PRIME NUMBERS ALONG RUDIN–SHAPIRO SEQUENCES

CHRISTIAN MAUDUIT AND JOËL RIVAT

Abstract. For a large class of digital functions f, we estimate the sums $\sum_{n\leq x}\Lambda(n)f(n)$ (and $\sum_{n\leq x}\mu(n)f(n)$) where Λ denotes the von Mangoldt function (and μ the Möbius function). We deduce from these estimates a Prime Number Theorem (and a Möbius randomness principle) for sequences of integers with digit properties including the Rudin-Shapiro sequence and some of its generalizations.

Contents

1. Introduction

We denote by $\mathbb{N}$ the set of integers $n\geq 0$, by $\mathbb{U}$ the set of complex numbers of modulus 1, by $\mathcal{P}$ the set of prime numbers and for any $a\in\mathbb{Z}$ and $m\in\mathbb{N}$ with $m\geq 1$, by $\mathcal{P}(a,m)$ the set of prime numbers $p\equiv a \bmod m$. For $n\in\mathbb{N}$, $n\geq 1$, we denote by $\tau(n)$ the number of divisors of n, by $\omega(n)$ the number of distinct prime factors of n, by $\Lambda(n)$ the von Mangoldt function (defined by $\Lambda(n)=\log p$ if $n=p^k$ with $k\in\mathbb{N}$, $k\geq 1$ and $\Lambda(n)=0$ otherwise) and by $\mu(n)$ the Möbius function (defined by $\mu(n)=(-1)^{\omega(n)}$ if n is squarefree and $\mu(n)=0$ otherwise). For $x\in\mathbb{R}$ we denote by $\|x\|$ the distance of x to the nearest integer, by $\pi(x)$ the number of prime numbers less or equal to x and we set $\mathrm{e}(x)=\exp(2i\pi x)$.

Throughout this work we denote by q an integer greater or equal to 2. Any $n\in\mathbb{N}$ can be written in base q as $n=\sum_{j\geq 0}\varepsilon_j(n)q^j$ with $\varepsilon_j(n)\in\{0,\dots,q-1\}$ for all $j\in\mathbb{N}$.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers of modulus at most 1 generated by a simple algorithm. Many recent works are devoted to the proof that special sequences $(u_n)_{n\in\mathbb{N}}$ satisfy the Möbius randomness principle (*i.e.* that $\sum_{n\leq x}\mu(n)u_n=o(x)$, see [13, page 338]) or a Prime Number Theorem (*i.e.* an asymptotic formula for the more difficult to handle sum $\sum_{n\leq x}\Lambda(n)u_n$), see [6], [7], [12], [20]. These works are related to the Sarnak conjecture (see [25]) which asserts that if $(u_n)_{n\in\mathbb{N}}$ is produced by a zero topological entropy dynamical system, then $\sum_{n\leq x}\mu(n)u_n=o(x)$. In the case of sequences $(u_n)_{n\in\mathbb{N}}$ such that u_n is defined by a digital property of the integer n, Dartyge and Tenenbaum proved in [7], using Daboussi's convolution

Date: June 8, 2013.
2010 *Mathematics Subject Classification.* Primary: 11A63, 11B85, 11N05, Secondary: 11L20,11N60.
Key words and phrases. Rudin-Shapiro sequence, prime numbers, Möbius function, exponential sums.
This work was supported by the Agence Nationale de la Recherche project ANR-10-BLAN 0103 MUNUM.

method, that for any real number α we have

$$\sum_{n\le x}\mu(n)\,\mathrm{e}\left(\alpha\sum_{j\ge 0}\varepsilon_j(n)\right)=O\left(\frac{x}{\log\log x}\right).$$

In [20] we proved that, for any real number α such that $(q-1)\alpha\notin\mathbb{Z}$, there exists a real number $\eta(\alpha)<1$ such that

$$\sum_{n\le x}\Lambda(n)\,\mathrm{e}\left(\alpha\sum_{j\ge 0}\varepsilon_j(n)\right)=O\left(x^{\eta(\alpha)}\right), \tag{1}$$

answering a question due to Gelfond in [10] (see [9] for an explicit value of $\eta(\alpha)$) and [18] for an extension to more general digital functions). The proof of (1), based on Vaughan's identity ([13, (13.39)]) and the estimate of type I and type II bilinear sums, can be applied to the Möbius function μ using [13, (13.40)] and this shows that, for any real number α such that $(q-1)\alpha\notin\mathbb{Z}$, there exists a real number $\eta(\alpha)<1$ such that

$$\sum_{n\le x}\mu(n)\,\mathrm{e}\left(\alpha\sum_{j\ge 0}\varepsilon_j(n)\right)=O\left(x^{\eta(\alpha)}\right).$$

Kalai asked in [14] and [15] a series of questions concerning the computational complexity of μ that can be translated in proving a Möbius randomness principle for some specific binary sequences. In [3] Bourgain proved that

$$\max_{S\subseteq\{0,\dots,\nu-1\}}\left|\sum_{n<2^\nu}\mu(n)(-1)^{\sum_{i\in S}\varepsilon_i(n)}\right|=O(2^{\nu-\nu^{1/10}}), \tag{2}$$

showing both a Möbius randomness principle and a Prime Number Theorem for these sequences (see [11] for a related result showing that μ is orthogonal to any Boolean function computable by constant depth and polynomial size circuits). Studying more precisely the distribution of the Fourier-Walsh coefficients $\sum_{n<2^\nu}\mu(n)(-1)^{\sum_{i\in S}\varepsilon_i(n)}$, Bourgain proved in [5] that μ is orthogonal to any monotone Boolean function (see [4] for a lower bound for the number of primes captured by these functions). The estimate (2) means that for any polynomial $P\in\mathbb{Z}[X_0,\dots,X_{\nu-1}]$ of degree at most 1 we have

$$\sum_{n<2^\nu}\mu(n)(-1)^{P(\varepsilon_0(n),\dots,\varepsilon_{\nu-1}(n))}=O(2^{\nu-\nu^{1/10}}),$$

but the question asked by Kalai in [16] concerning the case of polynomials of degree greater than 1 is open. The simplest case of polynomial of degree 2 is given by the Rudin-Shapiro sequence

$$\left((-1)^{\sum_{i\ge 1}\varepsilon_{i-1}(n)\varepsilon_i(n)}\right)_{n\in\mathbb{N}} \tag{3}$$

introduced independently by Shapiro in [26] and by Rudin in [24] for which Tao suggests in [16] a strategy to prove a Möbius randomness principle, *i.e.*

$$\sum_{n\le x}\mu(n)(-1)^{\sum_{i\ge 1}\varepsilon_{i-1}(n)\varepsilon_i(n)}=o(x).$$

In this paper we will obtain as a special case in Theorem 3 a quantitative Prime Number Theorem (and a Möbius randomness principle) for the sequences

$$\left((-1)^{\sum_{i\ge\delta+1}\varepsilon_{i-\delta-1}(n)\varepsilon_i(n)}\right)_{n\in\mathbb{N}}$$

for all integers $\delta\ge 0$ (including the Rudin-Shapiro sequence for $\delta=0$), and in Theorem 4 a quantitative Prime Number Theorem (and a Möbius randomness principle) for the sequences

$$\left((-1)^{\sum_{i\ge d-1}\varepsilon_{i-d+1}(n)\cdots\varepsilon_{i-1}(n)\varepsilon_i(n)}\right)_{n\in\mathbb{N}}$$

for all integers $d \geq 2$, providing an answer to Kalai's question for the simplest case of polynomial of degree d.

2. Statement of the results

One of the main ingredients in our proof of a Prime Number Theorem for the sequence $\left(\exp\left(\alpha \sum_{i\geq 0} \varepsilon_i(n)\right)\right)_{n\in\mathbb{N}}$ in [20] was to establish that the L^1 norm of the Discrete Fourier Transform of this sequence is very small. Unfortunately this property is generally not true for other digital sequences and in particular for the Rudin-Shapiro sequence (3). Such a difference in the behaviour of the Fourier transforms is not surprising if we remember that the sequences $\left((-1)^{\sum_{i\geq 0}\varepsilon_i(n)}\right)_{n\in\mathbb{N}}$ and $\left((-1)^{\sum_{i\geq 1}\varepsilon_{i-1}(n)\varepsilon_i(n)}\right)_{n\in\mathbb{N}}$ have quite different spectral properties: the correlation measure of the first one is a singular measure, namely the Riesz product $\prod_{n\geq 0}(1-\cos 2^n t)$ (see [23, section 3.3.3] or [17]), while for the second one it is the Lebesgue measure (see [23, corollary 8.5]).

For $f : \mathbb{N} \to \mathbb{U}$ and any $\lambda \in \mathbb{N}$, let us denote by f_λ the q^λ-periodic function defined by

$$\forall n \in \{0,\dots,q^\lambda - 1\},\ \forall k \in \mathbb{Z},\ f_\lambda(n + kq^\lambda) = f(n). \tag{4}$$

Definition 1. *A function $f : \mathbb{N} \to \mathbb{U}$ has the* carry property *if, uniformly for $(\lambda,\kappa,\rho) \in \mathbb{N}^3$ with $\rho < \lambda$, the number of integers $0 \leq \ell < q^\lambda$ such that there exists $(k_1,k_2) \in \{0,\dots,q^\kappa - 1\}^2$ with*

$$f(\ell q^\kappa + k_1 + k_2)\,\overline{f(\ell q^\kappa + k_1)} \neq f_{\kappa+\rho}(\ell q^\kappa + k_1 + k_2)\,\overline{f_{\kappa+\rho}(\ell q^\kappa + k_1)} \tag{5}$$

is at most $O(q^{\lambda-\rho})$ where the implied constant may depend only on q and f.

We introduce a set of functions with uniformly small Discrete Fourier Transforms:

Definition 2. *Given a non decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{\lambda\to+\infty}\gamma(\lambda) = +\infty$ and $c > 0$ we denote by $\mathcal{F}_{\gamma,c}$ the set of functions $f : \mathbb{N} \to \mathbb{U}$ such that for $(\kappa,\lambda) \in \mathbb{N}^2$ with $\kappa \leq c\lambda$ and $t \in \mathbb{R}$:*

$$\left| q^{-\lambda} \sum_{0\leq u<q^\lambda} f(uq^\kappa)\,\mathrm{e}(-ut) \right| \leq q^{-\gamma(\lambda)}. \tag{6}$$

For example, for any α such that $(q-1)\alpha \in \mathbb{R}\setminus\mathbb{Z}$, it follows from [19, Lemme 9] that the function $f(n) = \mathrm{e}(\alpha \sum_{i\geq 0}\varepsilon_i(n))$ verifies Definition 1 and $f \in \mathcal{F}_{\gamma,c}$ in Definition 2 for any $c > 0$ and γ such that for $\lambda \geq 2$

$$\gamma(\lambda) = \frac{\pi^2}{12\log q}\left(1 - \frac{2}{q+1}\right)\|(q-1)\alpha\|^2\,\lambda - \frac{\pi^2}{48\,\log q}.$$

The goal of this paper is to present a new method which allows to prove a Prime Number Theorem for a large class of sequences with digit properties including the Rudin-Shapiro sequence and some of its generalizations. Roughly speaking, we prove that if we control the carry properties of a function $f : \mathbb{N} \to \mathbb{U}$ (Definition 1) for which the discrete Fourier transform is uniformly small (Definition 2), then we have a Prime Number Theorem (Theorem 1) (and a Möbius randomness principle (Theorem 2)) for f. This general result can be applied in many situations. In part 11 we will apply it to the case of Rudin-Shapiro sequences.

Theorem 1. *Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a non decreasing function satisfying $\lim_{\lambda\to+\infty}\gamma(\lambda) = +\infty$, and $f : \mathbb{N} \to \mathbb{U}$ be a function satisfying Definition 1 and $f \in \mathcal{F}_{\gamma,c}$ for some $c \geq 10$ in Definition 2. Then for any $\vartheta \in \mathbb{R}$ we have*

$$\left| \sum_{n\leq x} \Lambda(n) f(n)\,\mathrm{e}(\vartheta n) \right| \ll c_1(q)(\log x)^{c_2(q)}\,x\,q^{-\gamma(2\lfloor(\log x)/80\log q\rfloor)/20}, \tag{7}$$

with $c_1(q)=\max(\tau(q),\log^2 q)^{1/4}\,(\log q)^{-2-\frac14\max(\omega(q),2)}$ *and* $c_2(q)=\frac94+\frac14\max(\omega(q),2)$.

Remark 1. *Theorem 1 gives a non trivial result if*

$$\liminf_{\lambda\to\infty}\frac{\gamma(\lambda)}{\log\lambda}>\frac{20\,c_2(q)}{\log q}. \tag{8}$$

Corollary 1. *Let* $b:\mathbb{N}\to\mathbb{N}$ *be such that, for any* $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, *the function* $f(n)=\mathrm{e}(\alpha b(n))$ *satisfies Definition 1 and* $f\in\mathcal{F}_{\gamma,c}$ *in Definition 2 for some* $c\geq 10$ *and* γ *satisfying* (8). *Then for any* $a\in\mathbb{Z}$, $m\in\mathbb{N}$, $m\geq 1$ *with* $\gcd(a,m)=1$, *the sequence* $(\alpha b(p))_{p\in\mathcal{P}(a,m)}$ *is uniformly distributed modulo* 1 *if and only if* $\alpha\in\mathbb{R}\setminus\mathbb{Q}$.

Corollary 2. *Let* $b:\mathbb{N}\to\mathbb{N}$ *and* $(m,m')\in\mathbb{N}^2$, $m,m'\geq 1$ *be such that, for any integer* j', $1\leq j'<m'$, *the function* $f(n)=\mathrm{e}(\frac{j'}{m'}b(n))$ *satisfies Definition 1 and* $f\in\mathcal{F}_{\gamma,c}$ *in Definition 2 for some* $c\geq 10$ *and* γ *satisfying* (8). *Then for any* $(a,a')\in\mathbb{Z}^2$ *such that* $\gcd(a,m)=1$ *we have for* $x\to+\infty$

$$\mathrm{card}\{p\leq x,\ p\in\mathcal{P}(a,m),\ b(p)\equiv a' \bmod m'\}=(1+o(1))\,\frac{\pi(x;a,m)}{m'}.$$

Corollary 3. *Let* $b:\mathbb{N}\to\mathbb{N}$ *and* $(m,m')\in\mathbb{N}^2$, $m,m'\geq 1$ *be such that, for any integer* j', $1\leq j'<m'$, *the function* $f(n)=\mathrm{e}(\frac{j'}{m'}b(n))$ *satisfies Definition 1 and* $f\in\mathcal{F}_{\gamma,c}$ *in Definition 2 for some* $c\geq 10$ *and* γ *satisfying* (8). *Then for any* $(a,a')\in\mathbb{Z}^2$ *such that* $\gcd(a,m)=1$ *the sequence* $(\vartheta p)_{p\in\mathcal{B}(a,m,a',m')}$ *is uniformly distributed modulo* 1 *if and only if* $\vartheta\in\mathbb{R}\setminus\mathbb{Q}$, *where* $\mathcal{B}(a,m,a',m')=\{p\in\mathcal{P}(a,m),\ b(p)\equiv a' \bmod m'\}$.

In order to estimate sums of the form $\sum_n\Lambda(n)F(n)$ in Theorem 1 by using a combinatorial identity like Vaughan's identity (see (13.39) of [13]), it is sufficient to estimate bilinear sums of the form

$$\sum_m\sum_n a_m b_n F(mn)$$

(we have described this method in details in [20]). These sums are said of type I if b_n is a smooth function of n. Otherwise they are said of type II. The key of this approach is that for type I sums the summation over the smooth variable n is of significant length, while for type II sums both summations have a significant length.

Using (13.40) instead of (13.39) of [13] we obtain a similar result for the Möbius function μ (a better exponent of the factor $\log x$ might be obtained with some extra work):

Theorem 2. *Let* $\gamma:\mathbb{R}\to\mathbb{R}$ *be a non decreasing function satisfying* $\lim_{\lambda\to+\infty}\gamma(\lambda)=+\infty$, $c\geq 10$ *and* $f:\mathbb{N}\to\mathbb{U}$ *be a function satisfying Definition 1 and* $f\in\mathcal{F}_{\gamma,c}$ *in Definition 2. Then for any* $\vartheta\in\mathbb{R}$ *we have*

$$\left|\sum_{n\leq x}\mu(n)f(n)\,\mathrm{e}(\vartheta n)\right|\ll c_1(q)\,(\log x)^{\frac94+\frac14\max(\omega(q),2)}\ x\,q^{-\gamma(2\lfloor(\log x)/80\log q\rfloor)/20}, \tag{9}$$

with $c_1(q)$ *and* $c_2(q)$ *defined in Theorem 1.*

3. Notations and preliminary lemmas

For $a\in\mathbb{Z}$ and $\kappa\in\mathbb{N}$ we denote by $\mathrm{r}_\kappa(a)$ the unique integer $r\in\{0,\ldots,q^\kappa-1\}$ such that $a\equiv r \bmod q^\kappa$. More generally for integers $0\leq\kappa_1\leq\kappa_2$ we denote by $\mathrm{r}_{\kappa_1,\kappa_2}(a)$ the unique integer $u\in\{0,\ldots,q^{\kappa_2-\kappa_1}-1\}$ such that $a=kq^{\kappa_2}+uq^{\kappa_1}+v$ for some $v\in\{0,\ldots,q^{\kappa_1}-1\}$ and $k\in\mathbb{Z}$. We notice that we have $\mathrm{r}_{\kappa_1,\kappa_2}(a)=\left\lfloor\frac{\mathrm{r}_{\kappa_2}(a)}{q^{\kappa_1}}\right\rfloor$ and for any $u\in\{0,\ldots,q^{\kappa_2-\kappa_1}-1\}$,

$$\mathrm{r}_{\kappa_1,\kappa_2}(a)=u\iff\frac{a}{q^{\kappa_2}}\in\left[\frac{u}{q^{\kappa_2-\kappa_1}},\frac{u+1}{q^{\kappa_2-\kappa_1}}\right)+\mathbb{Z}. \tag{10}$$

For $a\geq 0$, $\mathrm{r}_\kappa(a)$ is the integer obtained from the κ least significant digits of a, while $\mathrm{r}_{\kappa_1,\kappa_2}(a)$ is the integer obtained using the digits of a of indexes $\kappa_1,\ldots,\kappa_2-1$.

The following lemma is a classical method to detect real numbers in an interval modulo 1 by means of exponential sums. For $\alpha \in \mathbb{R}$ with $0 \le \alpha < 1$ we denote by χ_α the characteristic function of the interval $[0, \alpha)$ modulo 1:

$$\chi_\alpha(x) = \lfloor x \rfloor - \lfloor x - \alpha \rfloor . \tag{11}$$

Lemma 1. *For all $\alpha \in \mathbb{R}$ with $0 \le \alpha < 1$ and all integer $H \ge 1$ there exist real valued trigonometric polynomials $A_{\alpha,H}(x)$ and $B_{\alpha,H}(x)$ such that for all $x \in \mathbb{R}$*

$$|\chi_\alpha(x) - A_{\alpha,H}(x)| \le B_{\alpha,H}(x), \tag{12}$$

where

$$A_{\alpha,H}(x) = \sum_{|h|\le H} a_h(\alpha, H)\,\mathrm{e}(hx), \quad B_{\alpha,H}(x) = \sum_{|h|\le H} b_h(\alpha, H)\,\mathrm{e}(hx), \tag{13}$$

with coefficients $a_h(\alpha, H)$ and $b_h(\alpha, H)$ satisfying

$$a_0(\alpha, H) = \alpha, \ \ |a_h(\alpha, H)| \le \min\left(\alpha, \tfrac{1}{\pi|h|}\right), \ \ |b_h(\alpha, H)| \le \tfrac{1}{H+1}. \tag{14}$$

Proof. In order to apply Theorem 19 of [27] we need to normalize χ_α: let us define for all $x \in \mathbb{R}$

$$\widetilde{\chi_\alpha}(x) = \lim_{t\to 0+} \tfrac{1}{2}\left(\chi_\alpha(x - t) + \chi_\alpha(x + t)\right).$$

Applying (7.24) of [27] we get

$$|\widetilde{\chi_\alpha}(x) - A_{\alpha,H}(x)| \le B_{\alpha,H}(x),$$

with the coefficients $a_h(\alpha, H)$ and $b_h(\alpha, H)$ defined by $a_0(\alpha, H) = \alpha$,

$$a_h(\alpha, H) = a^*_h(\alpha, H)\,\mathrm{e}\left(\tfrac{-h\alpha}{2}\right), \quad a^*_h(\alpha, H) = \tfrac{\sin \pi h\alpha}{\pi h}\left(\pi\tfrac{|h|}{H+1}\left(1 - \tfrac{|h|}{H+1}\right)\cot \pi\tfrac{|h|}{H+1} + \tfrac{|h|}{H+1}\right) \tag{15}$$

and

$$b_h(\alpha, H) = b^*_h(\alpha, H)\,\mathrm{e}\left(\tfrac{-h\alpha}{2}\right), \quad b^*_h(\alpha, H) = \tfrac{1}{H+1}\left(1 - \tfrac{|h|}{H+1}\right)\cos(\pi h\alpha). \tag{16}$$

In order to see that $A_{\alpha,H}(x)$ is real valued we notice that $a^*_{-h}(\alpha, H) = a^*_h(\alpha, H)$ and

$$\begin{aligned} A_{\alpha,H}(x) &= a_0(\alpha, H) + \sum_{h=1}^{H} a^*_h(\alpha, H)\left(\mathrm{e}\left(h\left(x - \tfrac{\alpha}{2}\right)\right) + \mathrm{e}\left(-h\left(x - \tfrac{\alpha}{2}\right)\right)\right) \\ &= a_0(\alpha, H) + 2\sum_{h=1}^{H} a^*_h(\alpha, H)\cos\left(2\pi h\left(x - \tfrac{\alpha}{2}\right)\right). \end{aligned}$$

Since $B_{\alpha,H}(x) \ge 0$, obviously $B_{\alpha,H}(x)$ is real valued. By the argument above we have

$$B_{\alpha,H}(x) = b_0(\alpha, H) + 2\sum_{h=1}^{H} b^*_h(\alpha, H)\cos\left(2\pi h\left(x - \tfrac{\alpha}{2}\right)\right).$$

Observing that for all $x \in \mathbb{R}$ we have $\chi_\alpha(x) = \lim_{t\to 0+} \widetilde{\chi_\alpha}(x + t)$, we obtain (12).

