THE TME-EMT PROJECT

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THE PDF NOTES

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Théorie Multiplicative Explicite des nombres

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Explicit Multiplicative number Theory

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November 5, 2020
Introduction

Fully explicit results in multiplicative number theory are often scattered through the literature. The aim of this site is to present an annotated bibliography in order to keep track of the current knowledge. By the way, the acronym TME-EMT stands for

\[
\text{Théorie Multiplicative Explicite des nombres}
\]

\[
\text{Explicit Multiplicative number Theory}
\]

Being up-to date is a difficulty, so please do not consider this site as presenting the best available, but as a first line on which to strengthen more specialized inquiries.
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### Bibliography

Notation

Notation is standard, except maybe for the following one: we write \( f = O^*(g) \) to say that \(|f| \leq g\). This is simply a Landau-bigO symbol with an implied constant equal to one. Furthermore, the letter \( p \) always denotes a prime variable.

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*The TME-EMT project, 2018*  
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Part I

Averages of arithmetical functions
Chapter 1

Explicit bounds on primes

Corresponding html file: ../Articles/Art01.html
Collecting references: [P. Dusart, 56], [P. Dusart, 61],

1.1 Bounds on primes, in special ranges

The paper [J. Rosser and L. Schoenfeld, 178], contains several bounds valid only when the variable is small enough.

In [J. Büthe, 25], the author proves the next theorem.

**Theorem (2016)** Assume the Riemann Hypothesis has been checked up to height $H_0$. Then when $x$ satisfies $\sqrt{x}/\log x \leq H_0/4.92$, we have

- $|\psi(x) - x| \leq \frac{\sqrt{x}}{8\pi} \log^2 x$ when $x > 59$,
- $|\theta(x) - x| \leq \frac{\sqrt{x}}{8\pi} \log^2 x$ when $x > 599$,
- $|\pi(x) - \text{li}(x)| \leq \frac{\sqrt{x}}{8\pi} \log x$ when $x > 2657$.

If we use the value $H_0 = 30610046000$ obtained by D. Platt in [D. J. Platt, 150], these bounds are thus valid for $x \leq 1.8 \cdot 10^{21}$.

In [J. Büthe, 24], the following bounds are also obtained.

**Theorem (2018)** We have

- $|\psi(x) - x| \leq 0.94\sqrt{x}$ when $11 < x \leq 10^{19}$,
- $0 < \text{li}(x) - \pi(x) \leq \frac{\sqrt{x}}{\log x} \left(1.95 + \frac{3.9}{\log x} + \frac{19.5}{\log^2 x}\right)$ when $2 \leq x \leq 10^{19}$.

[61] P. Dusart, 2016, “Estimates of $\psi, \theta$ for large values of $x$ without the Riemann hypothesis”.
1.2 Bounds on primes, without any congruence condition

The subject really started with the four papers [J. Rosser, 174], [J. Rosser and L. Schoenfeld, 178], [J. Rosser and L. Schoenfeld, 179] and [L. Schoenfeld, 184]. We recall the usual notation: \( \pi(x) \) is the number of primes up to \( x \) (so that \( \pi(3) = 2 \)), the function \( \psi(x) \) is the summatory function of the van Mangold function \( \Lambda \), i.e. \( \psi(x) = \sum_{n \leq x} \Lambda(n) \), while we also define \( \vartheta(x) = \sum_{p \leq x} \log p \).

Here are some elegant bounds that one can find in these papers.

**Theorem (1962)**

- For \( x > 0 \), we have \( \psi(x) \leq 1.03883x \) and the maximum of \( \psi(x)/x \) is attained at \( x = 113 \).
- When \( x \geq 17 \), we have \( \pi(x) > x/\log x \).
- When \( x > 1 \), we have \( \sum_{p \leq x} 1/p > \log \log x \).
- When \( x > 1 \), we have \( \sum_{p \leq x} (\log p)/p < \log x \).

There are many other results in these papers. In [P. Dusart, 59], on can find among other things the inequality

\[
\text{When } x \geq 17, \text{ we have } \pi(x) > \frac{x}{\log x - 1}.
\]

And also

**Theorem (1999)**

- When \( x \geq e^{22} \), we have \( \psi(x) = x + O^* \left( 0.006409 \frac{x}{\log x} \right) \).
- When \( x \geq 10544111 \), we have \( \vartheta(x) = x + O^* \left( 0.006788 \frac{x}{\log x} \right) \).
- When \( x \geq 3594641 \), we have \( \vartheta(x) = x + O^* \left( 0.2 \frac{x}{\log^2 x} \right) \).
- When \( x > 1 \), we have \( \vartheta(x) = x + O^* \left( 515 \frac{x}{\log^2 x} \right) \).

[179] J. Rosser and L. Schoenfeld, 1975, “Sharper bounds for the Chebyshev Functions \( \theta(X) \) and \( \psi(X) \)”.
[184] L. Schoenfeld, 1976, “Sharper bounds for the Chebyshev Functions \( \theta(X) \) and \( \psi(X) \)” II“.
[59] P. Dusart, 1999, “Inégalités explicites pour \( \psi(X) \), \( \theta(X) \), \( \pi(X) \) et les nombres premiers”.

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This is improved in [P. Dusart, 58], and in particular, it is shown that the 515 above can be replaced by 20.83 and also that

When \( x \geq 89967803 \), we have \( \vartheta(x) = x + O^*(\frac{x}{\log^2 x}) \).

Bounds of the shape \(|\psi(x) - x| \leq \epsilon x\) have started appearing in [J. Rosser and L. Schoenfeld, 178]. The latest paper is [L. Faber and H. Kadiri, 64] with its corrigendum [Kadiri-Faber*18], where the explicit density estimate from [H. Kadiri, 92] is put to contribution, even for moderate values of the variable. In particular

When \( x \geq 485165196 \), we have \( \psi(x) = x + O^*(0.00053699 x) \).

Refined bounds for \( \pi(x) \) are to be found in [L. Panaitopol, 146] and in [C. Axler, 2].

By comparing in an efficient manner with \( \psi(x) - x \), [O. Ramaré, 161], obtained the next two results.

**Theorem (2013)** For \( x > 1 \), we have \( \sum_{n \leq x} \Lambda(n)/n = \log x - \gamma + O^*(1.833/\log^2 x) \).

When \( x \geq 23 \), we can replace the error term by \( O^*(0.0067/\log x) \). Furthermore, when \( 1 \leq x \leq 10^{10} \), this error term can be replaced by \( O^*(1.31/\sqrt{x}) \).

**Theorem (2013)** For \( x \geq 8950 \), we have

\[
\sum_{n \leq x} \Lambda(n)/n = \log x - \gamma + \frac{\psi(x) - x}{x} + O^*\left(\frac{1}{2\sqrt{x}} + 1.75 \cdot 10^{-12}\right)
\]

[R. Mawia, 126] developed the method to incorporate more functions (and corrected the initial work), extending results of [J. Rosser and L. Schoenfeld, 178].

Here are some of his results.

