Algebraic power series in several variables

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Notations

- $\mathbb{K}$ will always denote a field.
- t single indeterminate, and $x = (x_1, \ldots, x_n)$ vector of indeterminates. We set $x' = (x_1, \ldots, x_{n-1})$.
- We denote by $\mathbb{K}\langle x \rangle$ the subset of $\mathbb{K}[\![x]\!]$ of power series algebraic over $\mathbb{K}[x]$:

$$\mathbb{K}\langle x \rangle := \{f \in \mathbb{K}[\![x]\!] \mid \exists P \in \mathbb{K}[x,t], P \neq 0, P(x, f(x)) = 0\}.$$

- Equivalently, a formal power series f is algebraic if the $\mathbb{K}(x)$-vector space generated by the powers of f is finite.

- Power series appear essentially only as:
  - Taylor expansions of functions.
  - Generating series of sequences.
- Algebraic power series are ”simple” power series that have good algebraic properties.
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$$\{\text{convergent}\}$$

$$\mathbb{K}[x] \subset \{\text{rational}\} \subset \mathbb{K}\langle x\rangle \begin{array}{l} \rotatebox{45}{$\subset$} \\ \rotatebox{-45}{$\subset$} \end{array}$$

$$\{\text{D-finite}\}$$

First algebraic properties

- $\mathbb{K}\langle x\rangle$ is a ring.
- $\mathbb{K}\langle x\rangle$ is a local ring. Its maximal ideal is $(x_1, \dots, x_n)$.
- $\mathbb{K}\langle x\rangle$ is Noetherian: every ideal is finitely generated.
- $\mathbb{K}\langle x\rangle$ is a Henselian local ring: ”it satisfies the implicit function theorem”.
- It is stable under derivation and composition.

Henselianity (implicit function theorem)

Let $P(x,t) \in \mathbb{K}\langle x\rangle[t]$ such that

$$P(0,0) = 0 \text{ and } \frac{\partial P}{\partial t}(0,0) \neq 0.$$

$$\Longrightarrow \exists! f(x) \in \mathbb{K}\langle x\rangle, \quad P(x,f(x)) = 0 \text{ and } f(0) = 0.$$

Weierstrass theorems (Lafon)

Let $f(x)$ be a power series and $d \in \mathbb{N}$. We say that f is x_n-regular of order d if

$$f(0,\ldots,0,x_n) = x_n^d u(x_n) \text{ with } u(0) \neq 0.$$

Weierstrass division theorem: Let $f \in \mathbb{K}\langle x\rangle$ be x_n-regular of order d and $g \in \mathbb{K}\langle x\rangle$. Then there is a unique $(q,r) \in \mathbb{K}\langle x\rangle \times \mathbb{K}\langle x'\rangle[x_n]$ such that

$$g = fq + r$$

$$\text{and } \deg_{x_n}(r) < d.$$

Weierstrass preparation theorem

Let $f \in \mathbb{K}\langle x \rangle$ be x_n-regular of order d. Then f can be written in a unique way as

$$f(x) = u(x)\left(x_n^d + a_1(x')x_n^{d-1} + \cdots + a_n(x')\right)$$

where $u(x) \in \mathbb{K}\langle x \rangle$, $a_i(x') \in \mathbb{K}\langle x' \rangle$,

$$u(0) \neq 0 \text{ and } a_i(0) = 0 \ \ \forall i.$$

Example: walks restricted to the quarter plane

Let $S \subset \{-1, 0, 1\}^2$. Set

$$a_{i,j,n} := \#\{\text{walks of length } n \text{ ending at } (i,j)\}.$$

Let $Q(x, y, t) := \sum_{i,j,n} a_{i,j,n} x^i y^j t^n$. It is solution of an equation the form

$$xy = K(x, y, t)Q(x, y, t) + R(x, t) + S(y, t) \qquad (1)$$

where $R(x, t)$ and $S(y, t)$ can explicitly be expressed in terms of $Q(x, 0, t)$ and $Q(0, y, t)$, and

$$K(x, y, t) := \left(xy - t \sum_{(a,b) \in \mathcal{S}} x^{a+1} y^{b+1} \right).$$

Take $S = \{SW, W, N, NE\}$.
Here
$K = xy - t(1 + y + xy^2 + x^2y^2) = z - t(1 + y + yz + z^2)$
where $z = xy$.

$$z = (z - t(1 + y + yz + z^2))Q(t, z, y) + R(t, y)$$

Therefore

$$xy = K(x, y, t)Q(t, xy, y) + R(t, y).$$

One variable case

Let $f(x) = \sum_{k\in\mathbb{N}} f_k x^k \in \mathbb{K}\langle x\rangle$.