The upper bound of $|a_h(\alpha, H)|$ given by (14) follows from Theorem 6 of [27] and the upper bound of $|b_h(\alpha, H)|$ given by (14) follows from (16). □

In dimension 2, we can detect points in a square (modulo 1) using the following:

Lemma 2. *For $\alpha_1, \alpha_2 \in [0, 1)$ and integers $H_1 \ge 1$, $H_2 \ge 1$, we have for all $(x, y) \in \mathbb{R}^2$*

$$\begin{aligned} &|\chi_{\alpha_1}(x)\chi_{\alpha_2}(y) - A_{\alpha_1,H_1}(x)A_{\alpha_2,H_2}(y)| \\ &\quad \le \ \chi_{\alpha_1}(x)B_{\alpha_2,H_2}(y) + B_{\alpha_1,H_1}(x)\chi_{\alpha_2}(y) + B_{\alpha_1,H_1}(x)B_{\alpha_2,H_2}(y) \end{aligned} \tag{17}$$

where $A_{\alpha,H}(.)$ and $B_{\alpha,H}(.)$ are the real valued trigonometric polynomials defined by (13).

Proof. For $(x,y)\in\mathbb{R}^2$ we have

$$\begin{aligned}&\chi_{\alpha_1}(x)\chi_{\alpha_2}(y)-A_{\alpha_1,H_1}(x)A_{\alpha_2,H_2}(y)\\ &\quad=\ \chi_{\alpha_1}(x)(\chi_{\alpha_2}(y)-A_{\alpha_2,H_2}(y))+(\chi_{\alpha_1}(x)-A_{\alpha_1,H_1}(x))\chi_{\alpha_2}(y)\\ &\qquad\qquad-(\chi_{\alpha_1}(x)-A_{\alpha_1,H_1}(x))(\chi_{\alpha_2}(y)-A_{\alpha_2,H_2}(y)).\end{aligned}$$

Since $\chi_{\alpha_1}(x)\geq 0$ and $\chi_{\alpha_2}(y)\geq 0$, by (12) we get (17). □

The following lemma is a generalization of van der Corput's inequality.

Lemma 3. *For all complex numbers $z_1,\ldots,z_N$ and all integers $k\geq 1$ and $R\geq 1$ we have*

$$\left|\sum_{1\leq n\leq N} z_n\right|^2 \leq \frac{N+kR-k}{R}\left(\sum_{1\leq n\leq N}|z_n|^2+2\sum_{1\leq r<R}\left(1-\frac{r}{R}\right)\sum_{1\leq n\leq N-kr}\Re\left(z_{n+kr}\overline{z_n}\right)\right) \tag{18}$$

where $\Re(z)$ denotes the real part of z.

Proof. See for example Lemma 17 of [19]. □

We will often make use of the following upper bound of geometric series of ratio $\mathrm{e}(\xi)$ for $(L_1,L_2)\in\mathbb{Z}^2$, $L_1\leq L_2$ and $\xi\in\mathbb{R}$:

$$\left|\sum_{L_1<\ell\leq L_2}\mathrm{e}(\ell\xi)\right|\leq\min(L_2-L_1,|\sin\pi\xi|^{-1}). \tag{19}$$

Lemma 4 and Lemma 5 allow to estimate on average the minimums arising from (19).

Lemma 4. *Let $(a,m)\in\mathbb{Z}^2$ with $m\geq 1$ and $b\in\mathbb{R}$. For any real number $U>0$ we have*

$$\sum_{0\leq n\leq m-1}\min\left(U,\left|\sin\pi\tfrac{an+b}{m}\right|^{-1}\right)\ll\gcd(a,m)\,U+m\log m. \tag{20}$$

Proof. It follows from Lemma 6 of [20]. □

Lemma 5. *Let $m\geq 1$ and $A\geq 1$ be integers and $b\in\mathbb{R}$. For any real number $U>0$ we have*

$$\frac{1}{A}\sum_{1\leq a\leq A}\sum_{0\leq n<m}\min\left(U,\left|\sin\pi\tfrac{an+b}{m}\right|^{-1}\right)\ll\tau(m)\;U+m\log m. \tag{21}$$

Proof. By (20) it is enough to observe that

$$\sum_{1\leq a\leq A}\gcd(a,m)=\sum_{\substack{d\,|\,m\\ d\leq A}}d\sum_{\substack{1\leq a\leq A\\ (a,m)=d}}1\leq\sum_{\substack{d\,|\,m\\ d\leq A}}d\sum_{\substack{1\leq a\leq A\\ d\,|\,a}}1=\sum_{\substack{d\,|\,m\\ d\leq A}}d\left\lfloor\frac{A}{d}\right\rfloor\leq A\;\tau(m),$$

which implies (21). □

The following lemma is a classical application of the large sieve inequality:

Lemma 6. *For any complex numbers $z_1,\ldots,z_N$ and $Q>0$ we have*

$$\sum_{q\leq Q}\sum_{\substack{a=1\\ (a,q)=1}}^{q}\left|\sum_{n=1}^{N}z_n\,\mathrm{e}\left(\frac{an}{q}\right)\right|^2\leq(N-1+Q^2)\sum_{n=1}^{N}|z_n|^2. \tag{22}$$

Proof. See Theorem 3 and Section 8 of [21]. □

Let $f:\mathbb{N}\to\mathbb{U}$, $\lambda\in\mathbb{N}$ and f_λ defined by (4). The Discrete Fourier Transform of f_λ is defined for $t\in\mathbb{R}$ by

$$\widehat{f_\lambda}(t)=\frac{1}{q^\lambda}\sum_{0\leq u<q^\lambda}f_\lambda(u)\,\mathrm{e}\left(-\frac{ut}{q^\lambda}\right)=\frac{1}{q^\lambda}\sum_{0\leq u<q^\lambda}f(u)\,\mathrm{e}\left(-\frac{ut}{q^\lambda}\right). \tag{23}$$

For $\lambda \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$\sum_{0 \le h < q^\lambda} \left| \widehat{f_\lambda}(h+t) \right|^2 = 1. \tag{24}$$

so that, if f satisfies (6), then

$$1 = \sum_{0 \le h < q^\lambda} \left| q^{-\lambda} \sum_{0 \le u < q^\lambda} f(uq^\kappa)\, \mathrm{e}\left(-\frac{u(h+t)}{q^\lambda}\right) \right|^2 \le \sum_{0 \le h < q^\lambda} q^{-2\gamma(\lambda)} = q^{\lambda - 2\gamma(\lambda)}$$

and

$$\gamma(\lambda) \le \frac{\lambda}{2}. \tag{25}$$

4. Carry propagation lemmas

Lemma 7. *Let $\mu \ge 1$, $\nu \ge 1$, $\mu' \ge 1$ be integers with $\mu' \le \mu + \nu$. For $\mathcal{B} \subseteq \{0, \ldots, q^{\mu+\nu-\mu'} - 1\}$, the number $\mathcal{N}$ of pairs $(m,n) \in \{q^{\mu-1}, \ldots, q^\mu - 1\} \times \{q^{\nu-1}, \ldots, q^\nu - 1\}$ such that $mn = a + q^{\mu'} b$ with $0 \le a < q^{\mu'}$ and $b \in \mathcal{B}$ satisfies*

$$\mathcal{N} \le \left(q^{\mu'} \log q + q^\mu - q^{\mu-1} + q^{\mu'-\mu+1} \right) \operatorname{card} \mathcal{B}.$$

Proof. For each $m \in \{q^{\mu-1}, \ldots, q^\mu - 1\}$, the number $\mathcal{N}_m$ of n such that $mn = a + q^{\mu'} b$ with $0 \le a < q^{\mu'}$ and $b \in \mathcal{B}$ satisfies

$$\mathcal{N}_m \le \sum_{b \in \mathcal{B}} \operatorname{card}\{a : \ 0 \le a < q^{\mu'}, a + q^{\mu'} b \equiv 0 \bmod m\}.$$

This gives

$$\mathcal{N}_m \le \sum_{b \in \mathcal{B}} \left(1 + \frac{q^{\mu'}}{m}\right) = \left(1 + \frac{q^{\mu'}}{m}\right) \operatorname{card} \mathcal{B}.$$

It follows

$$\mathcal{N} = \sum_{q^{\mu-1} \le m < q^\mu} \mathcal{N}_m \le \sum_{q^{\mu-1} \le m < q^\mu} \left(1 + \frac{q^{\mu'}}{m}\right) \operatorname{card} \mathcal{B},$$

so that

$$\mathcal{N} \le (q^\mu - q^{\mu-1}) \operatorname{card} \mathcal{B} + q^{\mu'} \left(\frac{1}{q^{\mu-1}} + \int_{q^{\mu-1}}^{q^\mu} \frac{dt}{t} \right) \operatorname{card} \mathcal{B}$$

and the result follows. □

Lemma 8. *If $f : \mathbb{N} \to \mathbb{U}$ satifies Definition 1 then for $(\mu, \nu, \rho) \in \mathbb{N}^3$ with $2\rho < \nu$ the set $\mathcal{E}$ of pairs $(m,n) \in \{q^{\mu-1}, \ldots, q^\mu - 1\} \times \{q^{\nu-1}, \ldots, q^\nu - 1\}$ such that there exists $k < q^{\mu+\rho}$ with $f(mn+k)\, \overline{f(mn)} \ne f_{\mu+2\rho}(mn+k)\, \overline{f_{\mu+2\rho}(mn)}$ satisfies*

$$\operatorname{card} \mathcal{E} \ll (\log q)\, q^{\mu+\nu-\rho}. \tag{26}$$

Proof. Applying Definition 1 with $\lambda = \nu - \rho$ and $\kappa = \mu + \rho$, let $\mathcal{B}$ be the set of $\ell < q^{\nu-\rho}$ such that there exists $(k_1, k_2) \in \{0, \ldots, q^\kappa - 1\}^2$ for which (5) is true. By Definition 1 we have $\operatorname{card} \mathcal{B} \ll q^{\nu-2\rho}$. We need to count the pairs $(m,n) \in \{q^{\mu-1}, \ldots, q^\mu - 1\} \times \{q^{\nu-1}, \ldots, q^\nu - 1\}$ such that mn is of the form $mn = k_1 + q^{\mu'} \ell$ with $\ell \in \mathcal{B}$. Applying Lemma 7 with $\mu' = \mu + \rho$ we get

$$\operatorname{card} \mathcal{E} \ll \left(q^{\mu+\rho} \log q + q^\mu - q^{\mu-1} + q^{\rho+1} \right) \operatorname{card} \mathcal{B} \ll (\log q)\, q^{\mu+\nu-\rho},$$

which gives (26). □

Lemma 9. *Let $f:\mathbb{N}\to\mathbb{U}$ satisfying Definition 1 and $(\mu,\nu,\mu_0,\mu_1,\mu_2)\in\mathbb{N}^5$ with $\mu_0\le\mu_1\le\mu\le\mu_2$, $\mu\le\nu$ and $2(\mu_2-\mu)\le\mu_0$. For $(a,b,c)\in\mathbb{N}^3$ the set $\mathcal{E}(a,b,c)$ of pairs $(m,n)\in\{q^{\mu-1},\dots,q^\mu-1\}\times\{q^{\nu-1},\dots,q^\nu-1\}$ such that*

$$\begin{aligned} f_{\mu_2}(mn+am+bn+c)\ &\overline{f_{\mu_2}(q^{\mu_0}\,\mathrm{r}_{\mu_0,\mu_2}(mn+am+bn+c))}\\ \neq\ \ f_{\mu_1}(mn+am+bn+c)\ &\overline{f_{\mu_1}(q^{\mu_0}\,\mathrm{r}_{\mu_0,\mu_2}(mn+am+bn+c))}\end{aligned}$$

satisfies

$$\operatorname{card}\mathcal{E}(a,b,c)\ll\max(\tau(q),\log q)\ \mu_2^{\omega(q)}q^{\mu+\nu+\mu_0-\mu_1}. \tag{27}$$

Proof. Let $\mathcal{B}$ be the set of $\ell\in\{0,\dots,q^{\mu_2-\mu_0}-1\}$ for which there exists $(k_1,k_2)\in\{0,\dots,q^{\mu_0}-1\}^2$ with

$$f_{\mu_2}(q^{\mu_0}\ell+k_1+k_2)\ \overline{f_{\mu_2}(q^{\mu_0}\ell+k_1)}\neq f_{\mu_1}(q^{\mu_0}\ell+k_1+k_2)\ \overline{f_{\mu_1}(q^{\mu_0}\ell+k_1)}.$$

For $0\le\ell\le q^{\mu_2-\mu_0}-2$ we have $0\le q^{\mu_0}\ell+k_1+k_2\le q^{\mu_2}-2$. Therefore we have $f_{\mu_2}(q^{\mu_0}\ell+k_1+k_2)=f(q^{\mu_0}\ell+k_1+k_2)$ and $f_{\mu_2}(q^{\mu_0}\ell+k_1)=f(q^{\mu_0}\ell+k_1)$ except possibly if $\ell=q^{\mu_2-\mu_0}-1$. Since f satisfies Definition 1 it follows that $\operatorname{card}\mathcal{B}=O(q^{\mu_2-\mu_0-(\mu_1-\mu_0)})=O(q^{\mu_2-\mu_1})$. Observing for $k=mn+am+bn+c$ that $k=\mathrm{r}_{0,\mu_0}(k)+q^{\mu_0}\,\mathrm{r}_{\mu_0,\mu_2}(k)+q^{\mu_2}k'$, we notice that $\mathcal{E}(a,b,c)\subseteq\mathcal{E}'(a,b,c)$ where $\mathcal{E}'(a,b,c)$ is the set of pairs (m,n) such that $\mathrm{r}_{\mu_0,\mu_2}(mn+am+bn+c)\in\mathcal{B}$. Then we can write

$$\operatorname{card}\mathcal{E}'(a,b,c)=\sum_{\ell\in\mathcal{B}}\operatorname{card}\{(m,n),\ \mathrm{r}_{\mu_0,\mu_2}(mn+am+bn+c)=\ell\},$$

which by (10) and (11) can be written

$$\operatorname{card}\mathcal{E}'(a,b,c)=\sum_{\ell\in\mathcal{B}}\sum_m\sum_n\chi_{q^{\mu_0-\mu_2}}\left(\frac{mn+am+bn+c}{q^{\mu_2}}-\frac{\ell}{q^{\mu_2-\mu_0}}\right).$$

Using Lemma 1 it follows that for any integer $H\ge1$ there exists $a_h(q^{\mu_0-\mu_2},H)$ and $b_h(q^{\mu_0-\mu_2},H)$ satisfying (14) such that

$$\begin{aligned}\operatorname{card}\mathcal{E}'(a,b,c)\ \le\ &\sum_{\ell\in\mathcal{B}}\sum_m\sum_n\sum_{|h|\le H}a_h(q^{\mu_0-\mu_2},H)\,\mathrm{e}\left(\frac{h(mn+am+bn+c)}{q^{\mu_2}}-\frac{h\ell}{q^{\mu_2-\mu_0}}\right)\\ &+\sum_{\ell\in\mathcal{B}}\sum_m\sum_n\sum_{|h|\le H}b_h(q^{\mu_0-\mu_2},H)\,\mathrm{e}\left(\frac{h(mn+am+bn+c)}{q^{\mu_2}}-\frac{h\ell}{q^{\mu_2-\mu_0}}\right).\end{aligned}$$

Taking $H=q^{\mu_2-\mu_0}$ the contribution of the terms $h=0$ in both sums is bounded by

$$q^{\mu+\nu+\mu_0-\mu_2}\operatorname{card}\mathcal{B}\ll q^{\mu+\nu+\mu_0-\mu_1}.$$

We handle both sums over h similarly, exchanging the order of summations and using the bounds $|a_h(q^{\mu_0-\mu_2},H)|\le q^{\mu_0-\mu_2}$ and $|b_h(q^{\mu_0-\mu_2},H)|\le H^{-1}=q^{\mu_0-\mu_2}$. We obtain the upper bound

$$\operatorname{card}\mathcal{E}'(a,b,c)\ll q^{\mu+\nu+\mu_0-\mu_1}+\frac{\operatorname{card}\mathcal{B}}{q^{\mu_2-\mu_0}}\sum_{1\le|h|\le q^{\mu_2-\mu_0}}\sum_n\left|\sum_m\mathrm{e}\left(\frac{h(mn+am+bn+c)}{q^{\mu_2}}\right)\right|.$$

This gives

$$\operatorname{card}\mathcal{E}'(a,b,c)\ll q^{\mu+\nu+\mu_0-\mu_1}+\frac{q^{\mu_2-\mu_1}}{q^{\mu_2-\mu_0}}\sum_{1\le|h|\le q^{\mu_2-\mu_0}}\sum_n\min\left(q^\mu,\left|\sin\pi\frac{h(n+a)}{q^{\mu_2}}\right|^{-1}\right).$$

The summation on n runs over at most $\lceil q^{\nu-\mu_2}\rceil$ periods modulo q^{μ_2}. By (21) it follows

$$\operatorname{card}\mathcal{E}'(a,b,c)\ll q^{\mu+\nu+\mu_0-\mu_1}+q^{\mu_2-\mu_1}(q^{\nu-\mu_2}+1)\left(\tau(q^{\mu_2})q^\mu+q^{\mu_2}\log q^{\mu_2}\right).$$

By multiplicativity of the function τ we have $\tau(q^{\mu_2})\le\tau(q)\mu_2^{\omega(q)}$ and we obtain

$$\operatorname{card}\mathcal{E}'(a,b,c)\ll q^{\mu+\nu+\mu_0-\mu_1}\left(1+q^{-\mu_0}(1+q^{\mu_2-\nu})\left(\tau(q)\mu_2^{\omega(q)}+q^{\mu_2-\mu}\log q^{\mu_2}\right)\right)$$

and using $\mu_1 \le \mu \le \nu$ we may replace μ and ν by μ_1 in the parentheses and this leads to

$$\operatorname{card} \mathcal{E}'(a,b,c) \ll q^{\mu+\nu+\mu_0-\mu_1} \left(1 + \mu_2^{\omega(q)} \max(\tau(q), \log q) q^{2\mu_2-2\mu-\mu_0}\right)$$

which, using the hypothesis $2(\mu_2 - \mu) \le \mu_0$, gives (27). □

5. Sums of type I

We take a non decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{\lambda\to+\infty} \gamma(\lambda) = +\infty$, $c \ge 2$ and $f : \mathbb{N} \to \mathbb{U}$ be a function satisfying Definition 1 and $f \in \mathcal{F}_{\gamma,c}$ in Definition 2. Let

$$1 \le M \le N \text{ such that } M \le (MN)^{1/3}. \tag{28}$$

Let μ and ν be the unique integers such that

$$q^{\mu-1} \le M < q^{\mu} \quad \text{and} \quad q^{\nu-1} \le N < q^{\nu}.$$

Let $\vartheta \in \mathbb{R}$, an interval $I(M,N) \subseteq [0, MN]$ and

$$S_I(\vartheta) = \sum_{\frac{M}{q} < m \le M} \left| \sum_{\substack{n \\ mn \in I(M,N)}} f(mn)\, \mathrm{e}(\vartheta mn) \right|.$$

Proposition 1. *Assuming* (28) *and with* $c \ge 2$, *we have uniformly for* $\vartheta \in \mathbb{R}$

$$S_I(\vartheta) \ll (\log q)^{5/2} (\mu+\nu)^2 \; q^{\mu+\nu-\frac{1}{2}\gamma(\frac{\mu+\nu}{3})}. \tag{29}$$

Proof. Let $0 \le \ell < q^{\mu+\nu}$. For $\frac{M}{q} < m \le M$, we have $\ell = mn$ with $mn \in I(M,N)$ if and only if $\ell \in I(M,N)$ and $\ell \equiv 0 \bmod m$. Therefore the inner sum (over n) in $S_I(\vartheta)$ is

$$\sum_{0 \le \ell < q^{\mu+\nu}} f(\ell)\, \mathrm{e}(\vartheta \ell) \sum_{u \in I(M,N)} \frac{1}{q^{\mu+\nu}} \sum_{0 \le h < q^{\mu+\nu}} \mathrm{e}\left(\frac{h(u-\ell)}{q^{\mu+\nu}}\right) \frac{1}{m} \sum_{0 \le k < m} \mathrm{e}\left(\frac{k\ell}{m}\right)$$

i.e.

