (a \otimes c) = 1_C f : a$. Since $C$ is a behaviour, it is possible to cancel $1_C$ by considering $c - 1_C 0 \in C$. Eventually, this gives that $f \otimes a$ is not in $A \otimes C \rightarrowtail B$ unless $B = T$ or $A = 0$.

Remark. This amounts to show that there are no maps from $C$ to $1$ when $C$ is a behaviour.

## 5 Denotational Semantics

### 5.1 A $*$-autonomous category

Let us first define the category of conducts. For this, we define $\psi_i : N \to N \times \{0, 1\}$ ($i = 0, 1$):

$$\psi_i : x \mapsto (x, i)$$

**Definition 51** (Objects and morphisms of $\text{Graph}_{MLL}$). We define the following category:

- **Obj** = \{ $A | A = A \downarrow \downarrow$ of carrier $X_A \subseteq N$ \}
- **Mor**[$A, B]$ = \{ $f \in \psi_0(A) \rightarrowtail \psi_1(B)$ \}

To define the composition of morphisms, we will use three copies of $N$. We thus define the following useful bijections:

$$\nu : N \times \{0, 2\} \to N \times \{0, 1\}, \quad (x, i) \mapsto (x, i/2)$$

**Definition 52** (Composition in $\text{Graph}_{MLL}$). Given two morphisms $f$ and $g$ in **Mor**[$A, B$] and **Mor**[$B, C$] respectively, we define

$$g \circ f = \nu(f \mu(g))$$

Then one can show that this is indeed a category [Sei12]. Notice the identities are defined by faxes (Definition 29) which are represented by finite graphs.

We now define a bifunctor $\otimes$, and for that we will use the functions $\phi : N \times \{0, 1\} \to N$ defined by $\phi((x, i)) = 2x + i$ and $\tau$

$$\tau : \begin{cases} N \times \{0, 1\} \to N \times \{0, 1\} \\ (2x + 1, 0) \mapsto (2x, 1) \\ (2x, 1) \mapsto (2x + 1, 0) \\ (x, i) \mapsto (x, i) \text{ otherwise} \end{cases}$$

**Definition 53.** Define on $\text{Graph}_{MLL}$ the bifunctor $\otimes$ is induced by the tensor product defined on objects by

$$A \otimes B = \phi(\psi_0(A) \otimes \psi_1(B))$$

and on morphisms as

$$f \otimes g = \tau(\phi(f) \otimes \psi_1(g))$$
Theorem 54. The category $\mathbf{Graph}_{MLL}$ is a $*$-autonomous category. More precisely, $(\mathbf{Graph}_{MLL}, \otimes, 1_\emptyset)$ is symmetric monoidal closed and the object $\bot_\emptyset = 1^\bot_\emptyset$ is dualizing.

Proof. A proof of this result for directed weighted graphs can be found in our earlier paper [Sei12]. The proof in this case (sliced graphs) is a straightforward adaptation of it. 

5.2 Products and Coproducts

Moreover, to have a denotational semantics for MALL, we need to define a product. The natural construction would be to define the category $\mathbf{Graph}_{MALL}$:

$$\Omega\mathfrak{Ob} = \{ A \mid A = A^{\bot\bot} \text{ behaviour of carrier } X_A \subseteq \mathbb{N} \}$$

and then define on this full subcategory of $\mathbf{Graph}_{MALL}$ the bifunctor $\&$ by $A, B \mapsto \phi(\psi_0(A) \& \psi_1(B))$ on objects, and:

$$\tau(\mathcal{Distr} :: \psi_0(\phi(f)) \& \psi_1(\phi(g)))$$

However, this does not define a categorical product. Indeed, as usual when dealing with geometry of interaction for additives, the problem lies in the elimination of the cut between additive connectives.