**Theorem (2017)** For \( x \geq 2 \), we have

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + B + O^*\left(\frac{4}{\log^2 x}\right).
\]

When \( x \geq 1000 \), one can replace the 4 in the error term by 2.3, and when \( x \geq 24284 \), by 1. The constant \( B \) is the expected one.

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**References**

[58] P. Dusart, 2018, “Estimates of some functions over primes”.
[64] L. Faber and H. Kadiri, 2015, “New bounds for \( \psi(x) \)”.
[146] L. Panaitopol, 2000, “A formula for \( \pi(x) \) applied to a result of Koninck-Ivić”.
[161] O. Ramaré, 2013, “Explicit estimates for the summatory function of \( \Lambda(n)/n \) from the one of \( \Lambda(n) \)”.

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Theorem (2017) For \( \log x \geq 4635 \), we have

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + B + O^* \left( 1.1 \exp \left( -\sqrt{0.175 \log x} \right) \frac{\log x}{(\log x)^{3/4}} \right).
\]

When truncating sums over primes, Lemma 3.2 of [O. Ramaré, 156] is handy.

Theorem (2016) Let \( f \) be a \( C^1 \) non-negative, non-increasing function over \([P, \infty]\), where \( P \geq 3\,600\,000 \) is a real number and such that \( \lim_{t \to \infty} tf(t) = 0 \). We have

\[
\sum_{p \geq P} f(p) \log p \leq (1 + \epsilon) \int_P^\infty f(t)dt + \epsilon Pf(P) + Pf(P)/(5 \log^2 P)
\]

with \( \epsilon = 1/914 \). When we can only ensure \( P \geq 2 \), then a similar inequality holds, simply replacing the last \( 1/5 \) by a \( 4 \).

The above result relies on (5.1*) of [L. Schoenfeld, 184] because it is easily accessible. However on using Proposition 5.1 of [P. Dusart, 61], one has access to \( \epsilon = 1/36260 \).

Here is a result due to [E. Treviño, 194].

Theorem (2012) For \( x \) a positive real number. If \( x \geq x_0 \) then there exist \( c_1 \) and \( c_2 \) depending on \( x_0 \) such that

\[
\frac{x^2}{2 \log x} + \frac{c_1 x^2}{\log^2 x} \leq \sum_{p \leq x} p \leq \frac{x^2}{2 \log x} + \frac{c_2 x^2}{\log^2 x}.
\]

The constants \( c_1 \) and \( c_2 \) are given for various values of \( x_0 \) in the next table.

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<th>( x_0 )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
</tr>
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<td>1054411</td>
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<td>0.29251</td>
</tr>
</tbody>
</table>

[184] L. Schoenfeld, 1976, “Sharper bounds for the Chebyshev Functions \( \theta(X) \) and \( \psi(X) \) II”.
[61] P. Dusart, 2016, “Estimates of \( \psi, \theta \) for large values of \( x \) without the Riemann hypothesis”.

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1.3 Bounds on the n-th prime

Denote by $p_n$ the $n$-th prime. Thus $p_1 = 2$, $p_2 = 3$, $p_4 = 5$, $\cdots$.

The classical form of prime number theorem yields easily $p_n \sim n \log n$.

[J. Rosser, 177] shows that this equivalence does not oscillate by proving that $p_n$ is greater than $n \log n$ for $n \geq 2$.

The asymptotic formula for $p_n$ can be developed as shown in [M. Cipolla, 31]:

$$p_n \sim n \left( \log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} - \frac{(\ln \ln n)^2 - 6 \log \log n + 11}{2 \log^2 n} + \cdots \right).$$

This asymptotic expansion is the inverse of the logarithmic integral $\text{Li}(x)$ obtained by series reversion.

But [J. Rosser, 177] also proved that for every $n > 1$:

$$n(\log n + \log \log n - 10) < p_n < n(\log n + \log \log n + 8).$$

He improves these results in [J. Rosser, 174] : for every $n \geq 55$,

$$n(\log n + \log \log n - 4) < p_n < n(\log n + \log \log n + 2).$$

This result was subsequently improved by Rosser and Schoenfeld [J. Rosser and L. Schoenfeld, 178] in 1962 to

$$n(\log n + \log \log n - 3/2) < p_n < n(\log n + \log \log n - 1/2),$$

for $n > 1$ and $n > 19$ respectively.

The constants were subsequently reduced by [G. Robin, 172]. In particular, the lower bound

$$n(\log n + \log \log n - 1.0072629) < p_n$$

is valid for $n > 1$ and the constant 1.0072629 can be replaced by 1 for $p_k \leq 10^{11}$. Then [J.-P. Massias and G. Robin, 124] showed that the lower bound constant equals to 1 was admissible for $p_k < e^{508}$ and $p_k > e^{1800}$. Finally, [P. Dusart, 60] showed that

$$n(\log n - \log \log n - 1) < p_n$$

for all $n > 1$, and also that

$$p_n < n(\log n + \log \log n - 0.9484)$$

[177] J. Rosser, 1938, “The $n$-th prime is greater than $n \log n$”.
[31] M. Cipolla, 1902, “La determinazione assintotica dell’$n$-imo numero primo”.
[172] G. Robin, 1983, “Estimation de la fonction de Tchebychef $\theta$ sur le k-ième nombres premiers et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de $n$”.
[60] P. Dusart, 1999, “The $k$th prime is greater than $k(\ln k + \ln \ln k - 1)$ for $k \geq 2$”.

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for \( n > 39017 \) i.e. \( p_n > 467473 \).

In [E. Carneiro, M. Milinovich, and K. Soundararajan, 26], the authors prove the next result.

**Theorem (2019)** Under the Riemann Hypothesis we have \( p_{n+1} - p - n \leq \frac{22}{25} \sqrt{p_n} \log p_n \).

### 1.4 Bounds on primes in arithmetic progressions


A consequence of Theorem 1.1 and 1.2 of [M. A. Bennett et al., 11] states that

**Theorem (2018)** We have

\[
\max_{3 \leq q \leq 10^4, x \geq 8 \times 10^9, 1 \leq a \leq q, (a, q) = 1} \left| \frac{\log x}{x} \sum_{n \leq x, \ n \equiv a \ (q)} \Lambda(n) - \frac{x}{\varphi(q)} \right| \leq \frac{1}{840}.
\]

When \( q \in (10^4, 10^5] \), we may replace \( 1/840 \) by \( 1/160 \) and when \( q \geq 10^5 \), we may replace \( 1/840 \) by \( \exp(0.03 \sqrt{q \log 3} q) \).

Furthermore, we may replace \( \sum_{n \leq x, \ n \equiv a \ (q)} \Lambda(n) \) by \( \sum_{p \leq x, \ p \equiv a \ (q)} \log p \) with no changes in the constants.