$$\sum_{0 \le h < q^{\mu+\nu}} \left(\sum_{u \in I(M,N)} \mathrm{e}\left(\frac{hu}{q^{\mu+\nu}}\right)\right) \frac{1}{m} \sum_{0 \le k < m} \frac{1}{q^{\mu+\nu}} \sum_{0 \le \ell < q^{\mu+\nu}} f(\ell)\, \mathrm{e}\left(\vartheta \ell - \frac{h\ell}{q^{\mu+\nu}} + \frac{k\ell}{m}\right).$$

This gives

$$S_I(\vartheta) \le \sum_{0 \le h < q^{\mu+\nu}} \min\left(q^{\mu+\nu}, \left|\sin \tfrac{\pi h}{q^{\mu+\nu}}\right|^{-1}\right) S'_I\left(h - \vartheta q^{\mu+\nu}\right),$$

where

$$S'_I(\vartheta') = \sum_{\frac{M}{q} < m \le M} \frac{1}{m} \sum_{0 \le k < m} \left| \widehat{f_{\mu+\nu}}\left(\vartheta' - \tfrac{k}{m} q^{\mu+\nu}\right)\right|. \tag{30}$$

Then we have uniformly for $\vartheta \in \mathbb{R}$

$$S_I(\vartheta) \le \left(\max_{\vartheta' \in \mathbb{R}} S'_I(\vartheta')\right) \sum_{0 \le h < q^{\mu+\nu}} \min\left(q^{\mu+\nu}, \left|\sin \tfrac{\pi h}{q^{\mu+\nu}}\right|^{-1}\right),$$

which gives

$$S_I(\vartheta) \ll \left(\max_{\vartheta' \in \mathbb{R}} S'_I(\vartheta')\right) q^{\mu+\nu} \log q^{\mu+\nu}. \tag{31}$$

It remains to estimate $S'_I(\vartheta')$ uniformly for $\vartheta' \in \mathbb{R}$. For any κ such that

$$1 \le \kappa \le \tfrac{2}{3}(\mu+\nu), \tag{32}$$

by (23) we can write

$$\widehat{f_{\mu+\nu}}(t) = \frac{1}{q^{\mu+\nu}} \sum_{0\le u<q^\kappa} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(u+vq^\kappa)\,\mathrm{e}\left(-\frac{(u+vq^\kappa)t}{q^{\mu+\nu}}\right).$$

This gives

$$\begin{aligned}\widehat{f_{\mu+\nu}}(t) &= \frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^\kappa)\,\mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}}\right)\\ &\qquad \frac{1}{q^\kappa}\sum_{0\le u<q^\kappa} f(u+vq^\kappa)\overline{f(vq^\kappa)}\,\mathrm{e}\left(-\frac{ut}{q^{\mu+\nu}}\right).\end{aligned}$$

Given ρ_1 such that

$$1 \le \rho_1 \le \mu+\nu-\kappa, \tag{33}$$

by Definition 1 the number of $v\in\{0,\dots,q^{\mu+\nu-\kappa}-1\}$ such that there exists $u\in\{0,\dots,q^\kappa-1\}$ for which

$$f(u+vq^\kappa)\overline{f(vq^\kappa)} \neq f_{\kappa+\rho_1}(u+vq^\kappa)\overline{f_{\kappa+\rho_1}(vq^\kappa)}$$

is at most $O(q^{\mu+\nu-\kappa-\rho_1})$. Hence the set $\widetilde{\mathcal{W}}_\kappa$ of pairs (u,v) with this property satisfies

$$\operatorname{card}\widetilde{\mathcal{W}}_\kappa \ll q^{\mu+\nu-\rho_1}. \tag{34}$$

Therefore for all $t\in\mathbb{R}$, all κ satisfying (32), all ρ_1 satisfying (33) we have

$$\widehat{f_{\mu+\nu}}(t) = G_{\kappa,1}(t) + G_{\kappa,2}(t), \tag{35}$$

with

$$\begin{aligned}G_{\kappa,1}(t) &= \frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^\kappa)\,\mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}}\right)\\ &\qquad \frac{1}{q^\kappa}\sum_{0\le u<q^\kappa} f_{\kappa+\rho_1}(u+vq^\kappa)\overline{f_{\kappa+\rho_1}(vq^\kappa)}\,\mathrm{e}\left(-\frac{ut}{q^{\mu+\nu}}\right)\end{aligned}$$

and

$$\begin{aligned}G_{\kappa,2}(t) &= \frac{1}{q^{\mu+\nu}} \sum_{(u,v)\in\widetilde{\mathcal{W}}_\kappa} f(vq^\kappa)\,\mathrm{e}\left(-\frac{(u+vq^\kappa)t}{q^{\mu+\nu}}\right)\\ &\qquad \left(f(u+vq^\kappa)\overline{f(vq^\kappa)} - f_{\kappa+\rho_1}(u+vq^\kappa)\overline{f_{\kappa+\rho_1}(vq^\kappa)}\right).\end{aligned}$$

Let us introduce in $G_{\kappa,1}(t)$ the residue w of v mod q^{ρ_1} in order to make the variables u and v independent:

$$\begin{aligned}G_{\kappa,1}(t) &= \sum_{0\le w<q^{\rho_1}} \frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^\kappa)\,\mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}}\right) \frac{1}{q^{\rho_1}}\sum_{0\le h<q^{\rho_1}} \mathrm{e}\left(h\frac{v-w}{q^{\rho_1}}\right)\\ &\qquad \frac{1}{q^\kappa}\sum_{0\le u<q^\kappa} f_{\kappa+\rho_1}(u+wq^\kappa)\overline{f_{\kappa+\rho_1}(wq^\kappa)}\,\mathrm{e}\left(-\frac{ut}{q^{\mu+\nu}}\right).\end{aligned}$$

Writing

$$c_{\kappa,\rho_1}(u,h) = \frac{1}{q^{\rho_1}} \sum_{0\le w<q^{\rho_1}} f_{\kappa+\rho_1}(u+wq^\kappa)\overline{f_{\kappa+\rho_1}(wq^\kappa)}\,\mathrm{e}\left(-\frac{hw}{q^{\rho_1}}\right), \tag{36}$$

this leads to

$$G_{\kappa,1}(t) = \sum_{0\le h<q^{\rho_1}} \left(\frac{1}{q^\kappa} \sum_{0\le u<q^\kappa} c_{\kappa,\rho_1}(u,h)\, \mathrm{e}\left(\frac{-ut}{q^{\mu+\nu}}\right)\right) \left(\frac{1}{q^{\mu+\nu-\kappa}} \sum_{0\le v<q^{\mu+\nu-\kappa}} f(vq^\kappa)\, \mathrm{e}\left(-\frac{vt}{q^{\mu+\nu-\kappa}} + \frac{hv}{q^{\rho_1}}\right)\right).$$

By (32) we have $\kappa \le 2(\mu+\nu-\kappa)$ and we may use (6) (with $c\ge 2$) for the sum over v with $\kappa=\kappa$ and $\lambda=\mu+\nu-\kappa$. We get

$$|G_{\kappa,1}(t)| \ll q^{-\gamma(\mu+\nu-\kappa)} \sum_{0\le h<q^{\rho_1}} \left| \frac{1}{q^\kappa} \sum_{0\le u<q^\kappa} c_{\kappa,\rho_1}(u,h)\, \mathrm{e}\left(\frac{-ut}{q^{\mu+\nu}}\right)\right|.$$

From (30) we can write

$$S_I'(\vartheta') \le \sum_{1\le d\le M} \sum_{\frac{M}{q}<m\le M} \frac{1}{m} \sum_{\substack{0\le k<m\\ (k,m)=d}} \left| \widehat{f_{\mu+\nu}}\left(\vartheta' - \tfrac{k}{m} q^{\mu+\nu}\right)\right|.$$

In order to estimate $S_I'(\vartheta')$, for each $1\le d\le M$, we will use (35) with κ_d defined to be the unique integer such that

$$q^{\kappa_d-1} < M^2/d^2 \le q^{\kappa_d}. \tag{37}$$

Hence

$$S_I'(\vartheta') \le S_{I,1}'(\vartheta') + S_{I,2}'(\vartheta'),$$

with

$$S_{I,1}'(\vartheta') = \sum_{1\le d\le M} \sum_{\frac{M}{q}<m\le M} \frac{1}{m} \sum_{\substack{0\le k<m\\ (k,m)=d}} \left| G_{\kappa_d,1}\left(\vartheta' - \tfrac{k}{m} q^{\mu+\nu}\right)\right|$$

and

$$S_{I,2}'(\vartheta') = \sum_{1\le d\le M} \sum_{\frac{M}{q}<m\le M} \frac{1}{m} \sum_{\substack{0\le k<m\\ (k,m)=d}} \left| G_{\kappa_d,2}\left(\vartheta' - \tfrac{k}{m} q^{\mu+\nu}\right)\right|.$$

Then

$$\left| G_{\kappa_d,1}\left(\vartheta' - \frac{k}{m} q^{\mu+\nu}\right)\right| \le q^{-\gamma(\mu+\nu-\kappa_d)} \sum_{0\le h<q^{\rho_1}} \frac{1}{q^{\kappa_d}} \left| \sum_{0\le u<q^{\kappa_d}} c_{\kappa_d,\rho_1}(u,h)\, \mathrm{e}\left(-\frac{u\vartheta'}{q^{\mu+\nu}} + \frac{uk}{m}\right)\right|.$$

Since κ_d is decreasing with d, by (37) and (28) we can check that (32) is satisfied:

$$1 \le \kappa_d \le \kappa_1 \le 2\mu \le \frac{2}{3}(\mu+\nu). \tag{38}$$

But γ is non decreasing, which implies

$$S_{I,1}'(\vartheta') \le q^{-\gamma(\frac{\mu+\nu}{3})} \sum_{1\le d\le M} \frac{S_{I,1}''(M,d)}{d\; q^{\kappa_d}}, \tag{39}$$

where

$$S_{I,1}''(M,d) = \sum_{0\le h<q^{\rho_1}} \sum_{\frac{M}{qd}<m'\le \frac{M}{d}} \frac{1}{m'} \sum_{\substack{0\le k'<m'\\ (k',m')=1}} \left| \sum_{0\le u<q^{\kappa_d}} c_{\kappa_d,\rho_1}(u,h)\, \mathrm{e}\left(-\frac{u\vartheta'}{q^{\mu+\nu}} + \frac{uk'}{m'}\right)\right|.$$

Since

$$\sum_{\frac{M}{qd}<m'\le \frac{M}{d}} \sum_{\substack{0\le k'<m'\\ (k',m')=1}} \frac{1}{m'^2} \le \sum_{\frac{M}{qd}<m'\le \frac{M}{d}} \frac{1}{m'} \ll \log q, \tag{40}$$

by the Cauchy-Schwarz inequality we get

$$\left|S''_{I,1}(M,d)\right|^2 \ll (\log q)\ q^{\rho_1} \sum_{0\le h<q^{\rho_1}} \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \sum_{\substack{0\le k'<m'\\(k',m')=1}} \left|\sum_{0\le u<q^{\kappa_d}} c_{\kappa_d,\rho_1}(u,h)\,\mathrm{e}\left(-\frac{u\vartheta'}{q^{\mu+\nu}}+\frac{uk'}{m'}\right)\right|^2.$$

By (22) it follows that

$$\left|S''_{I,1}(M,d)\right|^2 \ll (\log q)q^{\rho_1} \sum_{0\le h<q^{\rho_1}} \left(q^{\kappa_d}+\tfrac{M^2}{d^2}\right) \sum_{0\le u<q^{\kappa_d}} \left|c_{\kappa_d,\rho_1}(u,h)\right|^2.$$

But by (36) and (24) we have

$$\sum_{0\le h<q^{\rho_1}} \left|c_{\kappa_d,\rho_1}(u,h)\right|^2 = 1,$$

hence summing over u and using (37) we obtain

$$\left|S''_{I,1}(M,d)\right| \ll (\log q)^{1/2} q^{\kappa_d+\frac{\rho_1}{2}},$$

which lead by (39) to

$$S'_{I,1}(\vartheta') \ll (\log q)^{1/2} q^{-\gamma(\frac{\mu+\nu}{3})} \sum_{1\le d\le M} \frac{q^{\frac{\rho_1}{2}}}{d} \ll \mu\ (\log q)^{3/2} q^{\frac{\rho_1}{2}-\gamma(\frac{\mu+\nu}{3})}. \tag{41}$$

In order to estimate $S'_{I,2}(\vartheta')$ we denote by $\mathcal{W}_{\kappa_d}$ the set of integers $w=u+vq^{\kappa_d}$ such that $(u,v)\in\widetilde{\mathcal{W}}_{\kappa_d}$. Using the bijective correspondence between $\mathcal{W}_{\kappa_d}$ and $\widetilde{\mathcal{W}}_{\kappa_d}$ given by $w\mapsto(\mathrm{r}_{\kappa_d}(w),\mathrm{r}_{\kappa_d,\mu+\nu}(w))$ we can write

$$G_{\kappa_d,2}(t) = \frac{1}{q^{\mu+\nu}} \sum_{0\le w<q^{\mu+\nu}} c'_{\kappa_d,\rho_1}(w)\,\mathrm{e}\left(-\frac{wt}{q^{\mu+\nu}}\right),$$

where

$$c'_{\kappa_d,\rho_1}(w) = f(q^{\kappa_d}\,\mathrm{r}_{\kappa_d,\mu+\nu}(w))\left(f(w)\overline{f(q^{\kappa_d}\,\mathrm{r}_{\kappa_d,\mu+\nu}(w))} - f_{\kappa_d+\rho_1}(w)\overline{f_{\kappa_d+\rho_1}(q^{\kappa_d}\,\mathrm{r}_{\kappa_d,\mu+\nu}(w))}\right)$$

satisfies $\left|c'_{\kappa_d,\rho_1}(w)\right|\le 2$ for $0\le w<q^{\mu+\nu}$ and $c'_{\kappa_d,\rho_1}(w)=0$ for $w\notin\mathcal{W}_{\kappa_d}$. Then we write

$$S'_{I,2}(\vartheta') \le \sum_{1\le d\le M} \frac{S''_{I,2}(M,d)}{d\ q^{\mu+\nu}}, \tag{42}$$

where

$$S''_{I,2}(M,d) = \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \frac{1}{m'} \sum_{\substack{0\le k'<m'\\(k',m')=1}} \left|\sum_{0\le w<q^{\mu+\nu}} c'_{\kappa_d,\rho_1}(w)\,\mathrm{e}\left(-\frac{w\vartheta'}{q^{\mu+\nu}}+\frac{wk'}{m'}\right)\right|.$$

It follows from the Cauchy-Schwarz inequality and (40) that

$$\left|S''_{I,2}(M,d)\right|^2 \ll (\log q) \sum_{\frac{M}{qd}<m'\le\frac{M}{d}} \sum_{\substack{0\le k'<m'\\(k',m')=1}} \left|\sum_{0\le w<q^{\mu+\nu}} c'_{\kappa_d,\rho_1}(w)\,\mathrm{e}\left(-\frac{w\vartheta'}{q^{\mu+\nu}}+\frac{wk'}{m'}\right)\right|^2.$$

By (22) we get

$$\begin{aligned}\left|S''_{I,2}(M,d)\right|^2 &\ll (\log q)\left(q^{\mu+\nu}+\tfrac{M^2}{d^2}\right) \sum_{0\le w<q^{\mu+\nu}} \left|c'_{\kappa_d,\rho_1}(w)\right|^2\\ &\ll (\log q)\left(q^{\mu+\nu}+\tfrac{M^2}{d^2}\right) \sum_{w\in\mathcal{W}_{\kappa_d}} 2^2.\end{aligned}$$

By (34) it follows that

$$\left|S''_{I,2}(M,d)\right| \ll (\log q)^{1/2} q^{\mu+\nu-\frac{\rho_1}{2}},$$

which lead by (39) to

$$S'_{I,2}(\vartheta') \ll (\log q)^{1/2} \sum_{1\leq d\leq M} \frac{q^{-\rho_1/2}}{d} \ll \mu\, (\log q)^{3/2} q^{-\rho_1/2}. \tag{43}$$

Taking

$$\rho_1 = \gamma\left(\frac{\mu+\nu}{3}\right), \tag{44}$$

by (25) we have $\rho_1 \leq \frac{\mu+\nu}{6}$, so that by (38) ρ_1 satisfies (33). By (31), (41) and (43) it follows that uniformly for $\vartheta \in \mathbb{R}$ we get (29). □

6. Sums of type II

We take $\gamma : \mathbb{R} \to \mathbb{R}$ a non decreasing function satisfying $\lim_{\lambda\to+\infty} \gamma(\lambda) = +\infty$, $c \geq 10$ (this condition appears in (95)) and $f : \mathbb{N} \to \mathbb{U}$ a function satisfying Definition 1 and $f \in \mathcal{F}_{\gamma,c}$ in Definition 2. Let $1 \leq M \leq N$. We denote by μ and ν the unique integers such that

$$q^{\mu-1} \leq M < q^{\mu} \quad \text{and} \quad q^{\nu-1} \leq N < q^{\nu}.$$

Let us assume that

$$\tfrac{1}{4}(\mu+\nu) \leq \mu \leq \nu \leq \tfrac{3}{4}(\mu+\nu) \tag{45}$$

(replacing $(1/4, 3/4)$ by $(\xi, 1-\xi)$ with $1/4 < \xi < 1/3$ would provide a better exponent for q in (46)). We assume also that the multiplicative dependence of the variables in the type II sums has been removed by the classical method described (for example) in section 5 of [20]. Let $\vartheta \in \mathbb{R}$, $a_m \in \mathbb{C}$, $b_n \in \mathbb{C}$ with $|a_m| \leq 1$, $|b_n| \leq 1$ and

$$S_{II}(\vartheta) = \sum_m \sum_n a_m b_n f(mn)\, \mathrm{e}(\vartheta mn)$$

where we sum over $m \in (M/q, M]$ and $n \in (N/q, N]$. We will prove

Proposition 2. *Assuming* (45) *and* $c \geq 10$, *uniformly for* $|a_m| \leq 1$, $|b_n| \leq 1$ *and* $\vartheta \in \mathbb{R}$, *we have*

$$|S_{II}(\vartheta)| \ll \max(\tau(q)\log q, \log^3 q)^{1/4} (\mu+\nu)^{\frac{1}{4}(1+\max(\omega(q),2))} q^{\mu+\nu-\gamma(2\lfloor \mu/15\rfloor)/20}. \tag{46}$$

As often in this approach, the proof of this result is the most difficult part. The proof is quite long and complicated and will be developped over several sections and completed at formula (96). By the Cauchy-Schwarz inequality

$$|S_{II}(\vartheta)|^2 \leq M \sum_m \left| \sum_n b_n f(mn)\, \mathrm{e}(\vartheta mn)\right|^2.$$

Let ρ be an integer such that

$$1 \leq 7\rho \leq \mu \tag{47}$$

and let

$$R = q^{\rho} \tag{48}$$

so that

$$1 \leq R \ll N. \tag{49}$$

Applying Lemma 3 to the summation over n with $k = 1$ and then summing over m we get

$$|S_{II}(\vartheta)|^2 \ll \frac{M^2N^2}{R} + \frac{MN}{R} \sum_{1\leq r<R} \left(1 - \frac{r}{R}\right) \Re(S_1(r)),$$

with

$$S_1(r) = \sum_m \sum_{n\in I(N,r)} b_{n+r}\overline{b_n} f(mn+mr)\overline{f(mn)}\, \mathrm{e}(\vartheta mr),$$

where $I(N,r) = (N/q, N-r]$. Let

$$\mu_2 = \mu + 2\rho. \tag{50}$$

If f satisfies the carry property explained in Definition 1, then by Lemma 8 the number of pairs (m,n) for which $f(mn+mr)\overline{f(mn)} \neq f_{\mu_2}(mn+mr)\overline{f_{\mu_2}(mn)}$ is $O(q^{\mu+\nu-\rho})$. Hence

$$S_1(r) = S_1'(r) + O(q^{\mu+\nu-\rho}),$$

where

$$S_1'(r) = \sum_m \sum_{n\in I(N,r)} b_{n+r}\overline{b_n} f_{\mu_2}(mn+mr)\overline{f_{\mu_2}(mn)}\, \mathrm{e}(\vartheta mr).$$

Using again the Cauchy-Schwarz inequality for the summation over r, this leads to

$$\left|S_{II}(\vartheta)\right|^4 \ll \frac{M^4N^4}{R^2} + \frac{M^2N^2}{R^2} R \sum_{1\leq r<R} \left|S_1'(r)\right|^2. \tag{51}$$

It remains to give an upper bound for $\left|S_1'(r)\right|^2$. We reverse the order of summation in $S_1'(r)$ and obtain:

$$\left|S_1'(r)\right| \leq \sum_{n\in I(N,r)} \left| \sum_m f_{\mu_2}(mn+mr)\overline{f_{\mu_2}(mn)}\, \mathrm{e}(\vartheta mr) \right|.$$

We may extend the summation over n to $(N/q, N]$ and apply the Cauchy-Schwarz inequality:

$$\left|S_1'(r)\right|^2 \ll N \sum_{N/Q<n\leq N} \left| \sum_m f_{\mu_2}(mn+mr)\overline{f_{\mu_2}(mn)}\, \mathrm{e}(\vartheta mr) \right|^2.$$

Applying to the summation over m the Lemma 3 with positive integers $k = q^{\mu_1}$ and S such that

$$1 \leq q^{\mu_1} S \ll M \tag{52}$$

and then summing over n and r we get

$$\sum_{1\leq r<R} \left|S_1'(r)\right|^2 \ll \frac{M^2N^2R}{S} + \frac{MN}{S}\Re(S_2), \tag{53}$$

with

$$S_2 = \sum_{1\leq r<R} \sum_{1\leq s<S} \left(1 - \frac{s}{S}\right)\ \mathrm{e}(\vartheta q^{\mu_1} rs)\ S_2'(r,s)$$

and

$$S_2'(r,s) = \sum_m \sum_n f_{\mu_2}((m+sq^{\mu_1})(n+r))\overline{f_{\mu_2}(m(n+r)) f_{\mu_2}((m+sq^{\mu_1})n)} f_{\mu_2}(mn).$$

Using (51) and (53) we obtain uniformly for $\vartheta \in \mathbb{R}$:

$$\left|S_{II}(\vartheta)\right|^4 \ll \frac{M^4N^4}{R^2} + \frac{M^4N^4}{S} + \frac{M^3N^3}{RS} \sum_{1\leq r<R} \sum_{1\leq s<S} \left|S_2'(r,s)\right|. \tag{54}$$

Writing $f_{\mu_2} = f_{\mu_1}(f_{\mu_2}\overline{f_{\mu_1}})$ and observing that $f_{\mu_1}((m+sq^{\mu_1})(n+r)) = f_{\mu_1}(m(n+r))$ and $f_{\mu_1}((m+sq^{\mu_1})n) = f_{\mu_1}(mn))$ we get

$$S_2'(r,s) = \sum_m \sum_n f_{\mu_1,\mu_2}(mn+mr+q^{\mu_1}sn+q^{\mu_1}rs)\overline{f_{\mu_1,\mu_2}(mn+mr) f_{\mu_1,\mu_2}(mn+q^{\mu_1}sn)} f_{\mu_1,\mu_2}(mn),$$

with

$$f_{\mu_1,\mu_2} = f_{\mu_2}\overline{f_{\mu_1}}. \tag{55}$$

We take

$$\mu_1 = \mu - 2\rho, \tag{56}$$

and

$$S = R^2 = q^{2\rho}, \tag{57}$$

so that the condition (52) is fullfilled.

