Algebraic power series in several variables

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Notations

- \( \mathbb{K} \) will always denote a field.
- \( t \) single indeterminate, and \( x = (x_1, \ldots, x_n) \) vector of indeterminates. We set \( x' = (x_1, \ldots, x_{n-1}) \).
- We denote by \( \mathbb{K}\langle x \rangle \) the subset of \( \mathbb{K}[x] \) of power series algebraic over \( \mathbb{K}[x] \):

\[
\mathbb{K}\langle x \rangle := \{ f \in \mathbb{K}[x] \mid \exists P \in \mathbb{K}[x, t], P \neq 0, P(x, f(x)) = 0 \}.
\]

- Equivalently, a formal power series \( f \) is algebraic if the \( \mathbb{K}(x) \)-vector space generated by the powers of \( f \) is finite.
Power series appear essentially only as:
- Taylor expansions of functions.
- Generating series of sequences.
Algebraic power series are "simple" power series that have good algebraic properties.

\[ \{ \text{convergent} \} \subset \mathbb{K}[x] \subset \{ \text{rational} \} \subset \mathbb{K}\langle x \rangle \subset \{ \text{D-finite} \} \]
First algebraic properties

- $\mathbb{K}\langle x \rangle$ is a ring.
- $\mathbb{K}\langle x \rangle$ is a local ring. Its maximal ideal is $(x_1, \ldots, x_n)$.
- $\mathbb{K}\langle x \rangle$ is Noetherian: every ideal is finitely generated.
- $\mathbb{K}\langle x \rangle$ is a Henselian local ring: ”it satisfies the implicit function theorem”.
- It is stable under derivation and composition.
Henselianity (implicit function theorem)

Let $P(x, t) \in \mathbb{K}\langle x \rangle [t]$ such that

$$P(0, 0) = 0 \text{ and } \frac{\partial P}{\partial t}(0, 0) \neq 0.$$ 

$$\implies \exists ! f(x) \in \mathbb{K}\langle x \rangle, \quad P(x, f(x)) = 0 \text{ and } f(0) = 0.$$
Weierstrass theorems (Lafon)

Let $f(x)$ be a power series and $d \in \mathbb{N}$. We say that $f$ is $x_n$-regular of order $d$ if

$$f(0, \ldots, 0, x_n) = x_n^d u(x_n) \text{ with } u(0) \neq 0.$$ 

**Weierstrass division theorem:** Let $f \in \mathbb{K}\langle x \rangle$ be $x_n$-regular of order $d$ and $g \in \mathbb{K}\langle x \rangle$. Then there is a unique $(q, r) \in \mathbb{K}\langle x \rangle \times \mathbb{K}\langle x'\rangle[x_n]$ such that

$$g = fq + r$$

and $\deg x_n(r) < d$. 
Weierstrass preparation theorem

Let $f \in \mathbb{K}\langle x \rangle$ be $x_n$-regular of order $d$. Then $f$ can be written in a unique way as

$$f(x) = u(x) \left( x_n^d + a_1(x')x_n^{d-1} + \cdots + a_n(x') \right)$$

where $u(x) \in \mathbb{K}\langle x \rangle$, $a_i(x') \in \mathbb{K}\langle x' \rangle$,

$$u(0) \neq 0 \text{ and } a_i(0) = 0 \ \forall i.$$
Example: walks restricted to the quarter plane

Let $S \subset \{-1, 0, 1\}^2$. Set

$$a_{i,j,n} := \#\{\text{walks of length } n \text{ ending at } (i,j)\}.$$ 

Let $Q(x, y, t) := \sum_{i,j,n} a_{i,j,n} x^i y^j t^n$. It is solution of an equation the form

$$xy = K(x, y, t)Q(x, y, t) + R(x, t) + S(y, t) \quad (1)$$

where $R(x, t)$ and $S(y, t)$ can explicitly be expressed in terms of $Q(x, 0, t)$ and $Q(0, y, t)$, and

$$K(x, y, t) := \left( xy - t \sum_{(a,b) \in S} x^{a+1} y^{b+1} \right).$$
Take $S = \{SW, W, N, NE\}$.

Here

$K = xy - t(1 + y + xy^2 + x^2y^2) = z - t(1 + y + yz + z^2)$

where $z = xy$.

\[ z = (z - t(1 + y + yz + z^2))Q(t, z, y) + R(t, y) \]

Therefore

\[ xy = K(x, y, t)Q(t, xy, y) + R(t, y). \]
One variable case

Let $f(x) = \sum_{k \in \mathbb{N}} f_k x^k \in \mathbb{K}\langle x \rangle$.