Similarly, as a consequence of Theorem 1.3 of [M. A. Bennett et al., 11] states that

**Theorem (2018)** We have

\[
\max_{3 \leq q \leq 10^4, x \geq 8 \times 10^9, 1 \leq a \leq q, (a, q) = 1} \left| \frac{\log^2 x}{x} \sum_{p \leq x, \ p \equiv a \ (q)} 1 - \frac{\text{Li}(x)}{\varphi(q)} \right| \leq \frac{1}{840}.
\]

When \( q \in (10^4, 10^5] \), we may replace \( 1/840 \) by \( 1/160 \) and when \( q \geq 10^5 \), we may replace \( 1/840 \) by \( \exp(0.03 \sqrt{q \log 3} q) \).

Concerning estimates with a logarithmic density, in [O. Ramaré, 169] and

[127] K. McCurley, 1984, “Explicit estimates for \( \theta(x; 3, \ell) \) and \( \psi(x; 3, \ell) \)”.
[57] P. Dusart, 2002, “Estimates for \( \theta(x; k, \ell) \) for large values of \( x \)”.

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in [D. Platt and O. Ramaré, 149], estimates for the functions \( \sum_{n \leq x, \ n \equiv a[q]} \Lambda(n)/n \) are considered. Extending computations from the former, the latter paper Theorem 8.1 reads as follows.

**Theorem (2016)** We have \( \max_{q \leq 1000} \max_{x \leq 10^5} \max_{1 \leq a \leq q, (a,q)=1} \sqrt{x} \left| \sum_{n \leq x, \ n \equiv a[q]} \Lambda(n)/n - \frac{\log x}{\varphi(q)} - C(q,a) \right| \in (0.8533, 0.8534) \) and the maximum is attained with \( q = 17, x = 606 \) and \( a = 2 \).

The constant \( C(q,a) \) is the one expected, i.e. such that \( \sum_{n \leq x, \ n \equiv a[q]} \Lambda(n)/n - \frac{\log x}{\varphi(q)} - C(q,a) \) goes to zero when \( x \) goes to infinity.

When \( q \) belongs to "Rumely’s list", i.e. in one of the following set:

- \( \{ k \leq 72 \} \)
- \( \{ k \leq 112, k \text{ non premier} \} \)
- \( \{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, \)
  \( 169, 180, 216, 243, 256, 360, 420, 432 \} \)

Theorem 2 of [O. Ramaré, 169] gives the following.

**Theorem (2002)** When \( q \) belongs to Rumely’s list and \( a \) is prime to \( q \), we have

\[
\sum_{n \leq x, \ n \equiv a[q]} \frac{\Lambda(n)}{n} = \frac{\log x}{\varphi(q)} + C(q,a) + O^*(1) \text{ as soon as } x \geq 1.
\]

More precise bounds are given.

### 1.5 Least prime verifying a condition

[E. Bach and J. Sorenson, 3], [H. Kadiri, 95],

*Last updated on January 1st, 2019, by Olivier Ramaré*

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[149] D. Platt and O. Ramaré, 2017, “Explicit estimates: from \( \Lambda(n) \) in arithmetic progressions to \( \Lambda(n)/n \).”


*The TME-EMT project, 2018*
Chapter 2

Explicit bounds on the Moebius function

Corresponding html file: ../Articles/Art02.html

Collecting references: [H. Diamond and P. Erdős, 48], [M. Deléglise and J. Rivat, 45], [P. Borwein, R. Ferguson, and M. Mossinghoff, 21].

2.1 Bounds on $M(D) = \sum_{d \leq D} \mu(d)$

The first explicit estimate for $M(D)$ is due to [R. von Sterneck, 203] where the author proved that $|M(D)| \leq \frac{1}{2}D + 8$ for any $D \geq 0$. A popular estimate is the one of [R. Mac Leod, 122].

**Theorem (1967)** When $D \geq 0$, we have $|M(D)| \leq \frac{1}{80}D + 5$. When $D \geq 1119$, we have $|M(D)| \leq D/80$.

We mention at this level the annotated bibliography contained at the end of [F. Dress, 51]. [N. Costa Pereira, 37] shows that

**Theorem (1993)** When $D \geq 120,727$, we have $|M(D)| \leq D/1036$.

On elaborating on this method, [F. Dress and M. El Marraki, 52] showed that

[122] R. Mac Leod, 1967, “A new estimate for the sum $M(x) = \sum_{n \leq x} \mu(n)$”.
[37] N. Costa Pereira, 1989, “Elementary estimates for the Chebyshev function $\psi(X)$ and for the Möbius function $M(X)$”.

Theorem (1993) When $D \geq 617,973$, we have $|M(D)| \leq D/2360$.

One of the argument is the estimate from [F. Dress, 50]

Theorem (1993) When $33 \leq D \leq 10^{12}$, we have $|M(D)| \leq 0.571\sqrt{D}$.

This has been extended by [T. Kotnik and J. van de Lune, 100] to $10^{14}$ and recently in [G. Hurst, 83] to $10^{16}$, i.e.

Theorem (2018) When $33 \leq D \leq 10^{16}$, we have $|M(D)| \leq 0.571\sqrt{D}$.

Another tool is [H. Cohen and F. Dress, 34] where refined explicit estimates for the remainder term of the counting functions of the squarefree numbers in intervals are obtained.

The latest best estimate of this shape comes from [H. Cohen, F. Dress, and M. El Marraki, 35]. This preprint being difficult to get, it has been republished in [H. Cohen, F. Dress, and M. El Marraki, 36].

Theorem (1996) When $D \geq 2\,160\,535$, we have $|M(D)| \leq D/4345$.

These results are used in [F. Dress, 49] to study the discrepancy of the Farey series.

Concerning upper bounds that tend to 0, [L. Schoenfeld, 183] is the pioneer and shows among other estimates that

Theorem (1969) When $D > 0$, we have $|M(D)|/D \leq 2.9/\log D$.

[M. El Marraki, 62] improves that into

Theorem (1995) When $D \geq 685$, we have $|M(D)|/D \leq 0.10917/\log D$.

The latest bound coming from [O. Ramaré, 164] improves that:

Theorem (2012) When $D \geq 1\,100\,000$, we have $|M(D)|/D \leq 0.013/\log D$.

[83] G. Hurst, 2018, “Computations of the Mertens function and improved bounds on the Mertens conjecture”.
[34] H. Cohen and F. Dress, 1988, “Estimations numériques du reste de la fonction sommatoire relative aux entiers sans facteur carré”.
[164] O. Ramaré, 2013, “From explicit estimates for the primes to explicit estimates for the Möbius function”.

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In [O. Ramaré, 162], bounds including coprimality conditions are proved and here is a typical example.

**Theorem (2013)** When \( 1 \leq q < D \), we have \(| \sum_{d \leq D, (d, q) = 1} \mu(d)/d | \leq \frac{q}{\varphi(q)}/(1 + \log(D/q))\).

### 2.2 Bounds on \( m(D) = \sum_{d \leq D} \mu(d)/d \)

[R. Mac Leod, 122] shows that the sum \( m(D) \) takes its minimal value at \( D = 13 \). A folklore result is generalized in [A. Granville and O. Ramaré, 75] and reads

**Theorem (1996)** When \( D \geq 0 \) and for any integer \( r \geq 1 \), we have \(| \sum_{d \leq D, (d, r) = 1} \mu(d)/d | \leq 1\).