For $0 \le r < R$ and $0 \le s < S$ and $\mu_0 \le \mu_1$ let us denote by $\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s)$ the set of pairs (m,n) with $M/q < m \le M$ and $N/q < n \le N$ such that

$$f_{\mu_1,\mu_2}(mn + q^{\mu_1} sn + q^{\mu_1} rs) \neq f_{\mu_1,\mu_2}(q^{\mu_0}\, \mathrm{r}_{\mu_0,\mu_2}(mn + q^{\mu_1} sn + q^{\mu_1} rs)).$$

The set $\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s)$ is a set of exceptions: if μ_0 is taken sufficiently small, the function f_{μ_1,μ_2} will depend on the digits of index in $\mu_0, \ldots, \mu_2 - 1$, except for $(m,n) \in \mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s)$. Of course if $\mu_0 = 0$ we have $\mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s) = \emptyset$ but we want to choose μ_0 more carefully so that this set is still small enough. More precisely, let $\rho' \in \mathbb{N}$ to be chosen later such that

$$0 \le \rho' \le \rho. \tag{58}$$

Since $f : \mathbb{N} \to \mathbb{U}$ is a function satisfying Definition 1, we have by taking

$$\mu_0 = \mu_1 - 2\rho' \tag{59}$$

in Lemma 9:

$$\operatorname{card} \mathcal{E}_{\mu_0,\mu_1,\mu_2}(r,s) \ll \max(\tau(q), \log q)\ (\mu+\nu)^{\omega(q)}\ q^{\mu+\nu-2\rho'}. \tag{60}$$

Remark 2. *A direct argument depending on a better knowledge of f might permit to choose a greater value of μ_0, leading to a sharper final estimate for such a more specific function f.*

For $k \in \mathbb{Z}$ we define a $q^{\mu_2-\mu_0}$-periodic function g by

$$g(k) = f_{\mu_1,\mu_2}(q^{\mu_0} k). \tag{61}$$

Let us put $\mathrm{r}_{\mu_0,\mu_2}(mn) = u_0$ so that

$$\mathrm{r}_{\mu_0,\mu_2}(mn + q^{\mu_1} sn) = \mathrm{r}_{\mu_0,\mu_2}\left(q^{\mu_0} u_0 + q^{\mu_1} sn\right) = \mathrm{r}_{\mu_2-\mu_0}(u_0 + q^{\mu_1-\mu_0} sn)$$

and

$$f_{\mu_1,\mu_2}(q^{\mu_0}\, \mathrm{r}_{\mu_0,\mu_2}(mn + q^{\mu_1} sn)) = g(u_0 + q^{\mu_1-\mu_0} sn).$$

Similarly if we put $\mathrm{r}_{\mu_0,\mu_2}(mn + mr) = u_1$ then we have

$$\begin{aligned} \mathrm{r}_{\mu_0,\mu_2}(mn + mr + q^{\mu_1} sn + q^{\mu_1} sr) &= \mathrm{r}_{\mu_0,\mu_2}\left(q^{\mu_0} u_1 + q^{\mu_1} sn + q^{\mu_1} sr\right) \\ &= \mathrm{r}_{\mu_2-\mu_0}(u_1 + q^{\mu_1-\mu_0} sn + q^{\mu_1-\mu_0} sr) \end{aligned}$$

and

$$f_{\mu_1,\mu_2}(q^{\mu_0}\, \mathrm{r}_{\mu_0,\mu_2}(mn + mr + q^{\mu_1} sn + q^{\mu_1} sr)) = g(u_1 + q^{\mu_1-\mu_0} sn + q^{\mu_1-\mu_0} sr).$$

Using (60) and (10), we can write

$$S_2'(r,s) = S_3(r,s) + O(\max(\tau(q), \log q)\ (\mu+\nu)^{\omega(q)}\ q^{\mu+\nu-2\rho'}), \tag{62}$$

where

$$\begin{aligned} S_3(r,s) = \sum_m \sum_n \sum_{\substack{0 \le u_0 < q^{\mu_2-\mu_0} \\ 0 \le u_1 < q^{\mu_2-\mu_0}}} &\chi_{q^{\mu_0-\mu_2}}\left(\frac{mn}{q^{\mu_2}} - \frac{u_0}{q^{\mu_2-\mu_0}}\right) \chi_{q^{\mu_0-\mu_2}}\left(\frac{mn+mr}{q^{\mu_2}} - \frac{u_1}{q^{\mu_2-\mu_0}}\right) \\ &g(u_1 + q^{\mu_1-\mu_0} sn + q^{\mu_1-\mu_0} sr)\ \overline{g}(u_1)\ \overline{g}(u_0 + q^{\mu_1-\mu_0} sn)\ g(u_0), \end{aligned}$$

with $\chi_{q^{\mu_0-\mu_2}}$ defined by (11) and $\alpha = q^{\mu_0-\mu_2}$. Let H be an integer with $q^{\mu_2-\mu_0} \le H \le q^{\mu}$ to be chosen later. Using (17) with $\alpha_1 = \alpha_2 = q^{\mu_0-\mu_2}$ we have

$$S_3(r,s) = S_4(r,s) + O(E_4(r,0)) + O(E_4(0,r')) + O(E_4'(r)), \tag{63}$$

where

$$S_4(r,s) = \sum_m \sum_n \sum_{\substack{0\le u_0<q^{\mu_2-\mu_0}\\ 0\le u_1<q^{\mu_2-\mu_0}}} g(u_1+q^{\mu_1-\mu_0}sn+q^{\mu_1-\mu_0}sr)\ \overline{g}(u_1)\overline{g}(u_0+q^{\mu_1-\mu_0}sn)\ g(u_0)$$
$$A_{q^{\mu_0-\mu_2},H}\left(\frac{mn}{q^{\mu_2}}-\frac{u_0}{q^{\mu_2-\mu_0}}\right) A_{q^{\mu_0-\mu_2},H}\left(\frac{mn+mr}{q^{\mu_2}}-\frac{u_1}{q^{\mu_2-\mu_0}}\right),$$

$$E_4(r,r') = \sum_m \sum_n \sum_{\substack{0\le u_0<q^{\mu_2-\mu_0}\\ 0\le u_1<q^{\mu_2-\mu_0}}} B_{q^{\mu_0-\mu_2},H}\left(\frac{mn+mr}{q^{\mu_2}}-\frac{u_0}{q^{\mu_2-\mu_0}}\right) \chi_{q^{\mu_0-\mu_2}}\left(\frac{mn+mr'}{q^{\mu_2}}-\frac{u_1}{q^{\mu_2-\mu_0}}\right),$$

and

$$E_4'(r) = \sum_m \sum_n \sum_{\substack{0\le u_0<q^{\mu_2-\mu_0}\\ 0\le u_1<q^{\mu_2-\mu_0}}} B_{q^{\mu_0-\mu_2},H}\left(\frac{mn}{q^{\mu_2}}-\frac{u_0}{q^{\mu_2-\mu_0}}\right) B_{q^{\mu_0-\mu_2},H}\left(\frac{mn+mr}{q^{\mu_2}}-\frac{u_1}{q^{\mu_2-\mu_0}}\right).$$

6.1. **Estimate of $E_4(r,r')$.** Since

$$\sum_{0\le u_1<q^{\mu_2-\mu_0}} \chi_{q^{\mu_0-\mu_2}}\left(\tfrac{mn+mr'}{q^{\mu_2}}-\tfrac{u_1}{q^{\mu_2-\mu_0}}\right)=1,$$

we have

$$E_4(r,r') = \sum_m \sum_n \sum_{0\le u_0<q^{\mu_2-\mu_0}} B_{q^{\mu_0-\mu_2},H}\left(\tfrac{mn+mr}{q^{\mu_2}}-\tfrac{u_0}{q^{\mu_2-\mu_0}}\right),$$

which by (13) gives

$$E_4(r,r') = \sum_{|h_0|\le H} b_{h_0}(q^{\mu_0-\mu_2},H) \sum_m \sum_n \sum_{0\le u_0<q^{\mu_2-\mu_0}} \mathrm{e}\left(h_0\tfrac{mn+mr}{q^{\mu_2}}-h_0\tfrac{u_0}{q^{\mu_2-\mu_0}}\right).$$

By (14) we have $|b_{h_0}(q^{\mu_0-\mu_2},H)|\le \frac{1}{H}$. For $h_0\not\equiv 0 \bmod q^{\mu_2-\mu_0}$ we have $\sum_{0\le u_0<q^{\mu_2-\mu_0}} \mathrm{e}\left(\frac{-h_0u_0}{q^{\mu_2-\mu_0}}\right)=0$, while for $h_0\equiv 0 \bmod q^{\mu_2-\mu_0}$ this sum is equal to $q^{\mu_2-\mu_0}$. Hence writing $h_0=h_0'q^{\mu_2-\mu_0}$ we get

(64) $$|E_4(r,r')| \ll E_5,$$

with

(65) $$E_5 = \frac{q^{\mu_2-\mu_0}}{H} \sum_{|h_0'|\le H/q^{\mu_2-\mu_0}} \sum_m \left|\sum_n \mathrm{e}\left(\frac{h_0'mn}{q^{\mu_0}}\right)\right|.$$

After summation over n, we have

$$E_5 \ll \frac{q^{\mu_2-\mu_0}}{H} \sum_{|h'|\le H/q^{\mu_2-\mu_0}} \sum_m \min\left(N,\left|\sin\pi\tfrac{h'm}{q^{\mu_0}}\right|^{-1}\right).$$

The summation over m runs over at most $q^{\mu-\mu_0}$ periods q^{μ_0}, hence

$$E_5 \ll q^{\mu-\mu_0}\frac{q^{\mu_2-\mu_0}}{H} \sum_{|h'|\le H/q^{\mu_2-\mu_0}} \sum_{0\le m'<q^{\mu_0}} \min\left(N,\left|\sin\pi\tfrac{h'm'}{q^{\mu_0}}\right|^{-1}\right).$$

Using the trivial estimate for $h'=0$ and (21) when $h'\neq 0$, we obtain

$$E_5 \ll q^{\mu+\nu}\frac{q^{\mu_2-\mu_0}}{H} + q^{\mu-\mu_0}\left(\tau(q^{\mu_0})N + q^{\mu_0}\log q^{\mu_0}\right).$$

Choosing

(66) $$H = q^{\mu_2-\mu_0+2\rho},$$

this gives

$$E_5 \ll q^{\mu+\nu-2\rho} + q^{\mu+\nu-\mu_0}\tau(q^{\mu_0}) + q^{\mu}\log q^{\mu_0}.$$

By (47), (59) and (45), we have $\mu_0 \geq \mu - 4\rho \geq 2\rho$ and $\nu \geq 2\rho$ so that

$$E_5 \ll \max(\log q^{\mu_0}, \tau(q^{\mu_0}))\ q^{\mu+\nu-2\rho}. \tag{67}$$

6.2. **Estimate of $E_4'(r)$.** We have

$$\begin{aligned} E_4'(r) \;&=\; \sum_{|h_0|\leq H} \sum_{|h_1|\leq H} b_{h_0}(q^{\mu_0-\mu_2}, H) b_{h_1}(q^{\mu_0-\mu_2}, H) \sum_m \sum_n \\ &\sum_{\substack{0\leq u_0<q^{\mu_2-\mu_0}\\ 0\leq u_1<q^{\mu_2-\mu_0}}} \mathrm{e}\left(h_0 \frac{mn}{q^{\mu_2}} - h_0 \frac{u_0}{q^{\mu_2-\mu_0}}\right) \mathrm{e}\left(h_1 \frac{mn+mr}{q^{\mu_2}} - h_1 \frac{u_1}{q^{\mu_2-\mu_0}}\right). \end{aligned}$$

We observe that for $h_0 \not\equiv 0 \bmod q^{\mu_2-\mu_0}$ we have $\sum_{0\leq u_0<q^{\mu_2-\mu_0}} \mathrm{e}\left(-h_0 \frac{u_0}{q^{\mu_2-\mu_0}}\right) = 0$ and similarly for h_1. Hence we may assume $h_0 \equiv h_1 \equiv 0 \bmod q^{\mu_2-\mu_0}$. Writing $h_0 = h_0' q^{\mu_2-\mu_0}$ and $h_1 = h_1' q^{\mu_2-\mu_0}$ and using the upper bound $|b_h(q^{\mu_0-\mu_2}, H)| \leq \frac{1}{H}$ from (14) we get

$$|E_4'(r)| \ll \frac{q^{2(\mu_2-\mu_0)}}{H^2} \sum_{|h_0'|\leq H/q^{\mu_2-\mu_0}} \sum_{|h_1'|\leq H/q^{\mu_2-\mu_0}} \left| \sum_m \sum_n \mathrm{e}\left(\frac{(h_0'+h_1')mn + h_1' mr}{q^{\mu_0}}\right)\right|.$$

The contribution to $E_4'(r)$ of the terms for which $h_0' + h_1' = 0$, after summation over m, is bounded by

$$\frac{q^{2(\mu_2-\mu_0)}}{H^2} N \sum_{|h_1'|\leq H/q^{\mu_2-\mu_0}} \min\left(M, \left|\sin\pi \frac{h_1' r}{q^{\mu_0}}\right|^{-1}\right).$$

Since $1 \leq r < q^\rho$ and $H \leq q^\mu$, we have $|h_1' r| < q^{\mu-\mu_2+\mu_0+\rho} = q^{\mu_0-\rho}$ (by (50)) so that the values of $h_1' r$ are all distinct modulo q^{μ_0}. Therefore we conclude that the contribution to $E_4'(r)$ of the terms for which $h_0' + h_1' = 0$ is bounded by

$$\frac{q^{2(\mu_2-\mu_0)}}{H^2} N(M + q^{\mu_0} \log q^{\mu_0}) \ll \frac{q^{2(\mu_2-\mu_0)}}{H^2} q^{\mu+\nu} (1 + q^{\mu_0-\mu} \log q^{\mu_0}).$$

The contribution to $E_4'(r)$ of the terms for which $h_0' + h_1' \neq 0$, after summation over n, is bounded by

$$\frac{q^{2(\mu_2-\mu_0)}}{H^2} \sum_{h_0'+h_1'\neq 0} \sum_m \min\left(N, \left|\sin\pi \frac{(h_0'+h_1')m}{q^{\mu_0}}\right|^{-1}\right)$$

which, writing $h' = h_0' + h_1'$, is less than

$$\frac{q^{\mu_2-\mu_0}}{H} \sum_{1\leq|h'|\leq 2H/q^{\mu_2-\mu_0}} \sum_m \min\left(N, \left|\sin\pi \frac{h'm}{q^{\mu_0}}\right|^{-1}\right).$$

The summation over m runs over at most $q^{\mu-\mu_0}$ periods q^{μ_0}, which gives the upper bound

$$q^{\mu-\mu_0} \frac{q^{\mu_2-\mu_0}}{H} \sum_{1\leq h'\leq 2H/q^{\mu_2-\mu_0}} \sum_{0\leq m'<q^{\mu_0}} \min\left(N, \left|\sin\pi \frac{h'm'}{q^{\mu_0}}\right|^{-1}\right)$$

Using (21) this is bounded by

$$q^{\mu-\mu_0} \left(\tau(q^{\mu_0}) N + q^{\mu_0} \log q^{\mu_0}\right) \ll q^{\nu+\mu-\mu_0} \tau(q^{\mu_0}) + q^\mu \log q^{\mu_0}.$$

We conclude that

$$|E_4'(r)| \ll \frac{q^{2(\mu_2-\mu_0)}}{H^2} q^{\mu+\nu} (1 + q^{\mu_0-\mu} \log q^{\mu_0}) + q^{\nu+\mu-\mu_0} \tau(q^{\mu_0}) + q^\mu \log q^{\mu_0}.$$

With the choice (66) and by (47) and (59), we have $\mu_0 \geq \mu - 4\rho \geq 2\rho$, so that

$$|E_4'(r)| \ll \max(\log q^{\mu_0}, \tau(q^{\mu_0}))\ q^{\mu+\nu-2\rho}. \tag{68}$$

By (63), (64), (67) and (68) we have

(69) $$S_3(r,s) = S_4(r,s) + O(\max(\log q^{\mu_0}, \tau(q^{\mu_0}))\ q^{\mu+\nu-2\rho}).$$

6.3. **Fourier analysis of $S_4(r,s)$.** We write

$$u_0 + q^{\mu_1-\mu_0} sn \equiv u_2 \bmod q^{\mu_2-\mu_0}$$

and

$$u_1 + q^{\mu_1-\mu_0} sn + q^{\mu_1-\mu_0} sr \equiv u_3 \bmod q^{\mu_2-\mu_0}.$$

This gives

$$\begin{aligned} S_4(r,s) &= \sum_{|h_0|\le H} \sum_{|h_1|\le H} a_{h_0}(q^{\mu_0-\mu_2},H) a_{h_1}(q^{\mu_0-\mu_2},H) \frac{1}{q^{2(\mu_2-\mu_0)}} \sum_{0\le h_2<q^{\mu_2-\mu_0}} \sum_{0\le h_3<q^{\mu_2-\mu_0}} \\ &\quad \sum_{\substack{0\le u_0<q^{\mu_2-\mu_0}\\ 0\le u_1<q^{\mu_2-\mu_0}}} \mathrm{e}\left(-\tfrac{h_0u_0}{q^{\mu_2-\mu_0}}\right) \mathrm{e}\left(-\tfrac{h_1u_1}{q^{\mu_2-\mu_0}}\right) \overline{g}(u_1)\ g(u_0) \sum_{\substack{0\le u_2<q^{\mu_2-\mu_0}\\ 0\le u_3<q^{\mu_2-\mu_0}}} g(u_3)\ \overline{g}(u_2) \\ &\quad \sum_m \sum_n \mathrm{e}\left(\frac{h_0mn + h_1mn + h_1mr}{q^{\mu_2}}\right) \mathrm{e}\left(h_2 \frac{u_0 + q^{\mu_1-\mu_0}sn - u_2}{q^{\mu_2-\mu_0}}\right) \\ &\quad \mathrm{e}\left(h_3 \frac{u_1 + q^{\mu_1-\mu_0}sn + q^{\mu_1-\mu_0}sr - u_3}{q^{\mu_2-\mu_0}}\right). \end{aligned}$$

The discrete Fourier transform of g defined by (61) is

(70) $$\widehat{g}(h) = \frac{1}{q^{\mu_2-\mu_0}} \sum_{0\le u<q^{\mu_2-\mu_0}} g(u)\ \mathrm{e}\left(\frac{-uh}{q^{\mu_2-\mu_0}}\right).$$

It follows that

$$\begin{aligned} S_4(r,s) &= q^{2(\mu_2-\mu_0)} \sum_{|h_0|\le H} \sum_{|h_1|\le H} a_{h_0}(q^{\mu_0-\mu_2},H) a_{h_1}(q^{\mu_0-\mu_2},H) \sum_{0\le h_2<q^{\mu_2-\mu_0}} \sum_{0\le h_3<q^{\mu_2-\mu_0}} \mathrm{e}\left(\tfrac{h_3sr}{q^{\mu_2-\mu_1}}\right) \\ &\quad \widehat{g}(h_0-h_2)\ \overline{\widehat{g}(h_3-h_1)}\ \overline{\widehat{g}(-h_2)}\ \widehat{g}(h_3) \\ &\quad \sum_m \sum_n \mathrm{e}\left(\frac{(h_0+h_1)mn + h_1mr + (h_2+h_3)q^{\mu_1}sn}{q^{\mu_2}}\right). \end{aligned}$$

6.4. **Estimate of $S_4(r,s)$.** We write

(71) $$S_4(r,s) = S_4'(r,s) + S_4''(r,s),$$

where $S_4'(r,s)$ denotes the contribution of the terms for which $h_0+h_1 = 0$, while $S_4''(r,s)$ denotes the contribution of the terms for which $h_0+h_1 \neq 0$.