Here is what happens on an example. Let us take two projects $f, g$ in respectively $A \to B$ and $A \to C$. Suppose moreover that both projects have only one slice to simplify the following discussion; we think of $f, g$ as interpretations of two sequent calculus proofs $\pi_f$ and $\pi_g$. Then, $f \& g$ is a project with two slices in $A \to (B \& C)$, where the first slice contains the graph $F \cup \emptyset_{V'}$, and the second contains the graph $G \cup \emptyset_{V''}$. We want $f \& g$ to be the interpretation of the proof obtained from $\pi_f$ and $\pi_g$ by a $\&$ rule; we will denote this proof by $\pi_{f \& g}$. Now, let us take a project $\bar{h}$ (with only one slice) in $B \to D$, and let us think of it as the interpretation of a proof $\pi_h$. Then the project $\bar{h} \otimes \psi_{V''}$ is in $(B \& C) \to D$. This project will be the interpretation of the proof obtained from $\pi_h$ by applying a $\oplus$ rule, which we will denote by $\pi_{\bar{h} \otimes \psi_{V''}}$. Taking the cut between $f \& g$ and $\bar{h} \otimes \psi_{V''}$ then gives us the interpretation of the proof $\pi$ obtained by applying a cut rule between $\pi_{f \& g}$ and $\pi_{\bar{h} \otimes \psi_{V''}}$. Applying one step of the cut-elimination procedure on $\pi$ then gives us a proof $\pi'$ whose interpretation should be $\bar{f} :: \bar{h}$. This raises the question: is $\pi' = \bar{f} :: \bar{h}$ equal to $(f \& g) :: (\bar{h} \otimes \psi_{V''})$? One can see that it is not the case in general. Indeed, $(f \& g) :: \bar{h}$ is equal to $p = ((f \otimes \psi_{V''}) :: \bar{h}) \& ((g \otimes \psi_{V''}) :: \psi_{V''})$ (see the graphic representation in Figure 1). It is clear that $p$ is not equal to $p'$ since $p$ has two slices, while $p'$ has only one. Moreover, even though one slice of $p$ is equal to $p'$, the other one is not in general equal to the empty graph: in the execution $(G \cup (V^D, \emptyset)) :: (H \cup (V^C, \emptyset))$ one keeps the edges in $G$ whose source and target are in $V^A$ and the edges in $H$ whose source and target are in $V^D$.

So, the categories considered are not a denotational semantics for MALL, and the $\&$ seems to be a bad candidate for defining a product. However, we get the following result that tells us the $\&$ may not be so bad after all, and suggest keeping it but considering conducts up to an observational quotient.
Here an edge from $V^i$ to $V^j$ represents the set of edges whose sources are in $V^i$ and targets are in $V^j$. We did not represent those sets of edges that are necessarily empty (for instance from $V^C$ to $V^C$ in the graph of $f \otimes 0$). The circled dots represent the location of the cut, i.e. the vertices that disappear during the execution.

Figure 1: Graphic representation of the plugging of $f \otimes g$ and $h \otimes 0_{V^C}$.

### 5.3 Observational Equivalence

As we explained earlier, the $\&$ connective does not define a categorical product because, if $f \in A \rightarrow B$, $g \in A \rightarrow C$ and $b \in B^\lambda$, the computation of the cut $(f \& g) :: (b \otimes 0_{V^C})$ yields $f + \text{res}$ where $\text{res}$ is a residue equal to $(g \otimes 0_{V^B}) :: (b \otimes 0_{V^C})$. The following proposition shows, however, that this residue is not detected by the elements of $A^\lambda$, i.e. that $\ll \text{res}, a'' \gg = 0$ for all $a'' \in A^\lambda$. This means that $\&$ defines a categorical product up to observational equivalence.

**Proposition 55.** Let $f \in A \rightarrow B$ and $g \in A \rightarrow C$ be projects, and write $h = f \otimes g$. Then for all $b \in B^\lambda$, $f :: b \simeq_A h :: b \otimes 0_{V^C}$.