Question: How can we express the coefficients $f_k$? Are there formulas for the coefficients? What about the "relations" satisfied by the coefficients?

- (Flajolet-Soria, Hickel-Matusinski): there exist formulas for the coefficients.
- There are results about the relations if $\mathbb{K}$ is finite (or of positive characteristic). For instance, $\sum_k f_k x^k \in F_p [x]$ is algebraic iff the sequence $(f_n)_n$ is $p$-automatic (Christol-Kamae-Mendès France-Rauzy).
One variable case 2

- If $\mathbb{K}$ is a valued field, there are bounds on the size of the coefficients.
- Algebraicity implies some bounds on the gaps in the expansion of $f(x)$ (Schmidt).
- Eisenstein theorem: For a series $f(x) = \sum_k f_k x^k \in \mathbb{Q}\langle x \rangle$, there is an integer $a \in \mathbb{N}$ such that

$$a^{k+1} f_k \in \mathbb{Z}, \quad \forall k \in \mathbb{N}.$$
General Problems

Given an equation $G(x,f(x)) = 0$ admitting formal power series solutions $f(x) \in \mathbb{K}[x]^p$, does there exist algebraic power series solutions?

If there exist algebraic solutions, what about the "size" of the solutions?

What kind of equations: algebraic equations, differential equations, functional equations, with or without constraints in the solutions?

If there is no algebraic solution, what about the solutions?
Artin approximation theorem

Let $P(x, y) \in \mathbb{K}[x, y]^p$ with $y = (y_1, \ldots, y_m)$. Assume given $y(x) \in \mathbb{K}[x]^m$ such that

$$P(x, y(x)) = 0.$$ 

Fix $c \in \mathbb{N}$. Then there exists $\tilde{y}(x) \in \mathbb{K}\langle x \rangle^m$ such that

$$P(x, \tilde{y}(x)) = 0 \text{ and } \tilde{y}_i(x) - y_i(x) \in (x)^c \text{ \forall } i.$$
Euler differential equation

We consider the equation

\[ x^2 f' - f + x = 0. \] (2)

It has a unique power series solution

\[ f = \sum_{n \geq 0} n! x^{n+1}. \]

By considering the Taylor development of \( f(x + t) \) at \( x \):

\[ f(x + t) = f(x) + tf'(x) + O(t^2) \]

(2) \iff \left\{ \begin{array}{l}
  x^2 g(x) - f(x) + x = 0 \\
  f(x + t) = f(x) + tg(x) + t^2 h(x, t) = 0 \\
  l(u) = f(x) + tg(x) + t^2 h(x, t) + (x + t - u)k_1(x, t, u) = 0 \\
  l(u) - f(x) = (x - u)k_2(x, u) = 0 
\end{array} \right. \]
General division theorem

Let us consider the following monomial order on $\mathbb{N}^n$:

$\forall \alpha, \beta \in \mathbb{N}^n, \, \alpha \leq \beta$ if $(|\alpha|, \alpha_1, \ldots, \alpha_n) \leq_{\text{lex}} (|\beta|, \beta_1, \ldots, \beta_n)$.

Let $g_1, \ldots, g_s \in \mathbb{K}[x]$. We set

$\Delta_1 := \exp(g_1) + \mathbb{N}^n, \quad \Delta_i = (\exp(g_i) + \mathbb{N}^n) \setminus \bigcup_{j=1}^{i-1} \Delta_j$ for $i \geq 2$.

$\Delta_0 = \mathbb{N}^n \setminus \bigcup_{i=1}^s \Delta_i$.

Let $f \in \mathbb{K}[x]$. Then there exist unique $q_1, \ldots, q_s, r \in \mathbb{K}[x]$ such that

$f = g_1q_1 + \cdots + g_sq_s + r$

$\exp(g_i) + \text{Supp}(q_i) \subset \Delta_i$ and $\text{Supp}(r) \subset \Delta_0$. 
Kashiwara-Gabber example