6.4.1. *Contribution of $S_4'(r,s)$.*

Since $h_0+h_1 = 0$, the summations over m and n are independent. Noticing that by (48), (66) (59) and (47) we have $|h_1r| \le HR = q^{\mu_2-\mu_0+3\rho} \le q^{\mu_2}/2$, we deduce

$$\left|\sum_m \mathrm{e}\left(\frac{h_1mr}{q^{\mu_2}}\right)\right| \le \min\left(q^{\mu}, \left|\sin \frac{\pi h_1 r}{q^{\mu_2}}\right|^{-1}\right) \le \min\left(q^{\mu}, \frac{q^{\mu_2}}{r\,|h_1|}\right)$$

and

$$\left|\sum_n \mathrm{e}\left(\frac{(h_2+h_3)q^{\mu_1}sn}{q^{\mu_2}}\right)\right| \le \min\left(q^{\nu}, \left|\sin \frac{\pi(h_2+h_3)s}{q^{\mu_2-\mu_1}}\right|^{-1}\right).$$

Using the periodicity of $\widehat{g}$ modulo $q^{\mu_2-\mu_0}$ and writing $h = h_2+h_3$, we get

(72) $$|S_4'(r,s)| \le S_5(r,s),$$

with

$$S_5(r,s) = q^{2(\mu_2-\mu_0)} \sum_{|h_1|\le H} \left|a_{h_1}(q^{\mu_0-\mu_2},H)\right|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r\,|h_1|}\right) \sum_{0\le h<q^{\mu_2-\mu_0}} \min\left(q^{\nu}, \left|\sin \tfrac{\pi h s}{q^{\mu_2-\mu_1}}\right|^{-1}\right) S_6(h,h_1)$$

and

$$S_6(h,h_1) = \sum_{0\le h_3<q^{\mu_2-\mu_0}} \left|\widehat{g}(h_3-h_1-h)\ \widehat{g}(h_3-h_1)\ \widehat{g}(h_3-h)\ \widehat{g}(h_3)\right|.$$

By using the Cauchy-Schwarz inequality we have

$$S_6(h,h_1) \le \left(\sum_{h_3} \left|\widehat{g}(h_3-h_1-h)\ \widehat{g}(h_3-h)\right|^2\right)^{1/2} \left(\sum_{h_3} \left|\widehat{g}(h_3-h_1)\ \widehat{g}(h_3)\right|^2\right)^{1/2}.$$

The two quantities in the parentheses above are equal by periodicity modulo $q^{\mu_2-\mu_0}$, hence

$$S_6(h,h_1) \le S_7(h_1),$$

with

$$S_7(h_1) = \sum_{0\le h'<q^{\mu_2-\mu_0}} \left|\widehat{g}(h'-h_1)\ \widehat{g}(h')\right|^2. \tag{73}$$

By periodicity modulo $q^{\mu_2-\mu_1}$ and (21) we have

$$\begin{aligned} &\frac{1}{S}\sum_{1\le s<S}\ \sum_{0\le h<q^{\mu_2-\mu_0}} \min\left(q^{\nu}, \left|\sin \tfrac{\pi h s}{q^{\mu_2-\mu_1}}\right|^{-1}\right) \\ &= \frac{q^{\mu_1-\mu_0}}{S} \sum_{1\le s<S}\ \sum_{0\le h<q^{\mu_2-\mu_1}} \min\left(q^{\nu}, \left|\sin \tfrac{\pi h s}{q^{\mu_2-\mu_1}}\right|^{-1}\right) \\ &\ll q^{\mu_1-\mu_0}\left(q^{\nu}\,\tau\left(q^{\mu_2-\mu_1}\right) + q^{\mu_2-\mu_1}\log q^{\mu_2-\mu_1}\right). \end{aligned}$$

Hence

$$\frac{1}{S}\sum_{1\le s<S} S_5(r,s) \ll q^{\nu+\mu_1-\mu_0}\left(\tau\left(q^{\mu_2-\mu_1}\right) + q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1}\right) S_8(r), \tag{74}$$

with

$$S_8(r) = q^{2(\mu_2-\mu_0)} \sum_{|h_1|\le H} \left|a_{h_1}(q^{\mu_0-\mu_2},H)\right|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r\,|h_1|}\right) S_7(h_1).$$

Taking (66) into account we split the summation $S_8(r)$ in three parts

$$S_8(r) = S_8'(r) + S_8''(r) + S_8'''(r) \tag{75}$$

depending on the size of $|h_1|$: $|h_1| \le q^{2\rho}$, $q^{2\rho} < |h_1| \le q^{\mu_2-\mu_0}$, and $q^{\mu_2-\mu_0} < |h_1| \le H$. Using (14) in $S_8'(r)$ we have $|a_{h_1}(q^{\mu_0-\mu_2},H)| \le q^{\mu_0-\mu_2}$, thus

$$S_8'(r) = q^{2(\mu_2-\mu_0)} \sum_{|h_1|\le q^{2\rho}} \left|a_{h_1}(q^{\mu_0-\mu_2},H)\right|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r\,|h_1|}\right) S_7(h_1) \le q^{\mu} \sum_{|h_1|\le q^{2\rho}} S_7(h_1).$$

In order to prove that this short sum over h_1 is small, we will assume in this section that the following Lemma holds:

Lemma 10. *If*

$$\mu \le \left(2 + \tfrac{4}{3}c\right)\rho \tag{76}$$

then uniformly for $\lambda \in \mathbb{N}$ *with* $\frac{1}{3}(\mu_2-\mu_0) \le \lambda \le \frac{4}{5}(\mu_2-\mu_0)$ *we have*

$$\sum_{0\le h<q^{\mu_2-\mu_0}} \sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} \left|\widehat{g}(h+k)\ \widehat{g}(h)\right|^2 \ll q^{-\gamma_1(\lambda,\mu_1-\mu_0)}(\log q^{\mu_2-\mu_1})^2 \tag{77}$$

where

$$\gamma_1(\lambda,\mu_1-\mu_0) = \frac{\gamma(\lambda)-\mu_1+\mu_0}{2}. \tag{78}$$

Proof. Lemma 10 will be proved in section 7. □

By (77), (50), (56), (59), (58) we have $\mu_2-\mu_0 \le 6\rho$ so that $\mu_2-\mu_0-2\rho \le \frac{2}{3}(\mu_2-\mu_0) \le \frac{4}{5}(\mu_2-\mu_0)$ and

$$\sum_{|h_1|\le q^{2\rho}} S_7(h_1) \ll q^{-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)},$$

hence

$$S_8'(r) \ll q^{\mu-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)}. \tag{79}$$

Using (14) in $S_8''(r)$ we have $|a_{h_1}(q^{\mu_0-\mu_2},H)| \le q^{\mu_0-\mu_2}$, thus

$$\begin{aligned} S_8''(r) &= q^{2(\mu_2-\mu_0)} \sum_{q^{2\rho}<|h_1|\le q^{\mu_2-\mu_0}} \left|a_{h_1}(q^{\mu_0-\mu_2},H)\right|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r\,|h_1|}\right) S_7(h_1) \\ &\le \frac{q^{\mu_2}}{r} \sum_{q^{2\rho}<|h_1|\le q^{\mu_2-\mu_0}} \frac{S_7(h_1)}{|h_1|} \le \frac{q^{\mu_2-2\rho}}{r} \sum_{|h_1|\le q^{\mu_2-\mu_0}} S_7(h_1). \end{aligned}$$

As by (24) we have

$$\sum_{0\le h<q^{\mu_2-\mu_0}} \left|\widehat{g}(h)\right|^2 = 1, \tag{80}$$

we obtain

$$S_8''(r) \le \frac{q^{\mu_2-2\rho}}{r},$$

hence using (50) and (48),

$$\frac{1}{R}\sum_{1\le r<R} S_8''(r) \ll q^{\mu_2-2\rho}\frac{\log R}{R} = q^{\mu-\rho}\log q^{\rho}. \tag{81}$$

Using (14) in $S_8'''(r)$ we have $|a_{h_1}(q^{\mu_0-\mu_2},H)| \le \frac{1}{\pi|h_1|}$, thus

$$\begin{aligned} S_8'''(r) &= q^{2(\mu_2-\mu_0)} \sum_{q^{\mu_2-\mu_0}<|h_1|\le H} \left|a_{h_1}(q^{\mu_0-\mu_2},H)\right|^2 \min\left(q^{\mu}, \frac{q^{\mu_2}}{r\,|h_1|}\right) S_7(h_1) \\ &\ll q^{2(\mu_2-\mu_0)}\frac{q^{\mu_2}}{r} \sum_{q^{\mu_2-\mu_0}<|h_1|\le H} \frac{S_7(h_1)}{|h_1|^3}. \end{aligned}$$

Observing that $S_7(h_1)$ is $q^{\mu_2-\mu_0}$ periodic, we split the summation into $jq^{\mu_2-\mu_0} < |h_1| \le (j+1)q^{\mu_2-\mu_0}$ where $1 \le j < H/q^{\mu_2-\mu_0}$ and bound $|h_1|^{-3}$ by $j^{-3}q^{-3(\mu_2-\mu_0)}$:

$$S_8'''(r) \ll q^{2(\mu_2-\mu_0)}\frac{q^{\mu_2}}{r}\sum_{j\ge 1}\frac{1}{j^3 q^{3(\mu_2-\mu_0)}} \sum_{0\le h_1<q^{\mu_2-\mu_0}} S_7(h_1), \tag{82}$$

thus by (80) and (56)

$$S_8'''(r) \ll q^{-(\mu_2-\mu_0)}\frac{q^{\mu_2}}{r} = \frac{q^{\mu_0}}{r} \le \frac{q^{\mu-2\rho}}{r}.$$

It follows from (75), (79), (81), (82) that

$$\frac{1}{R}\sum_{1\le r<R} S_8(r) \ll q^{\mu-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)} + q^{\mu-\rho}\log q^{\rho},$$

and using (74) we obtain

$$\begin{aligned}&\frac{1}{RS}\sum_{1\le r<R}\sum_{1\le s<S} S_5(r,s)\\ &\ll\ q^{\nu+\mu_1-\mu_0}\left(\tau\left(q^{\mu_2-\mu_1}\right)+q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1}\right)\left(q^{\mu-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)}+q^{\mu-\rho}\log q^{\rho}\right).\end{aligned}$$

Finally by (72) we get

$$\begin{aligned}(83)\quad &\frac{1}{RS}\sum_{1\le r<R}\sum_{1\le s<S} |S_4'(r,s)|\\ &\ll\ q^{\mu+\nu+\mu_1-\mu_0}\left(q^{-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)}+q^{-\rho}\log q^{\rho}\right)\left(\tau\left(q^{\mu_2-\mu_1}\right)+q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1}\right).\end{aligned}$$

6.4.2. *Contribution of* $S_4''(r,s)$.

Since $h_0+h_1\neq 0$, after a summation over m, we get

$$\begin{aligned}S_4''(r,s)\ &\ll\ q^{2(\mu_2-\mu_0)}\sum_{|h_0|\le H}\sum_{h_1\neq -h_0}\left|a_{h_0}(q^{\mu_0-\mu_2},H)a_{h_1}(q^{\mu_0-\mu_2},H)\right|\\ &\sum_{0\le h_2<q^{\mu_2-\mu_0}}\left|\widehat{g}(h_0-h_2)\ \widehat{g}(-h_2)\ \right|\sum_{0\le h_3<q^{\mu_2-\mu_0}}\left|\widehat{g}(h_3-h_1)\ \widehat{g}(h_3)\right|\\ &\sum_n \min\left(q^{\mu},\left|\sin\pi\tfrac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right).\end{aligned}$$

Using (20) we have

$$\sum_n \min\left(q^{\mu},\left|\sin\pi\tfrac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right)\ll\left\lceil q^{\nu-\mu_2}\right\rceil\left((h_0+h_1,q^{\mu_2})q^{\mu}+q^{\mu_2}\log q^{\mu_2}\right),$$

and observing that $|h_0+h_1|\le 2H$ we get

$$\sum_n \min\left(q^{\mu},\left|\sin\pi\tfrac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right)\ll\left\lceil q^{\nu-\mu_2}\right\rceil\left(Hq^{\mu}+q^{\mu_2}\log q^{\mu_2}\right).$$

By (66) we have $Hq^{\mu}\ge q^{\mu+\mu_2-\mu_0}\ge q^{\mu_2}$,

$$\sum_n \min\left(q^{\mu},\left|\sin\pi\tfrac{(h_0+h_1)n+h_1r}{q^{\mu_2}}\right|^{-1}\right)\ll\left\lceil q^{\nu-\mu_2}\right\rceil H\ q^{\mu}\log q^{\mu_2}.$$

Moreover we have by the Cauchy-Schwarz inequality and (80)

$$\sum_{0\le h_2<q^{\mu_2-\mu_0}}\left|\widehat{g}(h_0-h_2)\ \widehat{g}(-h_2)\ \right|\le\left(\sum_{h_2}\left|\widehat{g}(h_0-h_2)\right|^2\right)^{1/2}\left(\sum_{h_2}\left|\widehat{g}(-h_2)\right|^2\right)^{1/2}=1,$$

and similarly

$$\sum_{0\le h_3<q^{\mu_2-\mu_0}}\left|\widehat{g}(h_3-h_1)\ \widehat{g}(h_3)\right|\le 1.$$

Furthermore

$$\sum_{|h|\le H}\left|a_h(q^{\mu_0-\mu_2},H)\right|\le\sum_{|h|\le q^{\mu_2-\mu_0}}\frac{1}{q^{\mu_2-\mu_0}}+\sum_{q^{\mu_2-\mu_0}<|h|\le H}\frac{1}{\pi\,|h|}\ll\log(H/q^{\mu_2-\mu_0})\ll\log q^{\rho}.$$

Finally we obtain

$$|S_4''(r,s)|\ll(\log q)^2\rho^2 q^{2(\mu_2-\mu_0)}\left\lceil q^{\nu-\mu_2}\right\rceil H\ q^{\mu}\log q^{\mu_2},$$

which gives with the choice of H defined by (66):

$$|S_4''(r,s)| \ll (\log q)^3(\mu+\nu)^3 q^{\mu+\nu+3(\mu_2-\mu_0)+2\rho}\ (q^{-\mu_2}+q^{-\nu}). \tag{84}$$

6.5. **Conclusion: estimate of $S_{II}(\vartheta)$.**

By (54), (62), (69), (71), (83) and (84), for all functions f satisfying Definition 1, $f \in \mathcal{F}_{\gamma,c}$ in Definition 2 with $c \geq 10$, (76) and (77) we obtain uniformly for $\vartheta \in \mathbb{R}$:

$$\begin{aligned}|S_{II}(\vartheta)|^4 \ll\ & q^{4\mu+4\nu+\mu_1-\mu_0}\left(q^{-\gamma_1(\mu_2-\mu_0-2\rho,\mu_1-\mu_0)}+q^{-\rho}\log q^{\rho}\right)\\ &\qquad\left(\tau\left(q^{\mu_2-\mu_1}\right)+q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1}\right)\\ &+(\log q)^3(\mu+\nu)^3 q^{4\mu+4\nu+3(\mu_2-\mu_0)+2\rho}\ (q^{-\mu_2}+q^{-\nu})\\ &+\max(\log q^{\mu_0},\tau(q^{\mu_0}))\ q^{4\mu+4\nu-2\rho}\\ &+\max(\tau(q),\log q)\ (\mu+\nu)^{\omega(q)}\ q^{4\mu+4\nu-2\rho'}.\end{aligned} \tag{85}$$

It remains to prove Lemma 10.

7. Distribution of the Discrete Fourier Transform

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a non decreasing function satisfying $\lim_{\lambda\to+\infty}\gamma(\lambda) = +\infty$, $f : \mathbb{N} \to \mathbb{U}$ be a function satisfying Definition 1 and $f \in \mathcal{F}_{\gamma,c}$ in Definition 2 for some $c \geq 10$. The discrete Fourier transform of the q^λ-periodic function $u \mapsto f_{\mu_1,\mu_2}(\mathrm{r}_\lambda(u)q^{\mu_0})$ is a q^λ-periodic function $G_{\mu_0,\lambda}$ defined for $t \in \mathbb{R}$ by

$$G_{\mu_0,\lambda}(t) = \frac{1}{q^\lambda}\sum_{0\leq u<q^\lambda} f_{\mu_1,\mu_2}(uq^{\mu_0})\,\mathrm{e}\left(-\frac{ut}{q^\lambda}\right). \tag{86}$$

By (24) we have for $t \in \mathbb{R}$ and $\lambda \in \mathbb{N}$,

$$\sum_{0\leq h<q^\lambda} |G_{\mu_0,\lambda}(h+t)|^2 = 1, \tag{87}$$

and by (61) and (70), for $h \in \mathbb{Z}$ we have

$$\widehat{g}(h) = G_{\mu_0,\mu_2-\mu_0}(h). \tag{88}$$

In order to prove Lemma 10 we will need the following:

Lemma 11. *If μ and ρ satisfy* (76) *then uniformly for $\lambda \in \mathbb{N}$ with*

$$\frac{1}{3}(\mu_2-\mu_0) \leq \lambda \leq \frac{4}{5}(\mu_2-\mu_0) \tag{89}$$

and $t \in \mathbb{R}$ we have

$$\sum_{0\leq k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0}(k+t)|^2 \ll q^{\frac{1}{2}(\mu_1-\mu_0-\gamma(\lambda))}(\log q^{\mu_2-\mu_1})^2.$$

Proof. For $0 \leq \lambda \leq \mu_2-\mu_0$ and $t \in \mathbb{R}$ we can write

$$G_{\mu_0,\mu_2-\mu_0}(t) = \frac{1}{q^{\mu_2-\mu_0}}\sum_{0\leq u<q^\lambda}\ \sum_{0\leq v<q^{\mu_2-\mu_0-\lambda}} f_{\mu_1,\mu_2}((u+vq^\lambda)q^{\mu_0})\,\mathrm{e}\left(-\frac{(u+vq^\lambda)t}{q^{\mu_2-\mu_0}}\right).$$

Hence for $\mu_1-\mu_0 \leq \lambda \leq \mu_2-\mu_0$, observing that $0 \leq u+vq^\lambda < q^{\mu_2-\mu_0}$ and $(u+vq^\lambda)q^{\mu_0} \equiv uq^{\mu_0} \bmod q^{\mu_1}$, using (55) we get for $0 \leq u < q^\lambda$ and $0 \leq v < q^{\mu_2-\mu_0-\lambda}$

$$\begin{aligned}f_{\mu_1,\mu_2}((u+vq^\lambda)q^{\mu_0}) &= f_{\mu_2}((u+vq^\lambda)q^{\mu_0})\overline{f_{\mu_1}((u+vq^\lambda)q^{\mu_0})}\\ &= f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_1}(uq^{\mu_0})},\end{aligned}$$

and this yields

$$\begin{aligned}
&G_{\mu_0,\mu_2-\mu_0}(t)\\
&\quad= \frac{1}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\left(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}\right)\\
&\qquad \frac{1}{q^{\lambda}} \sum_{0\le u<q^{\lambda}} f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f(vq^{\mu_0+\lambda})f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\left(-\frac{ut}{q^{\mu_2-\mu_0}}\right).
\end{aligned}$$

Given $\rho_3\in\mathbb{N}$ such that

$$1\le \rho_3\le \mu_2-\mu_0-\lambda, \tag{90}$$

by Definition 1 the number of $v\in\{0,\dots,q^{\mu_2-\mu_0-\lambda}-1\}$ such that there exists $u\in\{0,\dots,q^{\lambda}-1\}$ for which

$$f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f(vq^{\mu_0+\lambda})}\neq f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(vq^{\mu_0+\lambda})}$$

is $O(q^{\mu_2-\mu_0-\lambda-\rho_3})$. Hence the set $\widetilde{\mathcal{W}}_\lambda$ of pairs (u,v) with this property satisfies

$$\operatorname{card}\widetilde{\mathcal{W}}_\lambda \ll q^{\mu_2-\mu_0-\rho_3}. \tag{91}$$

This leads for all $t\in\mathbb{R}$ to write

$$G_{\mu_0,\mu_2-\mu_0}(t)=G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)+G_{\mu_0,\mu_2-\mu_0,\lambda,2}(t),$$

with

$$\begin{aligned}
&G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)\\
&\quad= \frac{1}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\left(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}\right)\\
&\qquad \frac{1}{q^{\lambda}} \sum_{0\le u<q^{\lambda}} f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(vq^{\mu_0+\lambda})f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\left(-\frac{ut}{q^{\mu_2-\mu_0}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&G_{\mu_0,\mu_2-\mu_0,\lambda,2}(t)\\
&\quad= \frac{1}{q^{\mu_2-\mu_0}} \sum_{(u,v)\in\widetilde{\mathcal{W}}_\lambda} f(vq^{\mu_0+\lambda})\overline{f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\left(-\frac{(u+vq^{\lambda})t}{q^{\mu_2-\mu_0}}\right)\\
&\qquad \left(f(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f(vq^{\mu_0+\lambda})}-f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+vq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(vq^{\mu_0+\lambda})}\right).
\end{aligned}$$