**Proof.** For any $a \in A$, we have $h :: (a_{V^B}) = f :: a + g :: 0_{V^B}$. Now, let $b \in B^\lambda$ be a project. We get $\ll h :: a_{V^B}, b \gg = \ll f :: a, b \gg + \ll g :: 0_{V^B}, b \gg$. Suppose that $\ll g :: 0_{V^B}, b \gg = \lambda \neq 0$. Since $f :: a \in B$, we have $\ll f :: a, b \gg = \mu \neq 0$. But, by the homothety lemma 31.1, we get $g :: a = -\frac{1}{\mu} f :: a$. We finally obtain $\ll \text{distr} :: (f \& g), b \gg = 0$, which is a contradiction.

Thus, for any $b \in B^\lambda$, we have $\ll g :: 0_{V^B}, b \gg = 0$ and the result is proven.

We thus would like to quotient the category $\mathcal{G}raph_{MLL}$ by the observational equivalence. For this, we need to show that the categorical structure we had do not collapse when taking the observational quotients. The following proposition — an easy consequence of the cyclic property (Theorem 19) — and its corollaries⁶ make sure of that.

**Proposition 56.** Let $f \simeq_A B$ and $g \in A \rightarrow C$ be projects. Then $f :: g \simeq_A C f' :: g$.

⁶They do more than just that, they also insure that the quotiented category inherits the monoidal structure of $\mathcal{G}raph_{MLL}$. 

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Proof. For all \( c \in C \to A \), we have \( \langle \bar{f} \cdot g, c \rangle = \langle f, g : c \rangle \). Therefore:

\[
\begin{align*}
\langle \bar{f} \cdot g, c \rangle &= \langle f, g : c \rangle \\
&= \langle f', g : c \rangle \\
&= \langle f' \cdot g, c \rangle
\end{align*}
\]

Corollary 56.1. Let \( f, f', g, g' \) be projects such that \( f \equiv_{A \to B} f' \) and \( g \equiv_{B \to C} g' \). Then \( f \cdot g \equiv_{A \to C} f' \cdot g' \).

Corollary 56.2. Let \( a \equiv_{A} a' \) and \( f \equiv_{A \to B} g \) be projects. Then \( f : g \equiv_{B} g : a' \).

Corollary 56.3. If \( a \equiv_{A} a' \) and \( b \equiv_{B} b' \), then \( a \odot b \equiv_{A \odot B} a' \odot b' \).

Corollary 56.1 allows us to define the following quotient category:

**Definition 57.** Define the category \( \mathsf{Cond} \) by

\[
\begin{align*}
\mathsf{Obj} &= \{ A \mid A = A^\perp \} \\
\mathsf{Mor}[A, B] &= \{ [f] \mid f \in \psi_0(A) \to \psi_1(B) \}
\end{align*}
\]

**Proposition 58.** The category \( \mathsf{Cond} \) inherits the *-autonomous structure of the category \( \mathsf{Graph}_{\mathsf{MLL}} \).

**Proof.** Notice that we quotient only the hom-sets. The three corollaries of Proposition 56 above insures that we have indeed defined a category, and that it inherits the monoidal structure of \( \mathsf{Graph}_{\mathsf{MLL}} \). To show that \( \mathsf{Cond} \) is closed, one shows that the isomorphism between \( \mathsf{Mor}[A, B \odot C] \) and \( \mathsf{Mor}[A \odot B, C] \) in \( \mathsf{Graph}_{\mathsf{MLL}} \) is compatible with the equivalence relation. This compatibility is however obvious: equivalence is preserved by delocations. The fact that \( \perp \) is dualizing is a direct consequence of the preservation of isomorphisms when one takes the quotient.

**Definition 59.** Define the category \( \mathsf{Behav} \) by

\[
\begin{align*}
\mathsf{Obj} &= \{ A \mid A \text{ behaviour of carrier } X_A \subset \mathbb{N} \} \\
\mathsf{Mor}[A, B] &= \{ [f] \mid f \in \psi_0(A) \to \psi_1(B) \}
\end{align*}
\]

**Proposition 60.** The category \( \mathsf{Behav} \) is a full subcategory of \( \mathsf{Cond} \) closed for the monoidal product, the internalisation of Hom-sets and duality which has products, coproducts and in which mix and weakening do not hold.