Question: How can we express the coefficients f_k? Are there formulas for the coefficients ? What about the "relations" satisfied by the coefficients ?

- (Flajolet-Soria, Hickel-Matusinski): there exist formulas for the coefficients.
- There are results about the relations if $\mathbb{K}$ is finite (or of positive characteristic). For instance, $\sum_k f_k x^k \in \mathbf{F}_p[\![x]\!]$ is algebraic iff the sequence $(f_n)_n$ is p-automatic (Christol-Kamae-Mendès France-Rauzy).

One variable case 2

- If $\mathbb{K}$ is a valued field, there are bounds on the size of the coefficients.
- Algebraicity implies some bounds on the gaps in the expansion of $f(x)$ (Schmidt).
- Eisenstein theorem: For a series $f(x) = \sum_k f_k x^k \in \mathbb{Q}\langle x \rangle$, there is an integer $a \in \mathbb{N}$ such that

$$a^{k+1} f_k \in \mathbb{Z}, \quad \forall k \in \mathbb{N}.$$

General Problems

- Given an equation $G(x, f(x)) = 0$ admitting formal power series solutions $f(x) \in \mathbb{K}[\![x]\!]^p$, does there exists algebraic power series solutions?
- If there exist algebraic solutions, what about the "size" of the solutions ?
- What kind of equations: algebraic equations, differential equations, functional equations, with or without constraints in the solutions?
- If there is no algebraic solution, what about the solutions ?

Artin approximation theorem

Let $P(x, y) \in \mathbb{K}[x, y]^p$ with $y = (y_1, \ldots, y_m)$. Assume given $y(x) \in \mathbb{K}[\![x]\!]^m$ such that

$$P(x, y(x)) = 0.$$

Fix $c \in \mathbb{N}$. Then there exists $\widetilde{y}(x) \in \mathbb{K}\langle x\rangle^m$ such that

$$P(x, \widetilde{y}(x)) = 0 \text{ and } \widetilde{y}_i(x) - y_i(x) \in (x)^c \quad \forall i.$$

Euler differential equation

We consider the equation

$$x^2 f' - f + x = 0. \qquad (2)$$

It has a unique power series solution $f = \sum_{n \geq 0} n! x^{n+1}$.
By considering the Taylor development of $f(x+t)$ at x:

$$f(x+t) = f(x) + tf'(x) + O(t^2)$$

$$(2) \iff \begin{cases} x^2 g(x) - f(x) + x = 0 \\ f(x+t) = f(x) + tg(x) + t^2 h(x,t) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2 g(x) - f(x) + x = 0 \\ l(u) = f(x) + tg(x) + t^2 h(x,t) + (x+t-u) k_1(x,t,u) = 0 \\ l(u) - f(x) = (x-u) k_2(x,u) = 0 \end{cases}$$

General division theorem

Let us consider the following monomial order on $\mathbb{N}^n$:

$$\forall \alpha, \beta \in \mathbb{N}^n, \ \ \alpha \leq \beta \text{ if } (|\alpha|, \alpha_1, \ldots, \alpha_n) \leq_{\text{lex}} (|\beta|, \beta_1, \ldots, \beta_n).$$

Let $g_1, \ldots, g_s \in \mathbb{K}[\![x]\!]$. We set

$$\Delta_1 := \exp(g_1) + \mathbb{N}^n, \ \ \Delta_i = (\exp(g_i) + \mathbb{N}^n) \setminus \cup_{j=1}^{i-1} \Delta_j \text{ for } i \geq 2.$$

$$\Delta_0 = \mathbb{N}^n \setminus \cup_{i=1}^{s} \Delta_i.$$

Let $f \in \mathbb{K}[\![x]\!]$. Then there exist unique $q_1, \ldots, q_s$, $r \in \mathbb{K}[\![x]\!]$ such that

$$f = g_1 q_1 + \cdots + g_s q_s + r$$

$$\exp(g_i) + \operatorname{Supp}(q_i) \subset \Delta_i \text{ and } \operatorname{Supp}(r) \subset \Delta_0.$$

Kashiwara-Gabber example

Let us perform the division of xy by

$$g := (x - y^2)(y - x^2) = xy - x^3 - y^3 + x^2y^2$$

in $\mathbb{C}[\![x, y]\!]$ with respect to the monomial xy:

$$xy - g(x, y)q(x, y) - r(x) - s(y) = 0. \tag{3}$$

By symmetry, we get $r(x) = s(x)$, and by substituting y by x^2 we get the Mahler equation $r(x^2) + r(x) - x^3 = 0$. Therefore

$$r(x) = \sum_{i=0}^{\infty} (-1)^i x^{3.2^i}$$

is not algebraic. This proves that (3) has a formal solution but no algebraic solution.