Let us introduce in $G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)$ the residue w of v modulo q^{ρ_3} in order to make the variables u and v independent:

$$\begin{aligned}
&G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)\\
&\quad= \sum_{0\le w<q^{\rho_3}} \frac{1}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\left(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}\right)\frac{1}{q^{\rho_3}}\sum_{0\le \ell<q^{\rho_3}}\mathrm{e}\left(\ell\frac{v-w}{q^{\rho_3}}\right)\\
&\qquad \frac{1}{q^{\lambda}} \sum_{0\le u<q^{\lambda}} f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+wq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(wq^{\mu_0+\lambda})f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\left(-\frac{ut}{q^{\mu_2-\mu_0}}\right).
\end{aligned}$$

This gives

$$G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)=\sum_{0\le \ell<q^{\rho_3}}\frac{\widetilde{c}_\ell(t)}{q^{\mu_2-\mu_0-\lambda}}\sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\left(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}+\frac{v\ell}{q^{\rho_3}}\right),$$

with
$$\widetilde{c_\ell}(t) = \frac{1}{q^{\rho_3}} \sum_{0\le w<q^{\rho_3}} c_\lambda(w,t)\,\mathrm{e}\left(-\frac{w\ell}{q^{\rho_3}}\right)$$
and
$$c_\lambda(w,t) = \frac{1}{q^\lambda}\sum_{0\le u<q^\lambda} f_{\mu_0+\lambda+\rho_3}(uq^{\mu_0}+wq^{\mu_0+\lambda})\overline{f_{\mu_0+\lambda+\rho_3}(wq^{\mu_0+\lambda}) f_{\mu_1}(uq^{\mu_0})}\,\mathrm{e}\left(-\frac{ut}{q^{\mu_2-\mu_0}}\right).$$
By the Cauchy-Schwarz inequality we get
$$\begin{aligned}&\left|G_{\mu_0,\mu_2-\mu_0,\lambda,1}(t)\right|^2\\ &\le \left(\sum_{0\le\ell<q^{\rho_3}} \left|\widetilde{c_\ell}(t)\right|^2\right)\sum_{0\le\ell<q^{\rho_3}}\left|\frac{1}{q^{\mu_2-\mu_0-\lambda}}\sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\,\mathrm{e}\left(-\frac{vt}{q^{\mu_2-\mu_0-\lambda}}+\frac{v\ell}{q^{\rho_3}}\right)\right|^2.\end{aligned}$$
But
$$\begin{aligned}\sum_{0\le\ell<q^{\rho_3}}\left|\widetilde{c_\ell}(t)\right|^2 &= \frac{1}{q^{2\rho_3}}\sum_{0\le w<q^{\rho_3}}\sum_{0\le w'<q^{\rho_3}} c_\lambda(w,t)\overline{c_\lambda(w',t)}\sum_{0\le\ell<q^{\rho_3}}\mathrm{e}\left(-\frac{(w-w')\ell}{q^{\rho_3}}\right)\\ &= \frac{1}{q^{\rho_3}}\sum_{0\le w<q^{\rho_3}}\left|c_\lambda(w,t)\right|^2.\end{aligned}$$
Since $f_{\mu_1}(u_0q^{\mu_0}+u_1q^{\mu_1}) = f_{\mu_1}(u_0q^{\mu_0})$ for $0\le u_0<q^{\mu_1-\mu_0}$ and $0\le u_1<q^{\lambda-\mu_1+\mu_0}$, we can write
$$\begin{aligned}&c_\lambda(w,t)\\ &= \overline{f_{\mu_0+\lambda+\rho_3}(wq^{\mu_0+\lambda})}\,\mathrm{e}\left(\frac{wq^\lambda t}{q^{\mu_2-\mu_0}}\right)\frac{1}{q^{\mu_1-\mu_0}}\sum_{0\le u_0<q^{\mu_1-\mu_0}}\overline{f_{\mu_1}(u_0q^{\mu_0})}\\ &\quad\frac{1}{q^{\lambda-\mu_1+\mu_0}}\sum_{0\le u_1<q^{\lambda-\mu_1+\mu_0}} f_{\mu_0+\lambda+\rho_3}\left(u_0q^{\mu_0}+u_1q^{\mu_1}+wq^{\mu_0+\lambda}\right)\mathrm{e}\left(-\frac{(u_0+u_1q^{\mu_1-\mu_0}+wq^\lambda)t}{q^{\mu_2-\mu_0}}\right).\end{aligned}$$
The sum over u_1 may be written as a sum over u' such that $0\le u'<q^{\lambda+\rho_3}$ and $u' = u_0 + u_1q^{\mu_1-\mu_0}+wq^\lambda$ for some u_1 with $0\le u_1<q^{\lambda-\mu_1+\mu_0}$. Hence this last line is
$$\begin{aligned}&\sum_{0\le\ell'<q^{\lambda+\rho_3}}\left(\frac{1}{q^{\lambda-\mu_1+\mu_0}}\sum_{0\le u_1<q^{\lambda-\mu_1+\mu_0}}\mathrm{e}\left(\ell'\frac{u_0+u_1q^{\mu_1-\mu_0}+wq^\lambda}{q^{\lambda+\rho_3}}\right)\right)\\ &\frac{1}{q^{\lambda+\rho_3}}\sum_{0\le u'<q^{\lambda+\rho_3}} f(u'q^{\mu_0})\,\mathrm{e}\left(-\frac{u't}{q^{\mu_2-\mu_0}}-\frac{\ell'u'}{q^{\lambda+\rho_3}}\right)\end{aligned}$$
In order to use (6) with $\kappa=\mu_0$ and λ replaced by $\lambda+\rho_3$ we need to check that $\mu_0\le c(\lambda+\rho_3)$. By (59) we have $\mu_0\le\mu_1$, so that by (89) a sufficient condition would be that $\mu_1\le\frac{c}{3}(\mu_2-\mu_1)$. By (56) and (50) this is equivalent to (76). We are now ready to use (6) with $\kappa=\mu_0$ and λ replaced by $\lambda+\rho_3$. Uniformly for $t\in\mathbb{R}$ and $0\le w<q^{\rho_3}$ we obtain:
$$|c_\lambda(w,t)| \ll \frac{q^{-\gamma(\lambda+\rho_3)}}{q^{\lambda-\mu_1+\mu_0}}\sum_{0\le\ell'<q^{\lambda+\rho_3}}\min\left(q^{\lambda-\mu_1+\mu_0},\left|\sin\pi\left(\frac{\ell' q^{\mu_1-\mu_0}}{q^{\lambda+\rho_3}}\right)\right|^{-1}\right).$$
The sum over ℓ' runs over $q^{\mu_1-\mu_0}$ periods modulo $q^{\lambda+\rho_3-\mu_1+\mu_0}$, thus
$$|c_\lambda(w,t)| \ll \frac{q^{-\gamma(\lambda+\rho_3)}}{q^{\lambda-\mu_1+\mu_0}}q^{\lambda+\rho_3}\log q^{\lambda+\rho_3-\mu_1+\mu_0} = q^{\rho_3+\mu_1-\mu_0-\gamma(\lambda+\rho_3)}\log q^{\lambda+\rho_3-\mu_1+\mu_0}.$$
Since the function γ is non decreasing and by (90) this gives uniformly for $t\in\mathbb{R}$ and $0\le w<q^{\rho_3}$:
$$|c_\lambda(w,t)| \ll q^{\rho_3+\mu_1-\mu_0-\gamma(\lambda)}\log q^{\mu_2-\mu_1}.$$

We deduce that

$$\sum_{0\le \ell<q^{\rho_3}} |\widetilde{c}_\ell(t)|^2 \ll q^{2\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)}(\log q^{\mu_2-\mu_1})^2$$

and

$$\begin{aligned}
&\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0,\lambda,1}(k+t)|^2\\
\ll\ & q^{2\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)}(\log q^{\mu_2-\mu_1})^2\\
&\quad \sum_{0\le \ell<q^{\rho_3}} \frac{1}{q^{2(\mu_2-\mu_0-\lambda)}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v'<q^{\mu_2-\mu_0-\lambda}} f(vq^{\mu_0+\lambda})\overline{f(v'q^{\mu_0+\lambda})}\\
&\quad \mathrm{e}\left(-\frac{(v-v')t}{q^{\mu_2-\mu_0-\lambda}}+\frac{(v-v')\ell}{q^{\rho_3}}\right) \sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} \mathrm{e}\left(-\frac{(v-v')k}{q^{\mu_2-\mu_0-\lambda}}\right)\\
\ll\ & q^{2\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)}(\log q^{\mu_2-\mu_1})^2 \sum_{0\le \ell<q^{\rho_3}} \frac{1}{q^{\mu_2-\mu_0-\lambda}} \sum_{0\le v<q^{\mu_2-\mu_0-\lambda}} 1\\
\ll\ & q^{3\rho_3+2(\mu_1-\mu_0)-2\gamma(\lambda)}(\log q^{\mu_2-\mu_1})^2.
\end{aligned}$$

Denoting by $\mathcal{W}_\lambda$ the set of integers $w=u+q^\lambda v$ such that $(u,v)\in\widetilde{\mathcal{W}}_\lambda$ we observe that

$$G_{\mu_0,\mu_2-\mu_0,\lambda,2}(t)=\frac{1}{q^{\mu_2-\mu_0}}\sum_{w<q^{\mu_2-\mu_0}} c'_\lambda(w)\,\mathrm{e}\left(-\frac{wt}{q^{\mu_2-\mu_0}}\right),$$

where $|c'_\lambda(w)|\le 2$ for $0\le w<q^{\mu_2-\mu_0}$ and $c'_\lambda(w)=0$ for $w\notin\mathcal{W}_\lambda$. Therefore for all $t\in\mathbb{R}$

$$\begin{aligned}
&\sum_{0\le k\le q^{\mu_2-\mu_0}} |G_{\mu_0,\mu_2-\mu_0,\lambda,2}(k+t)|^2\\
=\ & \frac{1}{q^{2(\mu_2-\mu_0)}} \sum_{w<q^{\mu_2-\mu_0}} \sum_{w'<q^{\mu_2-\mu_0}} c'_\lambda(w)\overline{c'_\lambda(w')}\,\mathrm{e}\left(-\frac{(w-w')t}{q^{\mu_2-\mu_0}}\right)\\
&\quad \sum_{0\le k\le q^{\mu_2-\mu_0}} \mathrm{e}\left(-\frac{(w-w')k}{q^{\mu_2-\mu_0}}\right)\\
=\ & \frac{1}{q^{\mu_2-\mu_0}} \sum_{w<q^{\mu_2-\mu_0}} |c'_\lambda(w)|^2 = \frac{1}{q^{\mu_2-\mu_0}} \sum_{w\in\mathcal{W}_\lambda} |c'_\lambda(w)|^2 \le \frac{1}{q^{\mu_2-\mu_0}} \sum_{w\in\mathcal{W}_\lambda} 2^2\\
\ll\ & q^{-\rho_3},
\end{aligned}$$

where we take

$$\rho_3=\max\left(1,\left\lfloor\tfrac{1}{2}(\gamma(\lambda)-\mu_1+\mu_0)\right\rfloor\right).$$

By (90) this is admissible at least if $\lambda+\frac{\gamma(\lambda)}{2}\le\mu_2-\mu_0$. By (25) a sufficient condition to ensure this inequality is that $\lambda\le\frac{4}{5}(\mu_2-\mu_0)$. Then we get

$$\begin{aligned}
&\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0}(k+t)|^2\\
\le\ & 2\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0,\lambda,1}(k+t)|^2 + 2\sum_{0\le k<q^{\mu_2-\mu_0-\lambda}} |G_{\mu_0,\mu_2-\mu_0,\lambda,2}(k+t)|^2\\
\ll\ & q^{\frac{1}{2}(\mu_1-\mu_0-\gamma(\lambda))}(\log q^{\mu_2-\mu_1})^2,
\end{aligned}$$

which completes the proof of Lemma 11. □

It follows from Lemma 11 and (80) that (77) holds with γ_1 defined by (78), which completes the proof of Lemma 10.

8. End of the estimate of the sums of type II

It follows from (85) that for all functions f satisfying Definition 1 and $f \in \mathcal{F}_{\gamma,c}$ in Definition 2 for some $c \geq 10$ that we have, under the condition (76), uniformly for any $\vartheta \in \mathbb{R}$:

$$\begin{aligned} |S_{II}(\vartheta)|^4 \ll\ & q^{4\mu+4\nu+\mu_1-\mu_0} \left(q^{\frac{1}{2}(\mu_1-\mu_0-\gamma(\lambda))} + q^{-\rho}\log q^{\rho}\right) \\ & \qquad \left(\tau\left(q^{\mu_2-\mu_1}\right) + q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1}\right) \\ & + (\log q)^3(\mu+\nu)^3 q^{4\mu+4\nu+3(\mu_2-\mu_0)+2\rho}\left(q^{-\mu_2}+q^{-\nu}\right) \\ & + \max(\log q^{\mu_0}, \tau(q^{\mu_0}))\ q^{4\mu+4\nu-2\rho} \\ & + \max(\tau(q), \log q)\ (\mu+\nu)^{\omega(q)}\ q^{4\mu+4\nu-2\rho'}. \end{aligned} \tag{92}$$

By (50), (56), (59), (58), (25) and since the function γ is non decreasing, $\gamma(\mu_2-\mu_0-2\rho) \geq \gamma(\mu_2-\mu_1-2\rho) = \gamma(2\rho)$ and $\rho \geq \gamma(2\rho)$. By multiplicativity of the function τ we have $\tau\left(q^{\mu_2-\mu_1}\right) \leq (\mu_2-\mu_1)^{\omega(q)}\tau(q)$. By (50), (56), (45) and (47) we have $\mu_2-\mu_1 = 4\rho \leq \mu-2\rho \leq \nu-2\rho$ so that $q^{\mu_2-\mu_1-\nu}\log q^{\mu_2-\mu_1} \ll 1$. This implies

$$\begin{aligned} |S_{II}(\vartheta)|^4 \ll\ & \tau(q)(\mu_2-\mu_1)^{\omega(q)} q^{4\mu+4\nu+\frac{3}{2}(\mu_1-\mu_0)-\frac{1}{2}\gamma(2\rho)}\log q^{\rho} \\ & + (\log q)^3(\mu+\nu)^3 q^{4\mu+4\nu+3(\mu_1-\mu_0)+14\rho-\mu} \\ & + \max(\log q^{\mu_0}, \tau(q^{\mu_0}))\ q^{4\mu+4\nu-2\rho} \\ & + \max(\tau(q), \log q)\ (\mu+\nu)^{\omega(q)}\ q^{4\mu+4\nu-2\rho'}. \end{aligned}$$

Taking

$$\rho' = \lfloor \gamma(2\rho)/10 \rfloor, \tag{93}$$

we have (58) by (25) and by (59) $\mu_1-\mu_0 \leq 2\rho' \leq \gamma(2\rho)/5 \leq \rho/5$ so that

$$\frac{3}{2}(\mu_1-\mu_0) - \frac{\gamma(2\rho)}{2} \leq \frac{3\gamma(2\rho)}{10} - \frac{\gamma(2\rho)}{2} \leq \frac{-\gamma(2\rho)}{5}.$$

Choosing

$$\rho = \lfloor \mu/15 \rfloor \tag{94}$$

we get for $\mu \geq 15 \times 75 = 1125$,

$$3(\mu_1-\mu_0) + 14\rho - \mu \leq \frac{3\rho}{5} + 14\rho - 15\rho + 15 \leq \frac{-\rho}{5}.$$

In order to ensure (76) it is sufficient to check that

$$\mu \leq \left(2 + \frac{4}{3}c\right)\left(\frac{\mu}{15} - 1\right),$$

which is true for μ large enough provided $c > 39/4$. For convenience and in order to avoid that the implied constants depend on c we take

$$c \geq 10, \tag{95}$$

so that the inequality above is valid for $\mu \geq 46 \times 15 = 690$. Finally we obtain

$$|S_{II}(\vartheta)|^4 \ll \max(\tau(q)\log q, \log^3 q)\,(\mu+\nu)^{1+\max(\omega(q),2)}\, q^{4\mu+4\nu-\gamma(2\lfloor \mu/15\rfloor)/5}, \tag{96}$$

which completes the proof of (46) and Proposition 2.

9. Proof of Theorems 1 and 2

Applying Lemma 1 of [20], or its analogue in the case of μ obtained using (13.40) instead of (13.39) of [13], our estimate of the sums of type I in Proposition 1 and our estimate of the sums of type II in Proposition 2 with the observation that (45) implies $\frac{\mu+\nu}{60} \leq \frac{\mu}{15}$ lead to

$$\left|\sum_{x/q<n\leq x} \Lambda(n) f(n)\,\mathrm{e}\,(\vartheta n)\right| \ll c_1(q)(\log x)^{\frac{9}{4}+\frac{1}{4}\max(\omega(q),2)}\, x\, q^{-\gamma(2\lfloor(\log x)/60\log q\rfloor)/20}.$$

and

$$\left|\sum_{x/q<n\leq x} \mu(n) f(n)\,\mathrm{e}\,(\vartheta n)\right| \ll c_1(q)(\log x)^{\frac{9}{4}+\frac{1}{4}\max(\omega(q),2)}\, x\, q^{-\gamma(2\lfloor(\log x)/60\log q\rfloor)/20}.$$

with $c_1(q) = \max(\tau(q), \log^2 q)^{1/4}\,(\log q)^{-2-\frac{1}{4}\max(\omega(q),2)}$. Now we will replace x by x/q^k and sum over k. Let $K \in \mathbb{N}$ such that $q^K \leq x^{1/4} < q^{K+1}$. Since γ is non decreasing we have

$$\begin{aligned}\sum_{k\leq K} \frac{x}{q^k} q^{-\gamma(2\left\lfloor \log(xq^{-k})/60\log q\right\rfloor)/20} &\leq q^{-\gamma(2\left\lfloor \log x^{3/4}/60\log q\right\rfloor)/20} \sum_{k\leq K}\frac{x}{q^k}\\ &\ll xq^{-\gamma(2\lfloor \log x/80\log q\rfloor)/20}\end{aligned}$$

while

$$\sum_{k>K} \frac{x}{q^k} q^{-\gamma(2\left\lfloor \log(xq^{-k})/60\log q\right\rfloor)/20} \leq \sum_{k\geq 0}\frac{x^{3/4}}{q^k} \ll x^{3/4} \ll xq^{-\gamma(2\lfloor \log x/80\log q\rfloor)/20}.$$

Then Theorem 1 and Theorem 2 follow.

10. Proof of Corollaries 1, 2 and 3

In order to prove Corollaries 1 and 2 we use a classical partial summation. Using (for example) Lemma 11 of [20], if $f : \mathbb{N} \to \mathbb{C}$ is such that $|f(n)| \leq 1$ for all $n \in \mathbb{N}$ then

$$\left|\sum_{p\leq x} f(p)\right| \leq \frac{2}{\log x} \max_{t\leq x}\left|\sum_{n\leq t} \Lambda(n) f(n)\right| + O(\sqrt{x}). \tag{97}$$

To prove Corollary 1 we observe first that if $\alpha \in \mathbb{Q}$, then the sequence $(\alpha b(p))_{p\in\mathcal{P}(a,m)}$ takes a finite number of values modulo 1, and therefore is not equidistributed modulo 1. If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, then for all $h \in \mathbb{Z}$ such that $h \neq 0$, we have $h\alpha \in \mathbb{R}\setminus\mathbb{Q}$, so that the function $n \mapsto \mathrm{e}(h\alpha b(n))$ satisfies Definition 1 and $f \in \mathcal{F}_{\gamma,c}$ in Definition 2 for some $c \geq 10$. By Theorem 1, we have for all $0 \leq j < m$

$$\sum_{n\leq x} \Lambda(n)\,\mathrm{e}\left(h\alpha b(n) + \frac{jn}{m}\right) = o(x),$$

hence

$$\sum_{\substack{n\leq x\\ n\equiv a \bmod m}} \Lambda(n)\,\mathrm{e}\,(h\alpha b(n)) = \frac{1}{m}\sum_{0\leq j<m} \mathrm{e}\left(-\frac{ja}{m}\right)\sum_{n\leq x}\Lambda(n)\,\mathrm{e}\left(h\alpha b(n)+\frac{jn}{m}\right) = o(x).$$

By Inequality (97) and the Prime Number Theorem in arithmetic progressions (see for example [2, Theorem 9.12]), we can write for $\gcd(a, m) = 1$

$$\sum_{\substack{p\leq x\\ p\equiv a \bmod m}} \mathrm{e}(h\alpha b(p)) = o\left(\frac{x}{\log x}\right) + O(\sqrt{x}) = o(\pi(x; a, m)),$$

which proves that the sequence $(\alpha b(p))_{p\in\mathcal{P}(a,m)}$ is equidistributed modulo 1 according to Weyl's criterion (see for example [22, Chapter 1,p. 1]).