**Proof.** It is sufficient to prove that \( \odot \) is a coproduct, since \( \mathsf{Behav} \) is close under taking the orthogonal. Let \( A, B, C \) be behaviours, and \( f \in A \to C, g \in B \to C \) be projects. Then the project \( f \odot g \) is a project in \( (A \odot B) \to C \). Define \( \iota_A \) (resp. \( \iota_B \)) as the identity on \( A \) (resp. on \( B \)) tensored with \( \phi_V \) (resp. \( \phi_{V^*} \)). Then, it is an easy consequence of Proposition 55 that, for any representant \( \mathsf{h} \) of \( [f \odot g] \) and any representant \( i \) of \( [\iota_A] \), we have \( \mathsf{h} : i \in [f] \). The verification concerning \( \iota_B \) is similar.

So the categorical model we obtain has two layers. The first layer consists in a non-degenerate (i.e. \( \otimes \neq \mathcal{G} \) and \( \mathbf{1} \neq \perp \) *-autonomous category \( \mathsf{Cond} \), hence a denotational model for MLL with units. The second layer is the full subcategory \( \mathsf{Behav} \) which does not contain the multiplicative units but is a non-degenerate model (i.e. \( \otimes \neq \mathcal{G}, \otimes \neq \& \) and \( \mathbf{0} \neq \top \)) of MALL with additive units.

Moreover, all the results up to this point are independent of the choice of the parameter (the function \( m \) that measures cycles), hence we did not define one, but a whole lot of such models. The last section is devoted to two instances of the construction.
6 Girard’s GoI(s)

In this section, we study two particular values of the parameter $m$. The first, $m(x) = -\log(1 - x)$, will give us a combinatorial version of Girard’s Geometry of Interaction in the Hyperfinite Factor (GoI5); the second, $m(x) = \infty$, will give us a refined version of (the multiplicative fragment) of more ancient versions of GoI [Gir89a, Gir88, Gir95a].

Let $H$ be a separable infinite-dimensional Hilbert space (for instance, the space $l^2(\mathbb{N})$ of square-summable sequences), and let $\{e_i \mid i \in \mathbb{N}\}$ be a base of $H$. For every finite subset $S \subset \mathbb{N}$ there is a projection on the subspace generated by $\{e_s \mid s \in S\}$ that we will denote by $p_S$.

### 6.1 Orthogonality as Nilpotency

**Definition 61.** Let $G$ be a graph. We define $M_{\text{conn}}^G$ to be the operator of $p_{V^G}B(H)p_{V^G} \subset B(H)$ whose matrix in the base $\{e_i \mid i \in V^G\}$ is the connectivity matrix of $G$.

The *connectivity matrix* of a sliced graph $G = \{G_i\}_{i \in I^G}$ is defined as the direct sum $M_{\text{conn}}^G = \bigoplus_{i \in I^G} M_{\text{conn}}^{G_i}$.

If $G, H$ are two sliced graphs, we define:

$$M_{\text{conn}}^G \star M_{\text{conn}}^H = \bigoplus_{(i,j) \in I^G \times I^H} M_{\text{conn}}^{G_i} M_{\text{conn}}^{H_j}$$

**Proposition 62.** Let $a, b$ be two projects of carrier $V$, and $m(x) = \infty$ for $x \in [0,1]$. Then:

$$a \perp b \Leftrightarrow \begin{cases} M_{\text{conn}}^G \star M_{\text{conn}}^H 	ext{ is nilpotent} \\ 1_A b + 1_B a \neq 0, \infty \end{cases}$$

In particular, if $A, B$ have only one slice, the product $M_{\text{conn}}^A \star M_{\text{conn}}^B$ is nilpotent.