Nested Artin approximation theorem

Let $P(x,y) \in \mathbb{K}[x,y]^p$ with $y = (y_1, \ldots, y_m)$. Assume given $y(x) \in \mathbb{K}[\![x]\!]^m$ such that

$$P(x, y(x)) = 0.$$

Assume moreover that $y_i(x) \in \mathbb{K}[\![x_1, \ldots, x_{\sigma_i}]\!]$ for some $\sigma_i \in \{1, \ldots, n\}$.

Fix $c \in \mathbb{N}$. Then there exists $\widetilde{y}(x) \in \mathbb{K}\langle x \rangle^m$ such that

$$P(x, \widetilde{y}(x)) = 0 \text{ and } \widetilde{y}_i(x) - y_i(x) \in (x)^c \quad \forall i$$

and

$$\widetilde{y}_i(x) \in \mathbb{K}\langle x_1, \ldots, x_{\sigma_i} \rangle.$$

Complexity of an algebraic power series

Let $f \in \mathbb{K}\langle x \rangle$ and $P(x,t) \in \mathbb{K}[x,t]$ its minimal polynomial. We set

$$\mathrm{Deg}(f) = \deg_t(P) \text{ and } \mathrm{ht}(f) = \deg_x(P).$$

complexity of f = $\mathrm{Deg}(f)$, $\mathrm{ht}(f)$.

This can be defined for every ”root” of a polynomial in $\mathbb{K}[x,t]$.

Complexity bounds (Adamczewski-Bell)

We consider $f_1, \ldots, f_k \in \mathbb{K}\langle x \rangle$, $p_1, \ldots, p_k \in \mathbb{K}[x]$.

$$\mathrm{Deg}(p_1 f_1 + \cdots + p_k f_k) \leq \prod_i \mathrm{Deg}(f_i)$$

$$\mathrm{ht}(\sum_i p_i f_i) \leq k \prod_i \mathrm{Deg}(f_i) \cdot (\max_j \mathrm{ht}(p_j) + \max_l \mathrm{ht}(f_l))$$

$$\mathrm{Deg}(f_1 \cdots f_k) \leq \mathrm{Deg}(f_1) \cdots \mathrm{Deg}(f_k)$$

$$\mathrm{ht}(f_1 \cdots f_k) \leq k \prod_i \mathrm{Deg}(f_i) \cdot \max_j \mathrm{ht}(f_j)$$

Complexity bounds 2 [R]

This allows us to give bounds for the derivations or compositions of alg. power series.

Let $f \in \mathbb{K}\langle x \rangle$ be x_n-regular of order d and $g \in \mathbb{K}\langle x \rangle$. Then there is a unique $(q, r) \in \mathbb{K}\langle x \rangle \times \mathbb{K}\langle x' \rangle[x_n]$ such that

$$g = fq + r \text{ and } \deg_{x_n}(r) < d,$$

$$\mathrm{ht}(r) \le 2^{2^{O(\mathrm{ht}(f)^{1+\varepsilon})}} \operatorname{Deg}(g)^{O(d^4 \operatorname{Deg}(g)^4)} (\mathrm{ht}(g) + 1),$$

$$\mathrm{ht}(q) \le 2^{2^{O(\mathrm{ht}(f)^{1+\varepsilon})}} \operatorname{Deg}(g)^{O(d^4 \operatorname{Deg}(g)^4)} \operatorname{Deg}(f) (\mathrm{ht}(g) + 1).$$

$$\operatorname{Deg}(r) \le \mathrm{ht}(f)! \operatorname{Deg}(g)^d,$$

$$\operatorname{Deg}(q) \le \mathrm{ht}(f)! \operatorname{Deg}(g)^{d+1} \operatorname{Deg}(f).$$

Ideal membership problem

Let $g, f_1, \ldots, f_p \in \mathbb{K}\langle x \rangle^q$. Assume that $g \in (f_1, \ldots, f_p)$, that is, there exist $a_1, \ldots, a_p \in \mathbb{K}\langle x \rangle$ such that

$$g = f_1 a_1 + \cdots + f_p a_p.$$

There is a computable bound $C = C(n, p, q, \mathrm{ht}(f_{i,j}), \mathrm{Deg}(f_{i,j}), \mathrm{Deg}(g_j))$, and $a_i \in \mathbb{K}\langle x \rangle$ such that

$$\mathrm{ht}(a_i) \leq C \cdot (\mathrm{ht}(g) + 1), \qquad \mathrm{Deg}(a_i) \leq C$$

and $g = f_1 a_1 + \cdots + f_p a_p$.