Let us perform the division of $xy$ by

$$g := (x - y^2)(y - x^2) = xy - x^3 - y^3 + x^2y^2$$

in $\mathbb{C}[x, y]$ with respect to the monomial $xy$:

$$xy - g(x, y)q(x, y) - r(x) - s(y) = 0. \quad (3)$$

By symmetry, we get $r(x) = s(x)$, and by substituting $y$ by $x^2$ we get the Mahler equation $r(x^2) + r(x) - x^3 = 0$. Therefore

$$r(x) = \sum_{i=0}^{\infty} (-1)^i x^{3 \cdot 2^i}$$

is not algebraic. This proves that (3) has a formal solution but no algebraic solution.
Nested Artin approximation theorem

Let $P(x, y) \in \mathbb{K}[x, y]^p$ with $y = (y_1, \ldots, y_m)$. Assume given $y(x) \in \mathbb{K}[x]^m$ such that

$$P(x, y(x)) = 0.$$ 

Assume moreover that $y_i(x) \in \mathbb{K}[x_1, \ldots, x_{\sigma_i}]$ for some $\sigma_i \in \{1, \ldots, n\}$.

Fix $c \in \mathbb{N}$. Then there exists $\tilde{y}(x) \in \mathbb{K}\langle x \rangle^m$ such that

$$P(x, \tilde{y}(x)) = 0 \text{ and } \tilde{y}_i(x) - y_i(x) \in (x)^c \quad \forall i$$

and

$$\tilde{y}_i(x) \in \mathbb{K}\langle x_1, \ldots, x_{\sigma_i} \rangle.$$
Complexity of an algebraic power series

Let $f \in \mathbb{K}\langle x \rangle$ and $P(x, t) \in \mathbb{K}[x, t]$ its minimal polynomial. We set

$$\text{Deg}(f) = \deg_t(P) \text{ and } \text{ht}(f) = \deg_x(P).$$

complexity of $f = \text{Deg}(f), \text{ht}(f)$.

This can be defined for every "root" of a polynomial in $\mathbb{K}[x, t]$. 
Complexity bounds (Adamczewski-Bell)

We consider \(f_1, \ldots, f_k \in \mathbb{K}\langle x \rangle, p_1, \ldots, p_k \in \mathbb{K}[x]\).

\[
\text{Deg}(p_1 f_1 + \cdots + p_k f_k) \leq \prod_i \text{Deg}(f_i)
\]

\[
\text{ht}\left(\sum_i p_i f_i\right) \leq k \prod_i \text{Deg}(f_i) \cdot \left(\max_j \text{ht}(p_j) + \max_l \text{ht}(f_l)\right)
\]

\[
\text{Deg}(f_1 \cdots f_k) \leq \text{Deg}(f_1) \cdots \text{Deg}(f_k)
\]

\[
\text{ht}(f_1 \cdots f_k) \leq k \prod_i \text{Deg}(f_i) \cdot \max_j \text{ht}(f_j)
\]
Complexity bounds 2 [R]

This allows us to give bounds for the derivations or compositions of alg. power series.

Let \( f \in \mathbb{K}\langle x \rangle \) be \( x_n \)-regular of order \( d \) and \( g \in \mathbb{K}\langle x \rangle \).

Then there is a unique \((q, r) \in \mathbb{K}\langle x \rangle \times \mathbb{K}\langle x' \rangle [x_n] \) such that

\[
g = fq + r \quad \text{and} \quad \deg_{x_n}(r) < d,
\]

\[
\ht(r) \leq 2^{2O(\ht(f)^{1+\epsilon})} \Deg(g)^{O(d^4 \Deg(g)^4)} (\ht(g) + 1),
\]

\[
\ht(q) \leq 2^{2O(\ht(f)^{1+\epsilon})} \Deg(g)^{O(d^4 \Deg(g)^4)} \Deg(f)(\ht(g) + 1).
\]

\[
\Deg(r) \leq \ht(f)! \Deg(g)^d,
\]

\[
\Deg(q) \leq \ht(f)! \Deg(g)^{d+1} \Deg(f).
\]
Ideal membership problem

Let \( g, f_1, \ldots, f_p \in \mathbb{K}\langle x \rangle^q \). Assume that \( g \in (f_1, \ldots, f_p) \), that is, there exist \( a_1, \ldots, a_p \in \mathbb{K}\langle x \rangle \) such that

\[
g = f_1 a_1 + \cdots + f_p a_p.
\]

There is a computable bound

\[
C = C(n, p, q, \text{ht}(f_{i,j}), \text{Deg}(f_{i,j}), \text{Deg}(g_j)), \text{ and } a_i \in \mathbb{K}\langle x \rangle
\]

such that

\[
\text{ht}(a_i) \leq C \cdot (\text{ht}(g) + 1), \quad \text{Deg}(a_i) \leq C
\]

and \( g = f_1 a_1 + \cdots + f_p a_p \).
Corollary [R]

Let $M = K[x]^s/N$ for some $K[x]$-module generated by elements of $K\langle x \rangle^s$.

