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ASYMPTOTIC BEHAVIOUR OF STANDARD BASES

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ABSTRACT. We prove that the elements of any standard basis of I^n, where I is an ideal of a Noetherian local ring and n is a positive integer, have order bounded by a linear function in n. We deduce from this that the elements of any standard basis of I^n in the sense of Grauert-Hironaka, where I is an ideal of the ring of power series, have order bounded by a polynomial function in n.

The aim of this paper is to study the growth of the orders of the elements of a standard basis of I^n, where I is an ideal of a Noetherian local ring. Here we show that the maximal order of an element of a standard basis of I^n is bounded by a linear function in n. For this we prove a linear version of the strong Artin-Rees lemma for ideals in a Noetherian ring. The main result of this paper is Theorem 3.

First we prove the following proposition inspired by Corollary 3.3 of [4]:

Proposition 1. *Let A be a Noetherian ring and let I and J be ideals of A. There exists an integer $\lambda \geq 0$ such that*

$$\forall x \in A, \forall n, m \in \mathbb{N},\ n \geq \lambda m,\quad (x^n) \cap (J + I^m) = ((x^{\lambda m}) \cap (J + I^m))(x^{n-\lambda m}).$$

Proof. Let $B := A/J$. By Theorem 3.4 of [5], there exists λ such that for any $m \geq 1$, there exists an irredundant primary decomposition $I^m = Q_1^{(m)} \cap \cdots \cap Q_r^{(m)}$ such that if $P_i^{(m)} := \sqrt{Q_i^{(m)}}$, then $(P_i^{(m)})^{\lambda m} \subset Q_i^{(m)}$ for $1 \leq i \leq m$. We denote by $\overline{Q}_i^{(m)}$ the image of $Q_i^{(m)}$ in $A/(J + I^m)$ for $1 \leq i \leq r$. We denote by $\mathfrak{P}_i^{(m)}$ the inverse image of $P_i^{(m)}$ in A, for $1 \leq i \leq r$.

Let $x \in A$. If $x \in \mathfrak{P}_i^{(m)}$, then $x^n \in (\mathfrak{P}_i^{(m)})^n$ and $(\overline{Q_i}^{(m)} : x^n) = A/(J + I^m)$ for any $n \geq \lambda m$. If $x \notin \mathfrak{P}_i^{(m)}$, then $x^n \notin (\mathfrak{P}_i^{(m)})^n$ and $(\overline{Q_i}^{(m)} : x^n) = \overline{Q}_i^{(m)}$ for any $n \geq \lambda m$. Thus, for any $n \geq \lambda m$,

$$(0_{A/(J+I^m)} : x^n) = \left(\bigcap_i \overline{Q}_i^{(m)} : x^n\right) = \bigcap_i \left(\overline{Q}_i^{(m)} : x^n\right) = \bigcap_{i\ /\ x \notin P_i} \overline{Q}_i^{(m)}.$$

Hence, by Remark 2 (1) of [4] and Theorem 2 of [4], we get the result. $\square$

Using the extended Rees algebra of $\mathfrak{a}$ to reduce to the principal case (as done in [5]), we prove the following corollary:

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Corollary 2. *Let A be a Noetherian ring and let I, J and $\mathfrak{a}$ be ideals of A. Then there exists $\lambda \geq 0$ such that*

$$(J+I^m)\cap\mathfrak{a}^n = ((J+I^m)\cap\mathfrak{a}^{\lambda m})\mathfrak{a}^{n-\lambda m}.$$

Proof. Let $B := A[\mathfrak{a}t, t^{-1}]$. Then $t^{-n}B\cap A = \mathfrak{a}^n$. By Proposition 1, there exists $\lambda \geq 1$ such that for any $n, m \in \mathbb{N}$, $n \geq \lambda m$,

$$(t^{-n})\cap(J+I^m) = ((t^{-\lambda m})\cap(J+I^m))(t^{-(n-\lambda m)}).$$

We have

$$(J+I^m)\cap\mathfrak{a}^n = ((t^{-m})\cap(J+I^m)B)\cap A = \Big(((t^{-\lambda m})\cap(J+I^m))(t^{-(n-\lambda m)})\Big)\cap A.$$

Thus $(J+I^m)\cap\mathfrak{a}^n \subset ((J+I^m)\cap\mathfrak{a}^{\lambda m})\mathfrak{a}^{n-\lambda m}$. The reverse inclusion is clear. □