In fact, Lemma 1 of [H. Davenport, 41] already contains the requisite material.

The next result is proved in [O. Ramaré, 162].

**Theorem (2013)** When \( D \geq 7 \), we have \(| \sum_{d \leq D} \mu(d)/d | \leq 1/10\). We can replace the couple \((7, 1/10)\) by \((41, 1/20)\) or \((694, 1/100)\).

This is further extended in [O. Ramaré, 168] where it is shown that

**Theorem (2012)** When \( D \geq 0 \) and for any integer \( r \geq 1 \) and any real number \( \varepsilon \geq 0 \), we have \(| \sum_{d \leq D, (d, r) = 1} \mu(d)/d^{1+\varepsilon} | \leq 1 + \varepsilon\).

Concerning upper bounds that tend to 0, [M. El Marraki, 63] is the first to get such an estimate.

**Theorem (1996)** When \( D \geq 33 \) we have \(|m(D)| \leq 0.2185/\log D\).

When \( D > 1 \) we have \(|m(D)| \leq 726/(\log D)^2\).

This second bound is improved in [O. Bordellès, 20].

**Theorem (2015)** When \( D > 1 \) we have \(|m(D)| \leq 546/(\log D)^2\).

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[162] O. Ramaré, 2015, “Explicit estimates on several summatory functions involving the Moebius function”.

[122] R. Mac Leod, 1967, “A new estimate for the sum \( M(x) = \sum_{n \leq x} \mu(n) \)”.


[168] O. Ramaré, 2013, “Some elementary explicit bounds for two mollifications of the Moebius function”.

[63] M. El Marraki, 1996, “Majorations de la fonction sommatoire de la fonction \( \frac{\mu(n)}{n} \)”.


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[O. Ramaré, 164] proves several bounds of the shape \( m(D) \ll 1 / \log D \). This is improved in [O. Ramaré, 162] by using [M. Balazard, 8], which provides us with a better manner to convert bounds on \( M(D) \) into bounds for \( m(D) \). Here is one result obtained.

**Theorem (2015)** When \( D \geq 463,421 \) we have \( |m(D)| \leq 0.0144 / \log D \).

We can for instance replace the couple \((463,421,0.0144)\) by any of \((96,955,1/69)\), \((60,298,1/65)\), \((1426,1/20)\) or \((687,1/12)\).

In [O. Ramaré, 163] and [O. Ramaré, 162], the problem of adding coprimality conditions is further addressed. Here is one of the results obtained.

**Theorem (2015)** When \( 1 \leq q < D \) we have \( \left| \sum_{d \leq D, (d,q)=1} \mu(d)/d \right| \leq \frac{q}{\varphi(q)} 0.78 / \log(D/q) \).

When \( D/q \geq 24,233 \), we can replace 0.78 by 17/125.

### 2.3 Bounds on \( \tilde{m}(D) = \sum_{d \leq D} \mu(d) \log(D/d)/d \)

The initial investigations on this function go back to [R. Daublebsky von Sterneck, 40]. In [O. Ramaré, 162] it is proved that

**Theorem (2015)** When \( 3846 \leq D \) we have \( |\tilde{m}(D) - 1| \leq 0.00257 / \log D \). When \( D > 1 \), we have \( |\tilde{m}(D) - 1| \leq 0.213 / \log D \).

This implies in particular that

**Theorem (2015)** When \( 222 \leq D \) we have \( |\tilde{m}(D) - 1| \leq 1/1250 \). When \( D > 1 \), the optimal bound 1 holds.

These bounds are a consequence of the identity:

\[
|\tilde{m}(D) - 1| \leq \frac{7}{D^2} \int_1^D |M(t)| dt + \frac{2}{D}.
\]

It is also proved that, for any \( D \geq 1 \), we have

\[
0 \leq \sum_{d \leq D, (d,q)=1} \mu(d) \log(D/d)/d \leq 1.00303q / \varphi(q).
\]

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[164] O. Ramaré, 2013, “From explicit estimates for the primes to explicit estimates for the Moebius function”.
[162] O. Ramaré, 2015, “Explicit estimates on several summatory functions involving the Moebius function”.

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2.4 Miscellanea

Here is an elegant wide ranging estimate, taken from Claim 3.1 of [E. Treviño, 193].

Theorem (2015) When $D \geq 1$ we have $|\sum_{d > D} \mu(d)/d^2| \leq 1/D$.

2.5 The Moebius function and arithmetic progressions

The results in this section are scarce. We mention a Theorem of [O. Bordellès, 20].

Theorem (2015) Let $\chi$ be a non-principal Dirichlet character modulo $q \geq 37$ and let $k \geq 1$ be some integer. Then

$$\left| \sum_{n \leq x, (n,k)=1} \frac{\mu(n)\chi(n)}{n} \right| \leq k \frac{2\sqrt{q}\log q}{\varphi(k) L(1,\chi)}.$$
Chapter 3

Averages of non-negative multiplicative functions

Corresponding html file: ../Articles/Art10.html

3.1 Asymptotic estimates

When looking for averages of functions that look like 1 or like the divisor function, Lemma 3.2 of [O. Ramaré, 167] offers an efficient easy path. The technique of comparison of two arithmetical function via their Dirichlet series is known as the Convolution method and is for instance decribed at length in [P. Berment and O. Ramaré, 13], and in the course that can be found here


1http://math.univ-lille1.fr/~ramare/CoursNouakchott/index.html

Theorem (1995) Let \((g_n)_{n \geq 1}\), \((h_n)_{n \geq 1}\) and \((k_n)_{n \geq 1}\) be three complex sequences. Let \(H(s) = \sum_n h_n n^{-s}\), and \(\Pi(s) = \sum_n |h_n| n^{-s}\). We assume that \(g = h \ast k\), that \(\Pi(s)\) is convergent for \(\Re(s) \geq -1/3\) and further that there exist four constants \(A\), \(B\), \(C\) and \(D\) such that

\[
\sum_{n \leq t} k_n = A \log^2 t + B \log t + C + O^*(Dt^{-1/3}) \quad \text{for} \ t > 0.
\]

Then we have for all \(t > 0\) :

\[
\sum_{n \leq t} g_n = u \log^2 t + v \log t + w + O^*(Dt^{-1/3}\Pi(-1/3))
\]

with \(u = AH(0)\), \(v = 2AH'(0) + BH(0)\) and \(w = AH''(0) + BH'(0) + CH(0)\). We have also

\[
\sum_{n \leq t} ng_n = Ut \log t + Vt + W + O^*(2.5Dt^{2/3}\Pi(-1/3))
\]
with

\[U = 2AH(0), V = -2AH(0) + 2AH'(0) + BH(0),\]
\[W = A(H''(0) - 2H'(0) + 2H(0)) + B(H'(0) - H(0)) + CH(0).\]

This Lemma says that one derives information from \(g_n\) from informations on the model \(k_n\). When this model is \(k_n = 1\), the values concerning \(A\), \(B\) and \(C\) are given by the first half of Lemma 3.3 of [O. Ramaré, 167]:

**Lemma (1995)** \(\sum_{n \leq t} 1/n = \log t + \gamma + O^*(0.9105t^{-1/3})\).

When this model is \(k_n = \tau(n)\), the number of divisors of \(n\), the values concerning \(A\), \(B\) and \(C\) are given by Corollary 2.2 of [D. Berkane, O. Bordellès, and O. Ramaré, 12]. Please note the "\(\gamma_2 - 2\gamma_1\)" which is wrongly typed as "\(\gamma_2 - \gamma_1\)" in the aforementioned paper (and thanks to Tim Trudgian and David Platt for spotting this typo):

**Lemma (2011)** \(\sum_{n \leq t} \tau(n)/n = \frac{1}{2} \log^2 t + 2\gamma \log t + \gamma^2 - 2\gamma_1 + O^*(1.16/t^{1/3})\)

where \(\gamma_1\) is the second Laurent-Stieltjes constant – for instance [R. Kreminski, 101] and [M. Coffey, 33]. In particular, we have \(\gamma_1 = -0.0728158454836758749013191377 + O^*(10^{-40})\).

The constants \(H(0), H'(0)\) and \(H''(0)\) are to be computed. In most cases, the Dirichlet series has an Euler product, in which case, (see section 3 of [O. Ramaré, 167]) we have

\[H(0) = \prod_p (1 + \sum_{m} h_{p^m}), \quad H'(0) = \sum_p \frac{\sum_m m h_{p^m}}{1 + \sum_m h_{p^m}} (-\log p),\]

and also

\[\frac{H''(0)}{H(0)} = \left(\frac{H'(0)}{H(0)}\right)^2 + \sum_p \left\{ \frac{\sum_m m^2 h_{p^m}}{1 + \sum_m h_{p^m}} - \left[ \frac{\sum_m m h_{p^m}}{1 + \sum_m h_{p^m}} \right]^2 \right\} \log^2 p.\]

It is sometimes more expedient to use the same convolution method but by comparing the function to the function \(q \mapsto q\). In such a case, the next lemma, Lemma 4.3 from [O. Ramaré, 156], is handy.

**Theorem (2015)** We have, for any real number \(x \geq 0\) and any real number \(c \in [1, 2]\), \(\sum_{q \leq x} q = \frac{1}{2} x^2 + O^*(x^c/2)\).

This leads to the next theorem.

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3.1 Asymptotic estimates

Theorem Let \((h_n)_{n \geq 1}\) be a complex sequences. Let \(H(s) = \sum_n h_n n^{-s}\), and \(\overline{H}(s) = \sum_n |h_n| n^{-s}\). We assume that \(\overline{H}(s)\) is convergent for \(\Re(s) \geq c\), for some \(c \in [1, 2]\). Then we have for all \(t > 0\):
\[
\sum_{n \leq t} \sum_{d \mid n} \frac{n}{d} h(d) = \frac{t^2}{2} H(2) + O^*(t^c \overline{H}(c)/2).
\]

A typical usage is to evaluate \(\sum_{n \leq t} \phi(n)\), with \(h(d) = \mu(d)\).

The convolution method has been brought one step further in [A. P. and O. Ramaré, 145] where the following theorem is proved.

Theorem (2017) Let \((g(m))_{m \geq 1}\) be a sequence of complex numbers such that both series \(\sum_{m \geq 1} g(m)/m\) and \(\sum_{m \geq 1} g(m)(\log m)/m\) converge. We define \(G^\sharp(x) = \sum_{m > x} g(m)/m\) and assume that \(\int_{1}^{\infty} |G^\sharp(t)| dt/t\) converges. Let \(A_0 \geq 1\) be a real parameter. We have
\[
\sum_{n \leq D} \frac{(g \ast \mathbb{1})(n)}{n} = \sum_{m \geq 1} \frac{g(m)}{m} \left(\log \frac{D}{m} + \gamma\right) + \int_{D/A_0}^{\infty} G^\sharp(t) \frac{dt}{t} + O^*(\mathfrak{R})
\]
where \(\mathfrak{R}\) is defined by
\[
\mathfrak{R} = \left| \sum_{1 \leq a \leq A_0} \frac{1}{a} \left( G^\sharp \left( \frac{D}{a} \right) + G^\sharp \left( \frac{D}{A_0} \right) \left( \log \frac{A_0}{[A_0]} - R([A_0]) \right) \right) \right| + \frac{6/11}{D} \sum_{m \leq D/A_0} |g(m)|
\]
where \([A_0]\) is the integer part of \(A_0\), while the remainder \(R\) is defined by \(R(X) = \sum_{n \leq X} 1/n - \log X - \gamma\).

The remainder \(R(X)\) is shown in Lemma to verify \(|R(X)| \leq \gamma/X\) for every \(X > 0\), and \(|R(X)| \leq (6/11)/X\) when \(X \geq 1\).

Theorem 21.1 of [O. Ramaré, 159] offers a fully explicit estimate for the average of a general non-negative multiplicative function, but it is often numerically rather poor. It relies on the technique developed by [B. Levin and A. Fainleib, 109].

Theorem (2009) Let \(g\) be a non-negative multiplicative function. Let \(A\) and \(\kappa\) be three positive real parameters such that, for any \(Q \geq 1\), one has
\[
\sum_{p^{\nu} \leq Q} \log(p^{\nu}) = \kappa \log Q + O^*(L)
\]


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and \( \sum_{p \geq 2} \sum_{\nu, k \geq 1} g(p^k)g(p^\nu) \log(p^\nu) \leq A. \) Then, when \( D \geq \exp(2(L + A)) \), we have
\[
\sum_{d \leq D} g(d) = C (\log D)^\kappa \left( 1 + O^*(B/\log D) \right)
\]
where \( B = 2(L + A)(1 + 2(\kappa + 1)e^{\kappa + 1}) \) and
\[
C = \frac{1}{\Gamma(\kappa + 1)} \prod_{p \geq 2} \left\{ \left( \sum_{\nu \geq 0} g(p^\nu) \right) \left( 1 - \frac{1}{p} \right)^\kappa \right\}.
\]

3.2 Upper bounds

When looking for an upper bound, it is common to compare sums to an Euler product, via,
\[
\sum_{n \leq y} f(n)/n \leq \prod_{p \leq y} \left( 1 + \frac{\sum_{1 \leq m \leq \log y/\log p} f(p^m)}{\log y/\log p} \right)
\]
valid when \( f \) is non-negative and multiplicative. Lemma 4 of [H. Daboussi and J. Rivat, 39] extends this. Let \( z \) be a parameter and \( v_z(n) \) be the characteristic function of those integers that have all their prime factors \( p \leq z \).