**Proof.** We show the left-to-right implication: $a \perp b$ implies $a 1_B + b 1_A + [A, B] \neq 0, \infty$. But, if there were a cycle in one of the $A_i \sqcup B_j$, the last term $[A, B]$ would be equal to $\infty$. Hence $\ll a, b \gg$ would be infinite, and the projects would not be orthogonal. Thus $M_{\text{conn}}^A \star M_{\text{conn}}^B$ is nilpotent and $[A, B] = 0$, which means that $1_B a + b 1_A \neq 0, \infty$.

The converse is straightforward. \hfill $\square$

**Remark.** This proposition is true as it is because we are working with finite graphs. What we are really proving is that $a \perp b$ if and only if $1_A b + 1_B a \neq 0, \infty$ and for all $i, j$, no cycles appear in $A_i \sqcup B_j$. In the case of infinite graphs, this condition would imply weak nilpotency (hence more in the style of the second version of GoI [Gir88]).

**Corollary 62.1.** Let $a = (a, A)$ be a project, and $a' = (a, A')$ be such that $M_{\text{conn}}^{A'} = M_{\text{conn}}^A$. Then $a \equiv_{A} a'$ for all conduct $A$ containing $a$.

The model we obtain when taking $m(x) = \infty$ can therefore be reduced, by working up to observational equivalence, to working with simple (at most one edge between two points) non-weighted directed graphs.
Notice however the differences between the first versions of GoI and our framework. The addition of the wager is a quite important improvement: without it, we would have $1 = \bot$. Moreover, the additive construction (the use of slices) — even though restricted since the weights of the slices do not matter anymore — allows us to define, as we have shown, a categorical model of MALL. Looking a little closer at this model, one can see, however, that it is not that exciting.

**Proposition 63.** Let $m(x) = \infty$ and $A$ be a behaviour. Then $A$ is either empty or the orthogonal of an empty conduct.

**Proof.** Notice that a proper behaviour and its orthogonal contain only project $\mathfrak{a} = (a, A)$ with $a = 0$. But two such projects cannot be orthogonal when $m(x) = \infty$. Hence there are no proper behaviours and using Proposition 40 we have the result.

Hence, the categorical model we get with an orthogonality defined by nilpotency is nothing more than a truth-value model.

### 6.2 GoI in the Hyperfinite Factor

**Definition 64.** If $G$ is a simple weighted graph, we define $\mathcal{M}_G$ to be the operator of $p_{V\alpha}B(\mathbb{H})p_{V\alpha} \subseteq B(\mathbb{H})$ whose matrix in the base $\{e_i \mid i \in V^G\}$ is the matrix of weights of $G$.

**Definition 65.** From a directed weighted graph $G$, we can define a simple graph $\hat{G}$ with weights in $\mathbb{R}^>_0 \cup \{\infty\}$:

\[
\begin{align*}
V^{\hat{G}} &= V^G \\
E^{\hat{G}} &= \{(v, w) \mid \exists e \in E^G, s^G(e) = v, t^G(e) = w\} \\
\omega^{\hat{G}} : (v, w) &\mapsto \sum_{e \in E^G(v, w)} \omega^G(e)
\end{align*}
\]

If the weights of $\hat{G}$ are in $\mathbb{R}^>_0$, we will say it is total.

**Definition 66.** An operator graph is a graph $G$ such that $\mathcal{M}_G$ is a hermitian (i.e. $\hat{G}$ is total and $G$ is symmetric) with $\|\mathcal{M}_G\| \leq 1$.

An operator project is a project $(a, A)$ where $A$ is an operator graph.

Girard defines his latest geometry of interaction in the hyperfinite factor $\mathcal{R}_{0,1}$ of type $\Pi_\infty$ with a fixed trace $tr$. This von Neumann algebra can be obtained as the tensor product of $B(\mathbb{H})$ with the hyperfinite factor of type $\Pi_1$, usually denoted $\mathcal{R}$. We will therefore work with operators in $B(\mathbb{H}) \otimes \mathcal{R}$ and the trace defined as the tensor product of the normalized trace on $B(\mathbb{H})$ (i.e. the trace of minimal projections is 1) and the normalized trace on $\mathcal{R}$ (i.e. the trace of the identity is 1).