Let $(A,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of A. Let us denote by $G(A/I)$ the associated graded ring of A/I with respect to $\mathfrak{m}$. Then $G(A/I) = G(A)/I^*$, where $I^* \subset G(A)$ is the graded ideal of $G(A)$ generated by the elements f^* with $f \in I$, where f^* is the leading form of f: if $\mathrm{ord}(f) := \sup\{k,\ f\in\mathfrak{m}^k\} = d$, then $f^* = f + \mathfrak{m}^{d+1}$. Finally $f_1, ..., f_p$ form a *(minimal) standard basis* of I if $f_1^*, ..., f_p^*$ form a (minimal) generating set of I^*. It is clear that $(I^*)^n$ is included in $(I^n)^*$, but both ideals are not equal in general. For example, if $I = (x^2, y^3 - xy) \subset \mathbb{k}[[x,y]]$ where $\mathbb{k}$ is a field, then $(I^n)^* = ((xy, x^2)^n, \{x^iy^{4n-3i+1}\}_{0\leq i\leq n-1})$; hence $y^{4n+1} \in (I^n)^*\backslash(I^*)^n$ [2]. Nevertheless we have the following theorem whose proof is inspired by the link made in [1] between the Artin-Rees lemma and the orders of the elements of a standard basis, with respect to a monomial order, of an ideal in the ring of formal power series over a field (see also [6]).

Theorem 3. *Let I be an ideal of a Noetherian local ring $(A,\mathfrak{m})$. Then there exists an integer $\lambda \geq 0$ such that for any integer $n \geq 0$ and any minimal standard basis $f_1, ..., f_{p_n}$ of I^n we have $\mathrm{ord}(f_i) \leq \lambda n$ for $1 \leq i \leq p_n$.*

Proof. The canonical morphism $A \longrightarrow \widehat{A}$ is injective and $G(A/I) = G(\widehat{A/I})$. Thus we may assume that A is complete. Then A is of the form B/J, where B is a regular local ring and J is an ideal of B. Hence we may assume that A is a regular local ring, I and J are ideals of A, and we need to prove that there exists $\lambda \geq 0$ such that for any minimal standard basis $f_1, ..., f_{p_n}$ of $J + I^n$ we have $\mathrm{ord}(f_i) \leq \lambda n$ for $1 \leq i \leq p_n$.

Let us assume that $I + J \neq (0)$. Let $n \in \mathbb{N}^*$ and let $f_1, ..., f_{p_n} \in J + I^n$ such that $f_1^*, ..., f_{p_n}^*$ form a minimal generating set of $(J + I^n)^*$ (in particular, $(f_1, ..., f_{p_n}) = J + I^n$). Let us denote by r_i the integer $\mathrm{ord}(f_i)$, $1 \leq i \leq p_n$, and let us assume that $r_1 \leq r_2 \leq \cdots \leq r_{p_n}$. Let $\lambda \geq 0$ satisfy Corollary 2 with $\mathfrak{a} = \mathfrak{m}$. Let $q \geq 0$ such that $r_i \leq \lambda n$ for $i \leq q$ and $r_i > \lambda n$ for $i > q$. It is enough to show that $q = p_n$. Let us assume that $q < p_n$. If $q = 0$, then $f_i \in (J + I^n)\cap\mathfrak{m}^{r_i} = ((J+I^n)\cap\mathfrak{m}^{\lambda n})\mathfrak{m}^{r_i-\lambda n} \subset (J + I^n)\mathfrak{m}$, $1 \leq i \leq p_n$. Hence $(J + I^n) = \mathfrak{m}(J + I^n)$, and $(J + I^n) = (0)$ by Nakayama, which is a contradiction. Thus $q \geq 1$. For $i > q$ we have $f_i \in (J+I^n)\cap\mathfrak{m}^{r_i} = ((J+I^n)\cap\mathfrak{m}^{\lambda n})\mathfrak{m}^{r_i-\lambda n}$. Thus, for $q + 1 \leq i \leq p_n$, $f_i = \sum_k \varepsilon_{i,k}g_{i,k}$ with $g_{i,k} \in (J + I^n)\cap\mathfrak{m}^{\lambda n}$ $\varepsilon_{i,k} \in \mathfrak{m}^{r_i-\lambda n}$ for $q < i \leq p_n$ and any k. Hence $f_i = \sum_k \varepsilon_{i,k}\left(\sum_{1\leq l\leq p_n}\eta_{i,k,l}f_l\right)$ with $\eta_{i,k,l} \in \mathfrak{m}^{\lambda n - r_l}$ for any i, k, l (because $f_1^*, ..., f_{p_n}^*$ generate $(J+I^n)^*$ and $G(A)$ is an integral domain).