Then there is a function $C : \mathbb{N} \rightarrow \mathbb{R}_+$ such that

$$\text{ord}_M(f) \leq C(\text{Deg}(f)) \cdot \text{ht}(f) \quad \forall f \in K\langle x \rangle^s \setminus N.$$
Problem: Describe an algebraic closure of $\mathbb{K}((x))$.
Assume $\text{char}(\mathbb{K}) = 0$. When $n = 1$: Newton-Puiseux theorem.
What about $n \geq 2$?
McDonald’s theorem

Let $P(x, t) \in \mathbb{K}[x][t]$ be a monic polynomial where $\mathbb{K} = \overline{\mathbb{K}}$ and $\text{char}(\mathbb{K}) = 0$. Then there exist a strongly convex rational cone $\sigma$, and $d \in \mathbb{N}^*$, such that

$$P(x, t) = \prod_{k=1}^{d} (t - \xi_k(x_1^{\frac{1}{d}} \ldots x_n^{\frac{1}{d}}))$$

where $\text{Supp}(\xi_k) \subset \sigma$.

Remark: $\xi(x_1^{\frac{1}{d}} \ldots x_n^{\frac{1}{d}})$ is algebraic over $\mathbb{K}((x))$ iff $\xi(x)$ is algebraic over $\mathbb{K}((x))$. 
Fix $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}_{>0}^n$. For $f \in K[x], f = \sum_\alpha f_\alpha x^\alpha$, we set

$$\nu_\omega(f) := \min\{\omega \cdot \alpha \mid f_\alpha \neq 0\}.$$ 

And $|f|_\omega := e^{-\nu_\omega(f)}$: ultrametric absolute value.

We denote by $K^\omega$ the completion of $K((x))$ for this absolute value.

Problem: determine the elements of $K^\omega$ algebraic over $K((x))$. 
Diophantine approximation theorem [R]

Let $\xi \in K^\omega \setminus K((x))$ be algebraic over $K((x))$. There exist $C > 0$ and $a \in \mathbb{N}$ such that

$$\forall f, g \in K[x], \quad \left| \xi - \frac{f}{g} \right|_\omega \geq C|g|^a.$$

Theorem [Aroca-Decaup-R]

Let $\xi$ be a Laurent series with support in $\gamma + \sigma$ where $\gamma \in \mathbb{Z}^n$ and $\sigma$ is a convex rational cone. If $\xi$ is algebraic over $K((x))$, there exist, $p(x) \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $\gamma_0 \in \mathbb{Z}^n$ and $\sigma_0$ convex rational cone such that

a) $\text{Supp}(\xi + p(x)) \subset \gamma_0 + \sigma_0$,

b) each face of $\gamma_0 + \sigma_0$ contains infinitely many monomials of $\xi + p(x)$. 
Gaps [Aroca-Decaup-R]

Let \( \xi \) be a Laurent series with support in \( \gamma + \sigma \) where \( \gamma \in \mathbb{Z}^n \) and \( \sigma \) is a convex rational cone. Assume that \( \xi \) is algebraic over \( \mathbb{K}((x)) \). Let \( \omega \in \mathbb{R}^n_+ \) be in the interior of \( \sigma_0^\vee \). Let us write \( \xi = \sum_{j \in \mathbb{N}} \xi_{k(j)} \) with

i) for every \( l \in \Gamma = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n \), \( \xi_l \) is a (finite) sum of monomials of the form \( cx^\alpha \) with \( \omega \cdot \alpha = l \),

ii) the sequence \( k(j) \) is a strictly increasing sequence of elements of \( \Gamma \),

iii) for every integer \( j \), \( \xi_{k(j)} \neq 0 \).

Then there exists a constant \( C > 0 \) such that

\[
\frac{k(i + 1)}{k(i)} \leq C \quad \forall i \in \mathbb{N}, \text{ if } \text{char}(\mathbb{K}) > 0
\]

\[
k(i + 1) - k(i) \leq C \quad \forall i \in \mathbb{N}, \text{ if } \text{char}(\mathbb{K}) = 0.
\]