**Theorem (2000)** Let \( z \geq 2, f \) a multiplicative function with \( f \geq 0 \) and \( S = \sum_{p \leq z} \frac{f(p)}{1 + f(p)} \log p. \) We assume that \( S > 0 \) and write \( K(t) = \log t - 1 - (1/t). \) For any \( y \) such that \( \log y \geq S \), we have
\[
\sum_{n > y} v_z(n)\mu^2(n)f(n) \leq \prod_{p \leq z} (1 + f(p)) \exp \left( -\frac{\log y}{\log z} K \left( \frac{\log y}{S} \right) \right)
\]
\[
\sum_{n \leq y} v_z(n)\mu^2(n)f(n) \geq \prod_{p \leq z} (1 + f(p)) \left\{ 1 - \exp \left( -\frac{\log y}{\log z} K \left( \frac{\log y}{S} \right) \right) \right\}
\]
and in particular, when \( \log y \geq 7S \), we have
\[
\sum_{n > y} v_z(n)\mu^2(n)f(n) \leq \prod_{p \leq z} (1 + f(p)) \exp \left( -\frac{\log y}{\log z} \right)
\]
\[
\sum_{n \leq y} v_z(n)\mu^2(n)f(n) \geq \prod_{p \leq z} (1 + f(p)) \left\{ 1 - \exp \left( -\frac{\log y}{\log z} \right) \right\}.
\]

It is sometimes required to compare a function close to 1 (or more generally to the divisor function \( \tau_k \)) to a function close to \( 1/n \) or \( \tau_k(n)/n. \) Theorem 01 of [R. Hall and G. Tenenbaum, 79] offers a fast way of doing so.


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3.3 Estimates of some special functions

Theorem (1988) Let \( f \) be a non-negative multiplicative function such that, for some \( A \) and \( B \),
\[
\sum_{p \leq y} f(p) \log p \leq Ay \quad (y \geq 0), \quad \sum_{p, \nu \geq 2} \frac{f(p^\nu)}{p^\nu} \log p^\nu \leq B.
\]
Then, for \( x > 1 \),
\[
\sum_{n \leq x} f(n) \leq (A + B + 1) \frac{x}{\log x} \sum_{n \leq x} \frac{f(n)}{n}.
\]

See also Section 4.6, and for instance Theorem 4.22, of [O. Bordellès, 17]. In particular, in case a further condition is assumed, we have Theorem 4.28 of [O. Bordellès, 17] at our disposal.

Theorem (2012) Let \( f \) be a non-negative multiplicative function such that, for every prime \( p \) and every non-negative power \( a \) the condition \( f(p^{a+1}) \geq f(p^a) \) holds, we have for \( x \geq 1 \)
\[
\sum_{n \leq x} f(n) \leq x \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \left( 1 + \sum_{a \geq 1} \frac{f(p^a)}{p^a} \right).
\]

The next lemma is handy to remove coprimality conditions. It originates from [J. van Lint and H. Richert, 110].

Theorem (1965) Let \( f \) be a non-negative multiplicative function and let \( d \) be a positive integer. We have for \( x \geq 0 \)
\[
\sum_{n \leq x} \mu^2(n)f(n) \leq \prod_{p \mid d} (1 + f(p)) \sum_{n \leq x, (n,d)=1} \mu^2(n)f(n) \leq \sum_{n \leq xd} \mu^2(n)f(n).
\]

Though it is somewhat difficult to get, this lemma has been further generalized in Lemma 4.1 of [O. Ramaré, 166].

3.3 Estimates of some special functions

[H. Cohen and F. Dress, 34] contains the following Theorem.

Theorem (1988) Let \( R(x) = \sum_{n \leq x} \mu^2(n) - 6x/\pi^2 \). We have \(|R(x+y) - R(x)| \leq 1.6749\sqrt{y} + 0.6212x/y \) and \(|R(x+y) - R(x)| \leq 0.7343y/x^{1/3} + 1.4327x^{1/3} \) for \( x, y \geq 1 \).

[34] H. Cohen and F. Dress, 1988, “Estimations numériques du reste de la fonction sommatoire relative aux entiers sans facteur carré”.

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See also [N. Costa Pereira, 37]. [L. Moser and R. A. MacLeod, 137] and [H. Cohen, F. Dress, and M. El Marraki, 36] contains:

**Theorem (2008)** We have $|\sum_{n\leq x} \mu^2(n) - 6x/\pi^2| \leq 0.02767\sqrt{x}$ for $x \geq 438653$. One can replace $(0.02767, 438653)$ by $(0.036438, 82005)$, by $(0.1333, 1004)$, or by $(1/2, 8)$ or by $(1, 1)$.

Lemma 3.4 of [O. Ramaré, 156] gives us:

**Theorem (2013)** We have $6/\pi^2 \log x + 0.578 \leq \sum_{n\leq x} \mu^2(n)/n \leq 6/\pi^2 \log x + 1.166$ for $x \geq 1$

When $x \geq 1000$, one can also replace the couple $(0.578, 1.166)$ by $(1.040, 1.048)$.

In fact, in the same paper, the asymptotic

$$\sum_{n\leq x} \frac{\mu^2(n)}{n} = \frac{6}{\pi^2} \left( \log x + 2 \sum_{p \geq 2} \frac{\log p}{p^2 - 1} + \gamma \right) + O^*(3/x^{1/3})$$

valid for $x \geq 1$ is proved. A script using SAGE and another one using GP/PARI are then displayed to explain how to cover the initial range in $x$. See also Lemma 1 of [L. Schoenfeld, 183] for an earlier version.

The main result [D. Berkane, O. Bordellès, and O. Ramaré, 12] reads as follows.

**Theorem (2012)** We define $\Delta(x) = \sum_{n\leq x} \tau(n) - x(\log x + 2\gamma - 1)$. We have

- When $x \geq 1$, we have $|\Delta(x)| \leq 0.961 x^{1/2}$.
- When $x \geq 1981$, we have $|\Delta(x)| \leq 0.482 x^{1/2}$.
- When $x \geq 5560$, we have $|\Delta(x)| \leq 0.397 x^{1/2}$.
- When $x \geq 5$, we have $|\Delta(x)| \leq 0.764 x^{1/3} \log x$.

For evaluation of the average of the divisor function on integers belonging to a fixed residue class modulo 6, see Corollary to Proposition 3.2 of [J.-M. Deshouillers and F. Dress, 46].

For more complicated sums and when $x$ is large with respect to $k$, one can use [C. Mardjanichvili, 123].

[37] N. Costa Pereira, 1989, “Elementary estimates for the Chebyshev function $\psi(X)$ and for the Möbius function $M(X)$”.
[137] L. Moser and R. A. MacLeod, 1966, “The error term for the squarefree integers”.

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3.3 Estimates of some special functions

Theorem (1939) Let \( k \) and \( \ell \) be two positive integers. We have for any real number \( x \geq 1 \)
\[
\sum_{m \leq x} \tau_k^\ell(m) \leq x \frac{k^\ell}{(k!)^{\frac{\ell}{k}}} (\log x + k^\ell - 1)^{\ell-1}.
\]

See [J.-M. Deshouil`les and F. Dress, 46] for some upper bounds linked with \( \tau_3 \).

[O. Bordell`es, 18] contains the following bounds, better than the above when \( x \) is small with respect to \( k \).

Theorem (2002) Let \( k \geq 1 \) be a positive integer.