In order to prove Corollary 2, we write

$$\begin{aligned}\sum_{\substack{p\le x\\ p\equiv a \bmod m\\ b(p)\equiv a' \bmod m'}} 1 &= \sum_{p\le x}\frac{1}{mm'}\sum_{\substack{0\le j<m\\ 0\le j'<m'}} \mathrm{e}\left(\frac{j}{m}(p-a)+\frac{j'}{m'}(b(p)-a')\right)\\ &= \frac{\pi(x;a,m)}{m'}+\frac{1}{mm'}\sum_{\substack{0\le j<m\\ 1\le j'<m'}} \mathrm{e}\left(-\frac{aj}{m}-\frac{a'j'}{m'}\right)\sum_{p\le x}\mathrm{e}\left(\frac{j}{m}p+\frac{j'}{m'}b(p)\right).\end{aligned}$$

Since for $1\le j'<m'$ the functions $n\mapsto \mathrm{e}(\frac{j'}{m'}b(n))$ satisfy Definition 1 and $f\in\mathcal{F}_{\gamma,c}$ in Definition 2 for some $c\ge 10$, by Theorem 1 and using (97), for all $0\le j<m$ and $1\le j'<m'$ we obtain

$$\sum_{p\le x}\mathrm{e}\left(\frac{j'}{m'}b(p)+\frac{j}{m}p\right)=o(\pi(x)).$$

As the integers m and m' are fixed, it follows by using the Prime Number Theorem in arithmetic progressions (see for example [2, Theorem 9.12]), that for $\gcd(a,m)=1$

$$\sum_{\substack{p\le x\\ p\equiv a \bmod m\\ b(p)\equiv a' \bmod m'}} 1=(1+o(1))\,\frac{\pi(x;a,m)}{m'},$$

which proves Corollary 2.

In order to prove Corollary 3 we observe first that if $\vartheta\in\mathbb{Q}$, then the sequence $(\vartheta p)_{p\in\mathcal{B}(a,m,a',m')}$ takes a finite number of values modulo 1, and therefore is not equidistributed modulo 1. If $\vartheta\in\mathbb{R}\setminus\mathbb{Q}$, we write

$$\begin{aligned}\sum_{\substack{p\le x\\ p\equiv a \bmod m\\ b(p)\equiv a' \bmod m'}} \mathrm{e}(h\vartheta p) &= \sum_{p\le x}\frac{1}{mm'}\sum_{\substack{0\le j<m\\ 0\le j'<m'}} \mathrm{e}\left(h\vartheta p+\frac{j}{m}(p-a)+\frac{j'}{m'}(b(p)-a')\right)\\ &= \frac{1}{mm'}\sum_{0\le j<m}\mathrm{e}\left(-\frac{aj}{m}\right)\sum_{p\le x}\mathrm{e}\left(\left(h\vartheta+\frac{j}{m}\right)p\right)\\ &\quad+\frac{1}{mm'}\sum_{\substack{0\le j<m\\ 1\le j'<m'}} \mathrm{e}\left(-\frac{aj}{m}-\frac{a'j'}{m'}\right)\sum_{p\le x}\mathrm{e}\left(\left(h\vartheta+\frac{j}{m}\right)p+\frac{j'}{m'}b(p)\right).\end{aligned}$$

It follows from [8] that for all $h\in\mathbb{Z}\setminus\{0\}$, $\vartheta\in\mathbb{R}\setminus\mathbb{Q}$, $0\le j<m$ we have

$$\sum_{p\le x}\mathrm{e}\left(\left(h\vartheta+\frac{j}{m}\right)p\right)=o(\pi(x)).$$

Since for $1\le j'<m'$ the functions $n\mapsto \mathrm{e}(\frac{j'}{m'}b(n))$ satisfy Definition 1 and $f\in\mathcal{F}_{\gamma,c}$ in Definition 2 for some $c\ge 10$, by Theorem 1 and using (97), for all $h\in\mathbb{Z}\setminus\{0\}$, $\vartheta\in\mathbb{R}$, $0\le j<m$ and $1\le j'<m'$ we obtain

$$\sum_{p\le x}\mathrm{e}\left(\frac{j'}{m'}b(p)+\left(h\vartheta+\frac{j}{m}\right)p\right)=o(\pi(x)).$$

It follows that

$$\sum_{\substack{p\le x\\ p\equiv a \bmod m\\ b(p)\equiv a' \bmod m'}} \mathrm{e}(h\vartheta p)=o(\pi(x)),$$

which, as the integers m and m' are fixed, proves that the sequence $(\vartheta p)_{p\in\mathcal{B}(a,m,a',m')}$ is equidistributed modulo 1 according to Weyl's criterion (see for example [22, Chapter 1,p. 1]).

11. Application to Rudin-Shapiro sequences

11.1. Rudin-Shapiro sequences of order δ. For any $n \in \mathbb{N}$ we denote by

$$n = \sum_{k \geq 0} \varepsilon_k(n)$$

its representation in base 2, where $\varepsilon_k(n)$ denotes the k-th least significant digit of n in base 2. Let $\delta \in \mathbb{N}$ and $\beta_\delta(n)$ the number of occurencies of patterns $1w1$ (where $w \in \{0,1\}^\delta$) in the representation of n in base 2:

$$\beta_\delta(n) = \sum_{k \geq \delta+1} \varepsilon_{k-\delta-1}(n)\, \varepsilon_k(n).$$

For $\alpha \in \mathbb{R}$ we consider in this section $f(n) = \mathrm{e}\left(\beta_\delta(n)\alpha\right)$. By (4) for all $\lambda \geq \delta + 2$ we have

$$f_\lambda = \mathrm{e}\left(\alpha \sum_{\delta+1 \leq i < \lambda} \varepsilon_{i-\delta-1}\, \varepsilon_i\right).$$

Therefore considering $f_{\kappa+\rho}$ in (5), the inequality may occur only by carry propagation when the digits of $\ell q^\kappa + k_1$ of indexes $\kappa, \ldots, \kappa + \rho - 1$ are equal to 1, *i.e.* for integers ℓ with $\gg 2^\rho$ least significant digits equal to 1. It follows that f satisfies Definition 1. For $\delta + 2 \leq \mu_1 < \mu_2$, by (55) we have

$$f_{\mu_1,\mu_2} = f_{\mu_2} \overline{f_{\mu_1}} = \mathrm{e}\left(\alpha \sum_{\mu_1 \leq i < \mu_2} \varepsilon_{i-\delta-1}\, \varepsilon_i\right).$$

It follows that $f_{\mu_1,\mu_2}(n)$ depends only on the digits of n of index $\mu_1 - \delta - 1, \mu_1, \ldots, \mu_2 - 1$. Therefore in (59) we can choose any value of μ_0 at most equal to $\mu_1 - \delta - 1$ and (60) will be satisfied for any value of ρ' at most equal to ρ, which makes the choice (93) admissible. The aim of Proposition 3 is to show that for any $\alpha \in \mathbb{R}$, the function $n \mapsto \mathrm{e}(\alpha\beta_\delta(n))$ belongs to some $\mathcal{F}_{\gamma,c}$ in Definition 2 (observe that $\beta_\delta(2^\kappa n) = \beta_\delta(n)$ for any $\kappa \in \mathbb{N}$).

Proposition 3. *For any $\delta \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\vartheta \in \mathbb{R}$ and $\lambda \in \mathbb{N}$ we have*

$$\left| 2^{-\lambda} \sum_{0 \leq n < 2^\lambda} \mathrm{e}(\alpha\beta_\delta(n) + \vartheta n) \right| \leq 2^{\frac{\delta+1}{2}} \left(\frac{1 + |\cos \pi\alpha|}{2} \right)^{\lambda/2}.$$

Remark 3. *This is Theorem 3.1 of* [1], *but we will present here a direct proof.*

Proof. For $0 \leq i < 2^{\delta+1}$ we write

$$\Gamma^{[i]}(n) = \sum_{k=0}^{\delta} \varepsilon_k(n)\, \varepsilon_{\delta-k}(i), \quad S_\lambda^{[i]}(\alpha,\vartheta) = \sum_{0 \leq n < 2^\lambda} \mathrm{e}\left(\alpha\Gamma^{[i]}(n) + \alpha\beta_\delta(n) + \vartheta n\right)$$

and

$$S_\lambda(\alpha,\vartheta) = \begin{pmatrix} S_\lambda^{[0]}(\alpha,\vartheta) \\ \vdots \\ S_\lambda^{[2^{\delta+1}-1]}(\alpha,\vartheta) \end{pmatrix}.$$

If $0 \leq i < 2^\delta$, we have $\varepsilon_\delta(i) = 0$, so that for any $n \in \mathbb{N}$ and $\varepsilon \in \{0,1\}$

$$\begin{aligned} \Gamma^{[i]}(2n+\varepsilon) &= \sum_{k=0}^{\delta} \varepsilon_k(2n+\varepsilon)\, \varepsilon_{\delta-k}(i) = \sum_{k=1}^{\delta} \varepsilon_k(2n+\varepsilon)\, \varepsilon_{\delta-k}(i) = \sum_{k=1}^{\delta} \varepsilon_{k-1}(n)\, \varepsilon_{\delta-k+1}(2i) \\ &= \sum_{k=0}^{\delta-1} \varepsilon_k(n)\, \varepsilon_{\delta-k}(2i) = \sum_{k=0}^{\delta} \varepsilon_k(n)\, \varepsilon_{\delta-k}(2i) = \Gamma^{[2i]}(n) \end{aligned}$$

and $\varepsilon_\delta(2^\delta+i)=1$, so that

$$\begin{aligned}\Gamma^{[2^\delta+i]}(2n+\varepsilon) &= \sum_{k=0}^{\delta}\varepsilon_k(2n+\varepsilon)\,\varepsilon_{\delta-k}(2^\delta+i) = \varepsilon_0(2n+\varepsilon)\cdot 1+\sum_{k=1}^{\delta}\varepsilon_k(2n+\varepsilon)\,\varepsilon_{\delta-k}(2^\delta+i)\\ &= \varepsilon+\sum_{k=1}^{\delta}\varepsilon_{k-1}(n)\,\varepsilon_{\delta-k}(i) = \varepsilon+\sum_{k=1}^{\delta}\varepsilon_{k-1}(n)\,\varepsilon_{\delta-k+1}(2i) = \varepsilon+\Gamma^{[2i]}(n).\end{aligned}$$

Furthermore

$$\begin{aligned}\Gamma^{[2i+1]}(n) &= \sum_{k=0}^{\delta}\varepsilon_k(n)\,\varepsilon_{\delta-k}(2i+1) = \varepsilon_\delta(n)\varepsilon_0(2i+1)+\sum_{k=0}^{\delta-1}\varepsilon_k(n)\,\varepsilon_{\delta-k}(2i+1)\\ &= \varepsilon_\delta(n)+\sum_{k=0}^{\delta-1}\varepsilon_k(n)\,\varepsilon_{\delta-k}(2i) = \varepsilon_\delta(n)+\Gamma^{[2i]}(n).\end{aligned}$$

It follows from the definition of β_δ that

$$\beta_\delta(2n) = \sum_{k\geq\delta+1}\varepsilon_{k-\delta-1}(2n)\,\varepsilon_k(2n) = \varepsilon_0(2n)\,\varepsilon_{\delta+1}(2n)+\sum_{k\geq\delta+2}\varepsilon_{k-\delta-2}(n)\,\varepsilon_{k-1}(n) = \beta_\delta(n)$$

and

$$\begin{aligned}\beta_\delta(2n+1) &= \sum_{k\geq\delta+1}\varepsilon_{k-\delta-1}(2n+1)\,\varepsilon_k(2n+1)\\ &= \varepsilon_0(2n+1)\,\varepsilon_{\delta+1}(2n+1)+\sum_{k\geq\delta+2}\varepsilon_{k-\delta-2}(n)\,\varepsilon_{k-1}(n) = \varepsilon_\delta(n)+\beta_\delta(n),\end{aligned}$$

so that for any $i\in\{0,\dots,2^\delta-1\}$ and $\lambda\in\mathbb{N}$ we have

$$\begin{aligned}S_{\lambda+1}^{[i]}(\alpha,\vartheta) &= \sum_{0\leq n<2^{\lambda+1}}\mathrm{e}\left(\alpha\Gamma^{[i]}(n)+\alpha\beta_\delta(n)+\vartheta n\right)\\ &= \sum_{0\leq n<2^{\lambda}}\mathrm{e}\left(\alpha\Gamma^{[i]}(2n)+\alpha\beta_\delta(2n)+2n\vartheta\right)\\ &\quad+\sum_{0\leq n<2^{\lambda}}\mathrm{e}\left(\alpha\Gamma^{[i]}(2n+1)+\alpha\beta_\delta(2n+1)+(2n+1)\vartheta\right)\\ &= \sum_{0\leq n<2^{\lambda}}\mathrm{e}\left(\alpha\Gamma^{[2i]}(n)+\alpha\beta_\delta(n)+2n\vartheta\right)\\ &\quad+\mathrm{e}(\vartheta)\sum_{0\leq n<2^{\lambda}}\mathrm{e}\left(\alpha\Gamma^{[2i]}(n)+\alpha\varepsilon_\delta(n)+\alpha\beta_\delta(n)+2n\vartheta\right)\\ &= S_\lambda^{[2i]}(\alpha,2\vartheta)+\mathrm{e}(\vartheta)\,S_\lambda^{[2i+1]}(\alpha,2\vartheta)\end{aligned}$$

and

$$\begin{aligned} S_{\lambda+1}^{[2^\delta+i]}(\alpha,\vartheta) &= \sum_{0\le n<2^{\lambda+1}} \mathrm{e}\left(\alpha\Gamma^{[2^\delta+i]}(n)+\alpha\beta_\delta(n)+\vartheta n\right)\\ &= \sum_{0\le n<2^{\lambda}} \mathrm{e}\left(\alpha\Gamma^{[2^\delta+i]}(2n)+\alpha\beta_\delta(2n)+2n\vartheta\right)\\ &\quad + \sum_{0\le n<2^{\lambda}} \mathrm{e}\left(\alpha\Gamma^{[2^\delta+i]}(2n+1)+\alpha\beta_\delta(2n+1)+(2n+1)\vartheta\right)\\ &= \sum_{0\le n<2^{\lambda}} \mathrm{e}\left(\alpha\Gamma^{[2i]}(n)+\alpha\beta_\delta(n)+2n\vartheta\right)\\ &\quad + \mathrm{e}(\vartheta)\sum_{0\le n<2^{\lambda}} \mathrm{e}\left(\alpha+\alpha\Gamma^{[2i]}(n)+\alpha\varepsilon_\delta(n)+\alpha\beta_\delta(n)+2n\vartheta\right)\\ &= S_\lambda^{[2i]}(\alpha,2\vartheta)+\mathrm{e}(\alpha+\vartheta)\,S_\lambda^{[2i+1]}(\alpha,2\vartheta). \end{aligned}$$

This yields

$$S_{\lambda+1}(\alpha,\vartheta)=M(\alpha,\vartheta)S_\lambda(\alpha,2\vartheta), \tag{98}$$

where $M(\alpha,\vartheta)$ denotes the $2^{\delta+1}\times 2^{\delta+1}$ matrix

$$M(\alpha,\vartheta)=\begin{pmatrix} 1 & \mathrm{e}(\vartheta) & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ 0 & 0 & 1 & \mathrm{e}(\vartheta) & 0 & \cdots & \cdots & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \mathrm{e}(\vartheta) & 0 & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \mathrm{e}(\vartheta)\\ 1 & \mathrm{e}(\alpha+\vartheta) & 0 & & & \cdots & & & 0\\ 0 & 0 & 1 & \mathrm{e}(\alpha+\vartheta) & 0 & \cdots & \cdots & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \mathrm{e}(\alpha+\vartheta) & 0 & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \mathrm{e}(\alpha+\vartheta) \end{pmatrix}.$$

Writing

$$A(\alpha,\vartheta)=\begin{pmatrix} 2 & \mathrm{e}(\vartheta)+\mathrm{e}(\alpha+\vartheta)\\ \mathrm{e}(-\vartheta)+\mathrm{e}(-\alpha-\vartheta) & 2 \end{pmatrix}$$

we observe that ${}^t\overline{M(\alpha,\vartheta)}M(\alpha,\vartheta)$ is the block matrix

$${}^t\overline{M(\alpha,\vartheta)}M(\alpha,\vartheta)=\begin{pmatrix} A(\alpha,\vartheta) & & 0\\ & \ddots & \\ 0 & & A(\alpha,\vartheta) \end{pmatrix}.$$

Denoting by $\rho(A(\alpha,\vartheta))$ the spectral radius of $A(\alpha,\vartheta)$, it follows that

$$\|M(\alpha,\vartheta)\|_2=\sqrt{\rho(A(\alpha,\vartheta))}=\sqrt{2+|\mathrm{e}(\vartheta)+\mathrm{e}(\alpha+\vartheta)|}=\sqrt{2(1+|\cos\pi\alpha|)}.$$

By (98) this permits to write

$$\|S_{\lambda+1}(\alpha,\vartheta)\|_2\le\|M(\alpha,\vartheta)\|_2\,\|S_\lambda(\alpha,2\vartheta)\|_2\le\sqrt{2(1+|\cos\pi\alpha|)}\,\|S_\lambda(\alpha,2\vartheta)\|_2\,.$$

By induction we get

$$\begin{aligned} \left|\sum_{0\le n<2^\lambda}\mathrm{e}(\alpha\beta_\delta(n)+\vartheta n)\right| = \left|S_\lambda^{[0]}(\alpha,\vartheta)\right| &\le \|S_\lambda(\alpha,\vartheta)\|_2\\ &\le (2+2\,|\cos\pi\alpha|)^{\lambda/2}\left\|S_0(\alpha,2^\lambda\vartheta)\right\|_2\\ &= 2^{\frac{\delta+1}{2}}(2+2\,|\cos\pi\alpha|)^{\lambda/2}, \end{aligned}$$

which gives Theorem 3. □

Observing that $\beta_\delta(u2^\kappa)=\beta_\delta(u)$, Proposition 3 shows that $f(n)=\mathrm{e}\left(\beta_\delta(n)\alpha\right)$ belongs to $\mathcal{F}_{\gamma,c}$ in Definition 2 for any $c>0$ and

$$\gamma(\lambda)=-\frac{\lambda}{2\log 2}\log\left(\frac{1+|\cos\pi\alpha|}{2}\right)-\frac{\delta+1}{2}. \tag{99}$$

Applying Theorem 1 and Theorem 2 we obtain

Theorem 3. *For any $\delta\in\mathbb{N}$, $\alpha\in\mathbb{R}$, $\vartheta\in\mathbb{R}$ and $x\geq 2$ we have*

$$\left|\sum_{n\leq x}\Lambda(n)\,\mathrm{e}\left(\beta_\delta(n)\alpha+\vartheta n\right)\right|\ll x\left(\log x\right)^{\frac{11}{4}}\,2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20} \tag{100}$$

and

$$\left|\sum_{n\leq x}\mu(n)\,\mathrm{e}\left(\beta_\delta(n)\alpha+\vartheta n\right)\right|\ll x\left(\log x\right)^{\frac{11}{4}}\,2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20}, \tag{101}$$

where γ is defined by (99).

11.2. **Rudin-Shapiro sequences of degree d.**

Let $d\in\mathbb{N}$, $d\geq 2$ and $b_d(n)$ the number of occurencies of blocks of d consecutive 1's in the representation of n in base 2:

$$b_d(n)=\sum_{k\geq d-1}\varepsilon_{k-d+1}(n)\cdots\varepsilon_k(n).$$

For $\alpha\in\mathbb{R}$ we consider in this section $f(n)=\mathrm{e}\left(b_d(n)\alpha\right)$. By (4) for all $\lambda\geq d$ we have

$$f_\lambda=\mathrm{e}\left(\alpha\sum_{d-1\leq i<\lambda}\varepsilon_{i-d+1}\cdots\varepsilon_{i-1}\varepsilon_i\right).$$

Therefore considering $f_{\kappa+\rho}$ in (5), the inequality may occur only by carry propagation when the digits of $\ell 2^\kappa+k_1$ of indexes $\kappa,\ldots,\kappa+\rho-1$ are equal to 1, *i.e.* for integers ℓ with $\gg 2^\rho$ least significant digits equal to 1. It follows that f satisfies Definition 1. For $d\leq\mu_1<\mu_2$, by (55) we have

$$f_{\mu_1,\mu_2}=f_{\mu_2}\overline{f_{\mu_1}}=\mathrm{e}\left(\alpha\sum_{\mu_1\leq i<\mu_2}\varepsilon_{i-d+1}\cdots\varepsilon_{i-1}\varepsilon_i\right).$$

It follows that $f_{\mu_1,\mu_2}(n)$ depends only on the digits of n of index $\mu_1-d+1,\ldots,\mu_2-1$. Given any ρ' satisfying $(d-1)/2\leq\rho'\leq\rho$ (which implies (58)), we can choose $\mu_0=\mu_1-d+1$ so that (59) and (60) are satisfied. This makes the choice (93) admissible.

The aim of Proposition 4 is to show that for any $\alpha\in\mathbb{R}$, the function $n\mapsto\mathrm{e}(\alpha b_d(n))$ belongs to some $\mathcal{F}_{\gamma,c}$ in Definition 2 (observe that $b_d(2^\kappa n)=b_d(n)$ for any $\kappa\in\mathbb{N}$).