Thus, for $q < i \leq p_n$,

$$f_i = (1 - \sum_k \varepsilon_{i,k}\eta_{i,k,i})^{-1} \sum_k \varepsilon_{i,k} \left(\sum_{l \neq i} \eta_{i,k,l} f_l \right).$$

Then $f_i \in \sum_{l \neq i} f_l \mathfrak{m}^{r_i - r_l}$ for $q+1 \leq i \leq p_n$. By Gaussian elimination we see that

$$f_i \in \sum_{l<i} f_l \mathfrak{m}^{r_i - r_l} \text{ for } q+1 \leq i \leq p_n.$$

This means that $f_i^* \in (f_1^*, ..., f_{i-1}^*)G(A)$, which contradicts the fact that $f_1^*, ..., f_{p_n}^*$ form a minimal generating set of $(J + I^n)^*$. □

Let $\mathcal{O}_s := \Bbbk[[x_1, ..., x_s]]$, where $\Bbbk$ is a field, or let $\mathcal{O}_s := \Bbbk\{x_1, ..., x_s\}$, where $\Bbbk$ is a valued field. We denote by $\mathfrak{m}$ its maximal ideal. For all $\alpha \in \mathbb{N}^s$ let us denote $|\alpha| := \alpha_1 + \cdots + \alpha_s$. We define a total order on $\mathbb{N}^s$ in the following way: $\alpha > \beta$ if $(|\alpha|, \alpha_1, ..., \alpha_s) >_{lex} (|\beta|, \beta_1, ..., \beta_s)$ for all $\alpha, \beta \in \mathbb{N}^s$. This induces a total order on the monomials of $\mathcal{O}_s$ in the following way: $x^\alpha > x^\beta$ if $\alpha > \beta$ for all $\alpha, \beta \in \mathbb{N}^s$. If $f = \sum_{\alpha \in \mathbb{N}^s} f_\alpha x^\alpha \in \mathcal{O}_s$, let us denote by $\text{in}_>(f)$ the element $f_\alpha x^\alpha$ such that $\alpha < \beta$ for all $\beta \neq \alpha$ such that $f_\beta \neq 0$. If $\text{in}_>(f) = f_\alpha x^\alpha$, let us denote by $\exp(f)$ the element $\alpha \in \mathbb{N}^s$. Let I be an ideal of $\mathcal{O}_s$; we say that $(f_1, ..., f_p)$ is a *(minimal) standard basis of I with respect to this order* if $\{\exp(f_1), ..., \exp(f_p)\}$ is a (minimal) set of generators of the semigroup $\{\exp(g), g \in I\}$ (in particular $(f_1, ..., f_p) = I$). We denote $\alpha_i := \exp(f_i)$ for all i. We may always assume that $|\alpha_1| \leq \cdots \leq |\alpha_p|$. In this case, for $l \in \mathbb{N}$ we define $q(l) \in \mathbb{N}$ by $\alpha_{q(l)} \leq l$ and $\alpha_{q(l)+1} > l$, where $q(l) = 0$ if $l < |\alpha_1|$ and $q(l) = p$ if $l \geq |\alpha_p|$. We have the following result:

Proposition 4 ([6]). *Let I be and ideal of $\mathcal{O}_s$. Then, with the previous notation,*

$$I \cap \mathfrak{m}^{m+l} = (I \cap \mathfrak{m}^l)\mathfrak{m}^m \textit{ for all } m \geq 0$$

if and only if $r(l) \geq 1$ and $f_j \in \mathfrak{m}^{|\alpha_j| - |\alpha_1|} f_1 + \cdots + \mathfrak{m}^{|\alpha_j| - |\alpha_{r(l)}|} f_{r(l)}$, for $j = r(l) + 1, ..., p$.

Corollary 5. *Let I be an ideal of $\mathcal{O}_s$. Then there exists a polynomial function in n, denoted by P, such that for all integers $n \geq 0$ and any minimal standard basis $f_1, ..., f_{p_n}$ of I^n with respect to $\text{in}_>$ we have $\text{ord}(f_i) \leq P(n)$ for $1 \leq i \leq p_n$.*

Proof. Let $f_1, ..., f_{p_n}$ be a minimal standard basis of I^n with respect to $\text{in}_>$. Let $\alpha_i := \exp(f_i)$, $1 \leq i \leq p_n$, and let us assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{p_n}$. The sequence $\alpha_1, ..., \alpha_{p_n}$ is uniquely determined by I^n. By applying Proposition 4 and Corollary 2, we see that there exists $\lambda \geq 0$, not depending on n, such that $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_r| \leq \lambda n < |\alpha_{r+1}| \leq \cdots \leq |\alpha_{p_n}|$ and

$$f_i \in \mathfrak{m}^{|\alpha_i| - |\alpha_1|} f_1 + \cdots + \mathfrak{m}^{|\alpha_i| - |\alpha_r|} f_r \quad \text{ for } r+1 \leq i \leq p_n.$$

In particular, $(f_1^*, ..., f_r^*)$ is a system of generators of $(I^n)^*$, and $(f_1^*, ..., f_{p_n}^*)$ is a Gröbner basis of the homogeneous ideal $(I^n)^*$ with respect to the graded lexicographic order. From [3], $\text{ord}(f_i^*)$ is bounded by a polynomial function in λn depending only on I and s, for $r+1 \leq i \leq p_n$. This proves the corollary. □

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