Corollary [R]

Let $M = \mathbb{K}[\![x]\!]^s/N$ for some $\mathbb{K}[\![x]\!]$-module generated by elements of $\mathbb{K}\langle x\rangle^s$.

Then there is a function $C : \mathbb{N} \longrightarrow \mathbb{R}_+$ such that

$$\text{ord}_M(f) \leq C(\text{Deg}(f)) \cdot \text{ht}(f) \qquad \forall f \in \mathbb{K}\langle x\rangle^s \setminus N.$$

Algebraic Laurent series

Problem: Describe an algebraic closure of $\mathbb{K}((x))$.

Assume $\mathrm{char}(\mathbb{K}) = 0$. When $n = 1$: Newton-Puiseux theorem.

What about $n \geq 2$?

McDonald's theorem

Let $P(x,t) \in \mathbb{K}[\![x]\!][t]$ be a monic polynomial where $\mathbb{K} = \overline{\mathbb{K}}$ and $\mathrm{char}(\mathbb{K}) = 0$. Then there exist a strongly convex rational cone σ, and $d \in \mathbb{N}^*$, such that

$$P(x,t) = \prod_{k=1}^{d} (t - \xi_k(x_1^{\frac{1}{d}} \dots x_n^{\frac{1}{d}}))$$

where $\mathrm{Supp}(\xi_k) \subset \sigma$.

Remark: $\xi(x_1^{\frac{1}{d}} \dots x_n^{\frac{1}{d}})$ is algebraic over $\mathbb{K}(\!(x)\!)$ iff $\xi(x)$ is algebraic over $\mathbb{K}(\!(x)\!)$.

Fix $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_{>0}$. For $f \in \mathbb{K}[\![x]\!]$, $f = \sum_\alpha f_\alpha x^\alpha$, we set

$$\nu_\omega(f) := \min\{\omega \cdot \alpha \mid f_\alpha \neq 0\}.$$

And $|f|_\omega := e^{-\nu_\omega(f)}$: ultrametric absolute value.

We denote by $\mathbb{K}^\omega$ the completion of $\mathbb{K}(\!(x)\!)$ for this absolute value.

Problem: determine the elements of $\mathbb{K}^\omega$ algebraic over $\mathbb{K}(\!(x)\!)$.

Diophantine approximation theorem [R]

Let $\xi \in \mathbb{K}^\omega \setminus \mathbb{K}(\!(x)\!)$ be algebraic over $\mathbb{K}(\!(x)\!)$. There exist $C > 0$ and $a \in \mathbb{N}$ such that

$$\forall f, g \in \mathbb{K}[\![x]\!], \quad \left| \xi - \frac{f}{g} \right|_\omega \geq C |g|_\omega^a.$$

Theorem [Aroca-Decaup-R]

Let ξ be a Laurent series with support in $\gamma + \sigma$ where $\gamma \in \mathbb{Z}^n$ and σ is a convex rational cone. If ξ is algebraic over $\mathbb{K}(\!(x)\!)$, there exist, $p(x) \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $\gamma_0 \in \mathbb{Z}^n$ and σ_0 convex rational cone such that

a) $\mathrm{Supp}(\xi + p(x)) \subset \gamma_0 + \sigma_0$,

b) each face of $\gamma_0 + \sigma_0$ contains infinitely many monomials of $\xi + p(x)$.

Gaps [Aroca-Decaup-R]

Let ξ be a Laurent series with support in $\gamma + \sigma$ where $\gamma \in \mathbb{Z}^n$ and σ is a convex rational cone. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. Let $\omega \in \mathbb{R}^n_+$ be in the interior of $\sigma_0^\vee$. Let us write $\xi = \sum_{j\in\mathbb{N}} \xi_{k(j)}$ with

i) for every $l \in \Gamma = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$, ξ_l is a (finite) sum of monomials of the form cx^α with $\omega \cdot \alpha = l$,

ii) the sequence $k(j)$ is a strictly increasing sequence of elements of Γ,

iii) for every integer j, $\xi_{k(j)} \neq 0$.

Then there exists a constant $C > 0$ such that

$$\frac{k(i+1)}{k(i)} \leq C \quad \forall i \in \mathbb{N}, \text{ if } \operatorname{char}(\mathbb{K}) > 0$$

$$k(i+1) - k(i) \leq C \quad \forall i \in \mathbb{N}, \text{ if } \operatorname{char}(\mathbb{K}) = 0.$$