- When \( x \geq 1 \) is a real number, we have \( \sum_{m \leq x} \tau_k(m) \leq x(\log x + \gamma + (1/x))^{k-1} \).
- When \( x \geq 6 \) is a real number, we have \( \sum_{m \leq x} \tau_k(m) \leq 2x(\log x)^{k-1} \).

In [K. Lapkova, 104], we find the next result.

Theorem (2015) Let \( b \) and \( c \) be two integers such that \( \delta = b^2 - c \) is non-zero, square-free and not congruent to 1 modulo 4. Assume further that the function \( n^2 + 2bn + c \) is positive and non-decreasing when \( n \geq 1 \). Then, for \( N \geq 1 \), we have
\[
\sum_{n \leq N} \tau(n^2 + 2bn + c) \leq C_1N \log N + C_2 + C_3
\]
where the constants \( C_1 \), \( C_2 \) and \( C_3 \) are defined as follows. We first define \( \xi = \sqrt{1 + 2|b| + |c|} \) and \( \kappa = \frac{1}{\pi^2} \sqrt{4|\delta|(\log(4|\delta|)) + 0.648} \). Then
\[
C_1 = \frac{12}{\pi^2}(\log \kappa + 1), C_2 = 2\left[\kappa + (\log \kappa + 1)\left(\frac{6}{\pi^2} \log \xi + 1.166\right)\right], C_3 = 2\kappa(\max(|b|, |c|^{1/2}) + 1).
\]

See [K. Lapkova, 105] for the number of divisors of a reducible quadratic polynomial.

Evaluations of Lemma 4.3 of [M. Cipu, 32] are improved in Lemma 12 of [T. Trudgian, 198]. Only upper bounds are given, but the proof given there gives the lower bounds as well. This gives the first two estimates, while the third one comes from Lemma 4.3 of [M. Cipu, 32].

Theorem (2015) Let \( x \geq 1 \) be a real number. We have

[18] O. Bordell`es, 2002, “Explicit upper bounds for the average order of \( d_n(m) \) and application to class number”.
[104] K. Lapkova, 2016, “Explicit upper bound for an average number of divisors of quadratic polynomials”.

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\[ 0.786x - 0.3761 - 8.14x^{2/3} \leq \sum_{n \leq x} 2^{\omega(n)} - \frac{6}{\pi^2} x \log x \leq 0.787x - 0.3762 + 8.14x^{2/3} \]

\[ \log x \]

\[ 1.3947 \log x + 0.4106 - 3.253x^{-1/3} \leq \sum_{n \leq x} \frac{2^{\omega(n)}}{n} - \frac{3}{\pi^2} (\log x)^2 \leq 1.3948 \log x + 0.4107 + 3.253x^{-1/3}, \]

\[ \sum_{n \leq x} \frac{2^{\omega(2n-1)}}{2n-1} \leq \frac{3}{2\pi^2} (\log x)^2 + 3.123 \log x + 3.569 + \frac{0.525}{x}. \]

We take the next lemma from [E. Treviño, 192], Lemma 2.

**Theorem (2015)** Let \( x \geq 1 \) be a real number. We have \( \sum_{n \leq x} \phi(n)/n \leq \frac{6}{4\pi} x + \log x + 1. \)

**Lemma 3** of the same paper is as follows.

**Theorem (2015)** Let \( x \geq 1 \) be a real number. We have \( \sum_{n \leq x} n\phi(n) \leq \frac{2}{\pi x^3} + \frac{1}{2} x^2 \log x + x^2. \)

Several estimates are proved in [A. P. and O. Ramaré, 145]. For instance Theorem 1.2 gives the following.

**Theorem (2017)** Let \( x \geq 1 \) be a real number. We have \( \sum_{n \leq x} \mu^2(n)/\phi(n) = \log x + c_0 + O^*(3.95/\sqrt{x}) \) where \( c_0 = \gamma + \sum_{p \geq 2} \frac{\log p}{p(p-1)}. \) When \( x > 1 \), this \( O^* \) can be replaced by \( O^*(21/\sqrt{x \log x}). \)

The function \( \sum_{n \leq x} \mu^2(n)/\phi(n) \) has been the subject of several estimates, see for instance Lemma 7 of [H. Montgomery and R. Vaughan, 133], Lemma 3.4-3.5 of [O. Ramaré, 167], the earlier paper [D. R. Ward, 204] and Lemma 4.5 of [J. Bütte, 23] where the error term \( O^*(58/\sqrt{x}) \) is achieved. The constant \( c_0 \) is evaluated precisely in (2.11) of [J. Rosser and L. Schoenfeld, 178].

[192] E. Treviño, 2015, “The Burgess inequality and the least kth power non-residue”.

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3.4 Euler products

[J. Rosser and L. Schoenfeld, 178] contains estimates regarding $\prod_{p \leq x} (1 - \frac{1}{p})$ and its inverse. In particular we find the next results therein.

**Theorem (1962)**

- When $x > 1$, we have $1 - \frac{1}{\log^2 x} \leq e^{-\gamma} \prod_{p \leq x} (1 - \frac{1}{p}) \leq 1 + \frac{1}{2 \log^2 x}$.

- When $x > 1$, we have $1 - \frac{1}{2 \log^2 x} \leq e^{-\gamma} \prod_{p \leq x} (1 - \frac{1}{p})^{-1} / \log x \leq 1 + \frac{1}{\log x}$.

Several other estimates are proven. In [P. Dusart, 58], it is proved that

**Theorem (2016)**

- When $x > 2278382$, we have $1 - \frac{1}{5 \log^2 x} \leq e^{-\gamma} \prod_{p \leq x} (1 - \frac{1}{p})^{-1} / \log x \leq 1 + \frac{1}{5 \log^2 x}$.

In [O. Bordellès, 19], the reader will find explicit upper bounds for $\prod_{p \leq x, p \equiv a \ [q]} (1 - \frac{1}{p})^{-1}$.

Theorem 5 of [R. Mawia, 126] contains the next result.

**Theorem (2017)** Let $\epsilon$ be a complex number such that $|\epsilon| \leq 2$. When $x \geq \exp(22)$, we have $\prod_{p \leq x} \left(1 + \frac{\epsilon}{p}\right) = e^{\gamma(\epsilon) + \epsilon B (\log x)} \left[1 + O\left(\frac{0.841}{\log^3 x}\right)\right]$ where $\gamma(\epsilon) = \sum_{p \geq 2} \sum_{n \geq 2} (-1)^{n+1} \frac{\epsilon^n}{np^n}$ and $B = \gamma + \sum_{p \geq 2} \log(1 - 1/p) + (1/p)$.

Equation (2.2) of [J. Rosser and L. Schoenfeld, 178] gives an approximate value for $B$.

Last updated on June 10th, 2019, by Olivier Ramaré

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[58] P. Dusart, 2018, “Estimates of some functions over primes”.