**Definition 67** (Girard’s project). A **Girard’s project** is a tuple $\mathfrak{a} = (p, a, A, \alpha, A)$ consisting of:

- a finite projection $p^* = p^2 = p \in \mathcal{R}_{0,1}$, the **carrier** of the project $\mathfrak{a}$;
• a finite and hyperfinite von Neumann algebra \( \mathcal{A} \), the *idiom* of \( a \);
• a normal hermitian tracial form \( \alpha \) on \( \mathcal{A} \), the *pseudo-trace* of \( a \);
• a real number \( a \in \mathbb{R} \cup \{ \alpha(1_A)\infty \} \), the *wager* of \( a \);
• an hermitian \( A \in (pR_{0,1}) \otimes \mathcal{A} \) such that \( \|A\| \leq 1 \).

As in Girard’s paper, we will denote such an object by \( a = a \cdot \alpha + A \).

We now define the embedding \( \phi \) that maps an operator graph \( G \) to the operator \( M_G \otimes 1_R \) in \( R_{0,1} \). This embedding can be extended to map operator projects to Girard projects as follows.

**Definition 68.** To an operator project \( a = (a,A) \) we associate the Girard project \( \phi(a) = a' \cdot + \bigoplus_{i \in I^A} \alpha^A_i + \bigoplus_{i \in I^A} \phi(M_{A_i}) \), where \( a' = -\infty \) if \( 1_A < 0 \) and \( a = \infty \), and \( a' = a \) otherwise.

**Theorem 69.** Let \( a, b \) be operator projects, and let \( m \) be the map \( x \mapsto -\log(1-x) \). We have the following equalities:

\[
\ll a, b \gg = \ll \phi(a), \phi(b) \gg \\
\phi(a \otimes b) = \phi(a) \otimes \phi(b) \\
\phi(a + \lambda b) = \phi(a) + \lambda \phi(b) \\
\phi(a : b) = \phi(a) :: \phi(b)
\]

provided \([A, B] \neq \infty\):

\[
\phi(a :: b) = \phi(a) :: \phi(b)
\]

*Proof.* The proof is quite involved and can be found in the earlier paper [Sei12], except for the equality \( \phi(a + \lambda b) = \phi(a) + \lambda \phi(b) \). The proof of this equality, however, is a direct consequence of the definitions of \( + \) in both Girard’s framework and this one. \( \square \)

### 7 Conclusion

Generalizing the first GoI model introduced by Girard [Gir87b], we were able to define a graph-theoretic geometry of interaction in which one can interpret the Multiplicative Additive fragment of Linear Logic. Contrarily to what happens in the two other versions of GoI dealing with additives [Gir95a, Gir11a], proofs of MALL are interpreted in our framework by *finite objects*.

We were able to define an internal notion of observational equivalence and show that it is possible to define, from our construction, a categorical model of MALL (with additive units) where no connectives and units are equal and in which neither the mix rule nor the weakening are satisfied. This model is moreover obtained as a subcategory of a \(*\)-autonomous category, i.e. a model of MLL with units.

All these constructions being parametrized by a choice of a "measuring map" from \([0, 1] \rightarrow R_{\geq 0} \cup \{ \infty \} \), we looked more closely at the construction in two particular cases. It can be shown that a first choice \((m(x) = -\log(1-x))\) defines a combinatorial version of the Multiplicative Additive fragment of Girard’s GoI5 [Gir11a]. It therefore gives insights on the notion of orthogonality used by Girard, and his use of *idioms* — which corresponds in our setting to slices —

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and pseudo-trace — which corresponds in our setting to the weights associated to the slices. On the other hand, a second choice of map \( m(x) = \infty \) defines a refined version of the Multiplicative fragment of the first versions of GoI where orthogonality was defined as nilpotency. However, this choice of parameter yields a trivial model of the additives. Nonetheless, our construction makes a bridge between “old-style” geometry of interaction and Girard’s most recent work [Gir11a].

References