Proposition 4. *For any $d\geq 2$, $\alpha\in\mathbb{R}$, $\vartheta\in\mathbb{R}$ and $\lambda\in\mathbb{N}$ we have*

$$\left|2^{-\lambda}\sum_{0\leq n<2^\lambda}\mathrm{e}\left(b_d(n)\alpha+n\vartheta\right)\right|\leq\left(1-2^{3-d}\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2\right)^{\lfloor\lambda/d\rfloor}.$$

Proof. For $k\in\mathbb{N}$ we define $\chi_k:\mathbb{N}\to\{0,1\}$ by $\chi_k(n)=1$ if the k least significant digits of n are 1's and $\chi_k(n)=0$ otherwise. This means that $\chi_0=1$ and for $k\geq 1$, $\chi_k(n)=\varepsilon_{k-1}(n)\cdots\varepsilon_0(n)$. In particular $\chi_k(n)=0$ for $k\geq 1$ and $n<2^{k-1}$.

For $n\in\mathbb{N}$ we define

$$\chi_d^{[1]}(n)=0$$

and for $i\in\{2,\ldots,d\}$,

$$\chi_d^{[i]}(n)=\chi_{d-i+1}(n)+\cdots+\chi_{d-1}(n).$$

We define

$$S_\lambda^{[i]}(\alpha,\vartheta) = \sum_{0\le n<2^\lambda} \mathrm{e}\left(\chi_d^{[i]}(n)\alpha + b_d(n)\alpha + n\vartheta\right)$$

and notice that we are interested by $\left|S_\lambda^{[1]}(\alpha,\vartheta)\right|$. For $n\in\mathbb{N}$ we have

$$b_d(2n) = b_d(n),\ b_d(2n+1) = b_d(n) + \chi_{d-1}(n)$$

and for $k\ge 1$,

$$\chi_k(2n) = 0,\ \chi_k(2n+1) = \chi_{k-1}(n),$$

so that for $i\in\{1,\dots,d\}$,

$$\chi_d^{[i]}(2n) = 0 = \chi_d^{[1]}(n), \quad \chi_d^{[1]}(2n+1) = 0 = \chi_d^{[2]}(n) - \chi_{d-1}(n)$$

and for $i\in\{2,\dots,d-1\}$

$$\chi_d^{[i]}(2n+1) = \chi_{d-i}(n) + \cdots + \chi_{d-2}(n) = \chi_d^{[i+1]}(n) - \chi_{d-1}(n),$$

while

$$\chi_d^{[d]}(2n+1) = \chi_0(n) + \cdots + \chi_{d-2}(n) = 1 + \chi_d^{[d]}(n) - \chi_{d-1}(n).$$

It follows that for $i = 1,\dots,d-1$

$$S_{\lambda+1}^{[i]}(\alpha,\vartheta) = S_\lambda^{[1]}(\alpha,2\vartheta) + \mathrm{e}(\vartheta)\, S_\lambda^{[i+1]}(\alpha,2\vartheta)$$

and

$$S_{\lambda+1}^{[d]}(\alpha,\vartheta) = S_\lambda^{[1]}(\alpha,2\vartheta) + \mathrm{e}(\alpha+\vartheta)\, S_\lambda^{[d]}(\alpha,2\vartheta).$$

Let us introduce the $d\times d$ matrix $M(\alpha,\vartheta)$ and the vector $S_\lambda(\alpha,\vartheta)$ defined by

$$M(\alpha,\vartheta) = \begin{pmatrix} 1 & \mathrm{e}(\vartheta) & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \mathrm{e}(\vartheta) & \cdots & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 1 & 0 & & 0 & \ddots & 0 \\ 1 & 0 & & & 0 & \mathrm{e}(\vartheta) \\ 1 & 0 & & & 0 & \mathrm{e}(\alpha+\vartheta) \end{pmatrix}, \quad S_\lambda(\alpha,\vartheta) = \begin{pmatrix} S_\lambda^{[1]}(\alpha,\vartheta) \\ \vdots \\ S_\lambda^{[d]}(\alpha,\vartheta) \end{pmatrix}.$$

We have

$$S_{\lambda+1}(\alpha,\vartheta) = M(\alpha,\vartheta)\, S_\lambda(\alpha,2\vartheta),$$

and writing

$$M^{[\lambda]}(\alpha,\vartheta) = M(\alpha,\vartheta)M(\alpha,2\vartheta)\cdots M(\alpha,2^{\lambda-1}\vartheta),$$

we get

$$S_\lambda(\alpha,\vartheta) = M^{[\lambda]}(\alpha,\vartheta)\, S_0(\alpha,2^\lambda\vartheta).$$

Therefore

$$\left|S_\lambda^{[1]}(\alpha,\vartheta)\right| \le \|S_\lambda(\alpha,\vartheta)\|_\infty \le \left\|M^{[\lambda]}(\alpha,\vartheta)\right\|_\infty,$$

so that Proposition 4 follows from the following Lemma. □

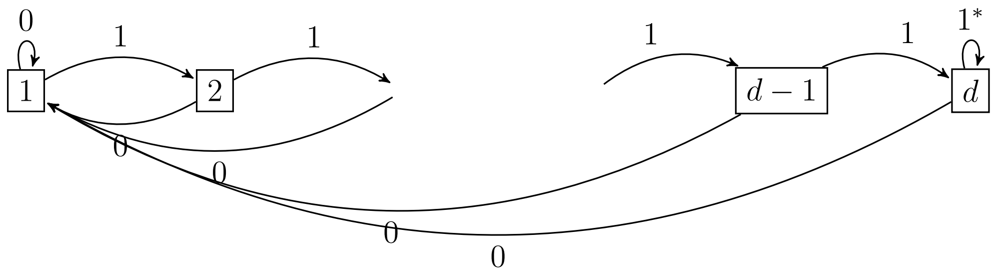
Lemma 12. *We have*

$$\left\|M^{[d]}(\alpha,\vartheta)\right\|_\infty \le 2^d - 8\left(\sin\frac{\pi\|\alpha\|}{4}\right)^2.$$

Proof. Let us first observe that $(M^{[d]}(0,0))_{i,j} = ((M(0,0))^d)_{i,j} = \max(2, 2^{d-j})$: indeed it easy to show by induction on k that for $1 \leq k \leq d-1$,

$$(M(0,0))^k = \begin{pmatrix} 2^{k-1} & 2^{k-2} & \dots & 2^0 & 1 & 0 & \dots & \dots & 0 \\ 2^{k-1} & 2^{k-2} & & 2^0 & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \ddots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots \\ 2^{k-1} & 2^{k-2} & \dots & 2^0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

We remark that for any $(i,j) \in \{1,\dots,d\}^2$ the coefficient $(M^{[d]}(\alpha,\vartheta))_{i,j}$ is a sum of complex numbers of modulus 1. The number of them is precisely $(M^{[d]}(0,0))_{i,j} = \max(2, 2^{d-j})$ while the argument of each of them being the coding of a path of length d going from the vertex $\boxed{\text{i}}$ to the vertex $\boxed{\text{j}}$ in the following graph:



The coding (given by the rules of matrix product) is the following: for any path of length d from $\boxed{\text{i}}$ to $\boxed{\text{j}}$ and for any $t \in \{1,\dots,d\}$:

- crossing an arc labeled by 0 at step t adds 0 to the argument;
- crossing an arc labeled by 1 at step t adds $2^{t-1}\vartheta$ to the argument;
- crossing an arc labeled by 1^* at step t adds $2^{t-1}\vartheta + \alpha$ to the argument.

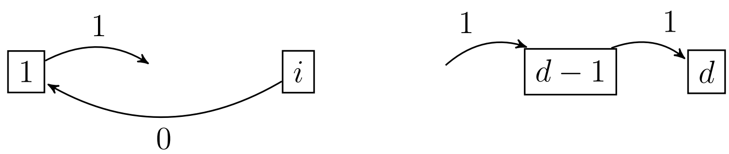
For any $i \in \{1,\dots,d\}$ there are exactly two paths of length d going from the vertex $\boxed{\text{i}}$ to the vertex $\boxed{\text{d}}$:

(1) the path

that corresponds to the coding

$$\vartheta + 2\vartheta + \cdots + 2^{d-i-1}\vartheta + (2^{d-i}\vartheta + \alpha) + \cdots + (2^{d-1}\vartheta + \alpha) = (2^d - 1)\vartheta + i\alpha;$$

(2) the path



that corresponds to the coding

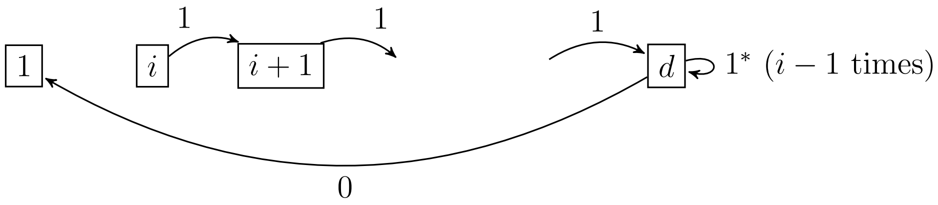
$$\vartheta + 2\vartheta + \cdots + 2^{d-1}\vartheta = (2^d - 2)\vartheta.$$

It follows that for any $i \in \{1,\dots,d\}$ we have

$$(M^{[d]}(\alpha,\vartheta))_{i,d} = \mathrm{e}\left((2^d - 2)\vartheta\right) + \mathrm{e}\left((2^d - 1)\vartheta + i\alpha\right). \tag{102}$$

For any $i \in \{1, \ldots, d\}$ there are exactly 2^{d-1} paths of length d going from the vertex $\boxed{i}$ to the vertex $\boxed{1}$ among which we have the two following ones:

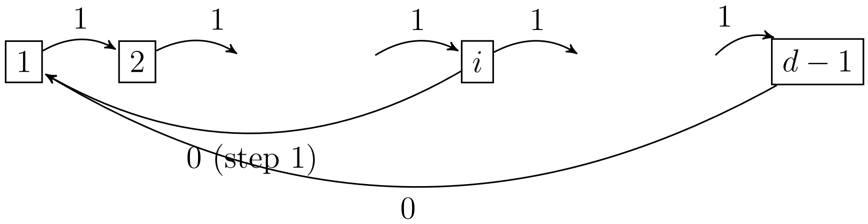
(1) the path



that corresponds to the coding

$$\vartheta + 2\vartheta + \cdots + 2^{d-i-1}\vartheta + (2^{d-i}\vartheta + \alpha) + \cdots + (2^{d-2}\vartheta + \alpha) + 0 = (2^{d-1} - 1)\vartheta + (i-1)\alpha;$$

(2) the path



that corresponds to the coding

$$0 + 2\vartheta + \cdots + 2^{d-2}\vartheta + 0 = (2^{d-1} - 2)\vartheta.$$

It follows that for any $i \in \{1, \ldots, d\}$ we have

$$(M^{[d]}(\alpha,\vartheta))_{i,1} = \mathrm{e}\left((2^{d-1}-2)\vartheta\right) + \mathrm{e}\left((2^{d-1}-1)\vartheta + (i-1)\alpha\right) + (2^{d-1}-2) \text{ terms of modulus } 1. \tag{103}$$

It follows from (102) and (103) that for any $i \in \{1, \ldots, d\}$ we have

$$\begin{aligned}\sum_{j=1}^{d} \left|(M^{[d]}(\alpha,\vartheta))_{i,j}\right| &\le \sum_{j=2}^{d-1} 2^{d-j} + (2^{d-1}-2) \\ &\quad + \left|\mathrm{e}\left((2^{d-1}-2)\vartheta\right) + \mathrm{e}\left((2^{d-1}-1)\vartheta + (i-1)\alpha\right)\right| \\ &\quad + \left|\mathrm{e}\left((2^{d}-2)\vartheta\right) + \mathrm{e}\left((2^{d}-1)\vartheta + i\alpha\right)\right|,\end{aligned}$$

which implies

$$\sum_{j=1}^{d} \left|(M^{[d]}(\alpha,\vartheta))_{i,j}\right| \le 2^d - 4 + 2\left|\cos\pi\left(\vartheta + (i-1)\alpha\right)\right| + 2\left|\cos\pi\left(\vartheta + i\alpha\right)\right|.$$

For any $x \in \mathbb{R}$ we have

$$|\cos\pi x| + |\cos\pi(x+\alpha)| = |\cos\pi x + \cos\pi(x+\alpha')| \le \left|e^{i\pi x} + e^{i\pi(x+\alpha')}\right| = 2\left|\cos\tfrac{\pi\alpha'}{2}\right|$$

with $\alpha' = \alpha$ if both cosinus have the same sign and $\alpha' = \alpha + 1$ otherwise. Hence

$$|\cos\pi x| + |\cos\pi(x+\alpha)| \le 2\max\left(\left|\cos\tfrac{\pi\alpha}{2}\right|, \left|\sin\tfrac{\pi\alpha}{2}\right|\right)$$

and observing that $\alpha = n \pm \|\alpha\|$ with $n \in \mathbb{Z}$, and $0 \le \|\alpha\| \le \frac{1}{2}$, we get

$$\max\left(\left|\cos\tfrac{\pi\alpha}{2}\right|, \left|\sin\tfrac{\pi\alpha}{2}\right|\right) = \max\left(\cos\tfrac{\pi\|\alpha\|}{2}, \sin\tfrac{\pi\|\alpha\|}{2}\right) = \cos\tfrac{\pi\|\alpha\|}{2} = 1 - 2\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2.$$

Taking $x = \vartheta + (i-1)\alpha$ we obtain

$$\sum_{j=1}^{d} \left|(M^{[d]}(\alpha,\vartheta))_{i,j}\right| \le 2^d - 8\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2.$$

which completes the proof of Lemma 12. □

We will now prove Proposition 4. We have

$$\left|2^{-\lambda}\sum_{0\le n<2^\lambda}\mathrm{e}\left(b_d(n)\alpha+n\vartheta\right)\right| = 2^{-\lambda}\left|S_\lambda^{[1]}(\alpha,\vartheta)\right| \le 2^{-\lambda}\left\|M^{[\lambda]}(\alpha,\vartheta)\right\|_\infty$$

and

$$\begin{aligned} M^{[\lambda]}(\alpha,\vartheta) &= M(\alpha,\vartheta)M(\alpha,2\vartheta)\cdots M(\alpha,2^{\lambda-1}\vartheta)\\ &= M^{[d]}(\alpha,\vartheta)M^{[d]}(\alpha,2^d\vartheta)\cdots M^{[d]}(\alpha,2^{d\lfloor\lambda/d\rfloor}\vartheta)\\ &\qquad M(\alpha,2^{d\lfloor\lambda/d\rfloor}\vartheta)\cdots M(\alpha,2^{\lambda-1}\vartheta).\end{aligned}$$

Using the sub-multiplicativity of the matrix norm, the bound $\left\|M(\alpha,\vartheta')\right\|_\infty = 2$ and Lemma 12, we get

$$\left\|M^{[\lambda]}(\alpha,\vartheta)\right\|_\infty \le 2^{\lambda-d\lfloor\lambda/d\rfloor}\left(2^d-8\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2\right)^{\lfloor\lambda/d\rfloor},$$

so that

$$\left|2^{-\lambda}\sum_{0\le n<2^\lambda}\mathrm{e}\left(b_d(n)\alpha+n\vartheta\right)\right| \le \left(1-2^{3-d}\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2\right)^{\lfloor\lambda/d\rfloor},$$

which completes the proof of Proposition 4. Taking into account that $b_d(n2^k)=b_d(n)$ for all $k\in\mathbb{N}$, $d\ge 2$ and $0\le\|\alpha\|\le 1/2$, it follows from Proposition 4 that for $f(n)=\mathrm{e}\left(b_d(n)\alpha\right)$ and for all $k\in\mathbb{N}$ and all $\vartheta\in\mathbb{R}$,

$$\begin{aligned}\left|2^{-\lambda}\sum_{0\le u<2^\lambda}f(u2^k)\,\mathrm{e}\left(u\vartheta\right)\right| &\le \left(1-2^{3-d}\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2\right)^{\lfloor\lambda/d\rfloor}\\ &\le \left(1-2^{3-d}\left(\sin\tfrac{\pi}{8}\right)^2\right)^{-1}\left(1-2^{3-d}\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2\right)^{\lambda/d},\\ &\le \sqrt{2}\left(1-2^{3-d}\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2\right)^{\lambda/d}.\end{aligned}$$

It follows that f belongs to $\mathcal{F}_{\gamma,c}$ in Definition 2 for all $c>0$ and

$$\gamma(\lambda)=\frac{-\lambda}{d\ \log 2}\log\left(1-2^{3-d}\left(\sin\tfrac{\pi\|\alpha\|}{4}\right)^2\right)-\frac{1}{2}. \tag{104}$$

Applying Theorem 1 and Theorem 2 we obtain

Theorem 4. *For $d\ge 2$, $\alpha\in\mathbb{R}$, $\vartheta\in\mathbb{R}$ and $x\ge 2$ we have*

$$\left|\sum_{n\le x}\Lambda(n)\,\mathrm{e}\left(b_d(n)\alpha+\vartheta n\right)\right| \ll x\left(\log x\right)^{\frac{11}{4}}\,2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20} \tag{105}$$

and

$$\left|\sum_{n\le x}\mu(n)\,\mathrm{e}\left(b_d(n)\alpha+\vartheta n\right)\right| \ll x\left(\log x\right)^{\frac{11}{4}}\,2^{-\gamma(2\lfloor(\log x)/80\log 2\rfloor)/20}, \tag{106}$$

where γ is defined by (104).

Acknowledgements. We thank Guy Barat for helpful comments concerning this work.

References

[1] J.-P. Allouche and P. Liardet, *Generalized Rudin-Shapiro sequences*, Acta Arith., 60 (1991), pp. 1–27.

[2] P. T. Bateman and H. G. Diamond, *Analytic number theory*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2004. An introductory course.

[3] J. Bourgain, *Moebius-Walsh correlation bounds and an estimate of Mauduit and Rivat*, Journal d'Analyse, (to appear).

[4] ——, *Monotone boolean functions capture their primes*, Journal d'Analyse, (to appear).

[5] ——, *On the Fourier-Walsh Spectrum on the Moebius Function*, Israel J. Math, (to appear).

[6] J. Bourgain, P. Sarnak., and T. Ziegler, *Disjointness of Mobius from horocycle flow*, in From Fourier Analysis and Number Theory to Radon Transforms and Geometry, vol. 28 of Developments in Mathematics, Springer-Verlag, 2013, pp. 67–83.

[7] C. Dartyge and G. Tenenbaum, *Sommes des chiffres de multiples d'entiers*, Ann. Inst. Fourier, 55 (2005), pp. 2423–2474.

[8] H. Davenport, *On some infinite series involving arithmetical functions. II.*, Q. J. Math., Oxf. Ser., 8 (1937), pp. 313–320.

[9] M. Drmota, C. Mauduit, and J. Rivat, *Primes with an Average Sum of Digits*, Compositio, 145 (2009), pp. 271–292.

[10] A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith., 13 (1967/1968), pp. 259–265.

[11] B. Green, *On (not) computing the Mobius function using bounded depth circuits*, Combinatorics, Probability and Computing, 21 (2012), pp. 942–951.

[12] B. Green and T. Tao, *The Möbius function is strongly orthogonal to nilsequences*, Ann. of Math. (2), 175 (2012), pp. 541–566.

[13] H. Iwaniec and E. Kowalski, *Analytic number theory*, vol. 53 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2004.

[14] G. Kalai, *The AC0 Prime Number Conjecture*, `http://gilkalai.wordpress.com/2011/02/21/the-ac0-prime-number-conjecture/`, (2011).

[15] ——, *Walsh Fourier Transform of the Möbius function*, `http://mathoverflow.net/questions/57543/walsh-fourier-transform-of-the-mobius-function`, (2011).

[16] ——, *Möbius Randomness of the Rudin-Shapiro Sequence*, `http://mathoverflow.net/questions/97261/mobius-randomness-of-the-rudin-shapiro-sequence`, (2012).

[17] M. Keane, *Generalized Morse sequences*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 10 (1968), pp. 335–353.

[18] B. Martin, C. Mauduit, and J. Rivat, *Théorème des nombres premiers pour les fonctions digitales*, soumis.

[19] C. Mauduit and J. Rivat, *La somme des chiffres des carrés*, Acta Mathematica, 203 (2009), pp. 107–148.

[20] ——, *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*, Annals of Mathematics, 171 (2010), pp. 1591–1646.

[21] H. L. Montgomery, *The analytic principle of the large sieve*, Bull. Amer. Math. Soc., 84 (1978), pp. 547–567.

[22] ——, *Ten lectures on the interface between analytic number theory and harmonic analysis*, vol. 84 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994.

[23] M. Queffelec, *Substitution Dynamical Systems – Spectral Analysis*, vol. 1294 of Lecture Notes in Math., Springer Verlag, New-York – Berlin, 1987.

[24] W. Rudin, *Some theorems on Fourier coefficients*, Proc. Amer. Math. Soc., 10 (1959), pp. 855–859.

[25] P. Sarnak, *Three Lectures on the Mobius function randomness and dynamics*, `http://publications.ias.edu/sarnak/paper/512`, (2011).

[26] H. S. Shapiro, *Extremal Problems for polynomials and Power Series*, PhD thesis, M.I.T., 1951.

[27] J. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc., 12 (1985), pp. 183–216.

E-mail address: `mauduit@iml.univ-mrs.fr`

E-mail address: `rivat@iml.univ-mrs.fr`

Institut de Mathématiques de Luminy, CNRS, Université d'Aix-Marseille, 163 avenue de Luminy, Case 907, 13288 MARSEILLE Cedex 9, France