The TME-EMT project, 2018
Chapter 4

Explicit upper bounds for some special arithmetic functions

Corresponding html file: ../Articles/Art12.html
The following bounds may be useful in applications.
From [G. Robin, 172]:

**Theorem (1983)** For any integer \( n \geq 3 \), the number of prime divisors \( \omega(n) \) of \( n \) satisfies:

\[
\omega(n) \leq 1.3841 \frac{\log n}{\log \log n}.
\]

From [J.-L. Nicolas and G. Robin, 142]:

**Theorem (1983)** For any integer \( n \geq 3 \), the number \( \tau(n) \) of divisors of \( n \) satisfies:

\[
\tau(n) \leq n^{1.538 \log 2 / \log \log n}.
\]

From page 51 of [G. Robin, 173]:

**Theorem (1983)** For any integer \( n \geq 3 \), we have

\[
\tau_3(n) \leq n^{1.59141 \log 3 / \log \log n}
\]

where \( \tau_3(n) \) is the number of triples \((d_1, d_2, d_3)\) such that \( d_1 d_2 d_3 = n \).

[172] G. Robin, 1983, “Estimation de la fonction de Tchebychef \( \theta \) sur le \( k \)-ième nombres premiers et grandes valeurs de la fonction \( \omega(n) \) nombre de diviseurs premiers de \( n \)”.


CHAPTER 4. EXPLICIT UPPER BOUNDS FOR SOME SPECIAL ARITHMETIC FUNCTIONS

The PhD memoir [J.-L. Duras, 54] contains result concerning the maximum of $\tau_k(n)$, i.e. the number of $k$-tuples $(d_1, d_2, \ldots, d_k)$ such that $d_1d_2\cdots d_k = n$, when $3 \leq k \leq 25$.


**Theorem (1999)** For any integer $n \geq 1$, any real number $s > 1$ and any integer $k \geq 1$, we have

$$\tau_k(n) \leq n^s \zeta(s)^{k-1}$$

where $\tau_k(n)$ is the number of $k$-tuples $(d_1, d_2, \cdots, d_k)$ such that $d_1d_2\cdots d_k = n$.

The same paper also announces the bound for $n \geq 3$ and $k \geq 2$

$$\tau_k(n) \leq n^{a_k \log k / \log \log k}$$

where $a_k = 1.53797 \log k / \log 2$ but the proof never appeared.

From [J.-L. Nicolas, 141]:

**Theorem (2008)** For any integer $n \geq 3$, we have

$$\sigma(n) \leq 2.59791 n \log \log(3\tau(n)),$$

$$\sigma(n) \leq n\{e^\gamma \log \log(e\tau(n)) + \log \log(e^{e\tau(n)}) + 0.9415\}.$$

The first estimate above is a slight improvement of the bound

$$\sigma(n) \leq 2.59n \log \log n \quad (n \geq 7)$$

obtained in [A. Ivić, 88]. In this same paper, the author proves that

$$\sigma^*(n) \leq \frac{28}{15} n \log \log n \quad (n \geq 31)$$

where $\sigma^*(n)$ is the sum of the unitary divisors of $n$, i.e. divisors $d$ of $n$ that are such that $d$ and $n/d$ are coprime.

Last updated on June 10th, 2019, by Olivier Ramaré

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[88] A. Ivić, 1977, “Two inequalities for the sum of the divisors functions”.

November 5, 2020
Part II

Exact computations
Chapter 5

Exact computations of the number of primes

Corresponding html file: ../Articles/Art03.html
Collecting references: [M. Deléglise and J. Rivat, 43], [M. Deléglise and J. Rivat, 44], [D. Platt, 147].

Last updated on July 14th, 2012, by Olivier Ramaré

Chapter 6

Computations of arithmetical constants

Corresponding html file: ../Articles/Art04.html
Collecting references: [J. Cazaran and P. Moree, 27].

6.1 Euler products of rational functions

The computation of Euler product of rational function is dealt with in [P. Moree, 134]. The reader may also consult the following web page1.

6.2 Some special sums over prime values that are derivatives

Last updated on July 14th, 2012, by Olivier Ramaré


1http://guests.mpim-bonn.mpg.de/moree/Moree.en.html
Part III

General analytical tools
Chapter 7

Tools on Fourier transforms

Corresponding html file: ../Articles/Art16.html

7.1 The large sieve inequality

The best version of the large sieve inequality from [H. Montgomery and R. Vaughan, 132] and [H. Montgomery and R. Vaughan, 133] (obtained at the same time by A. Selberg) is as follows.

**Theorem (1974)** Let $M$ and $N \geq 1$ be two real numbers. Let $X$ be a set of points of $[0,1)$ such that
$$\min_{x,y \in X} \min_{k \in \mathbb{Z}} |x - y + k| \geq \delta > 0.$$ 

Then, for any sequence of complex numbers $(a_n)_{M<n\leq M+N}$, we have
$$\left| \sum_{x \in X} \left| \sum_{M<n\leq M+N} a_n \exp(2i\pi nx) \right|^2 \right| \leq \sum_{M<n\leq M+N} |a_n|^2 (N - 1 + \delta^{-1}).$$

It is very often used with part of the Farey dissection.

**Theorem (1974)** Let $M$ and $N \geq 1$ be two real numbers. Let $Q \geq 1$ be a real parameter. For any sequence of complex numbers $(a_n)_{M<n\leq M+N}$, we have
$$\sum_{q \in Q} \sum_{a \mod q, (a,q)=1} \left| \sum_{M<n\leq M+N} a_n \exp(2i\pi na/q) \right|^2 \leq \sum_{M<n\leq M+N} |a_n|^2 (N - 1 + Q^2).$$

The summation over $a$ runs over all invertible classes $a$ modulo $q$.

Last updated on July 14th, 2013, by Olivier Ramaré

Chapter 8

Tools on Mellin transforms

Corresponding html file: ../Articles/Art17.html

8.1 Explicit truncated Perron formula

Here is Theorem 7.1 of [O. Ramaré, 160].

**Theorem (2007)** Let $F(z) = \sum_n a_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z > \kappa_a$, and let $\kappa > 0$ be strictly larger than $\kappa_a$. For $x \geq 1$ and $T \geq 1$, we have

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} F(z) \frac{x^z \,dz}{z} + O^* \left( \int_{1/T}^{\infty} \sum_{n \atop \{ \log(x/n) \} \leq u} |a_n| \frac{2x^u \,du}{n^{\kappa} T u^2} \right).$$

See [O. Ramaré, 165] for different versions.

8.2 $L^2$-means

We start with a majorant principle taken for instance from [H. Montgomery, 131], chapter 7, Theorem 3.

**Theorem** Let $\lambda_1, \ldots, \lambda_N$ be $N$ real numbers, and suppose that $|a_n| \leq A_n$ for all $n$. Then

$$\int_{-T}^{T} \left| \sum_{1 \leq n \leq N} a_n e(\lambda_n t) \right|^2 \,dt \leq 3 \int_{-T}^{T} \left| \sum_{1 \leq n \leq N} A_n e(\lambda_n t) \right|^2 \,dt.$$
The constant 3 has furthermore been shown to be optimal in [B. F. Logan, 112] where the reader will find an intensive discussion on this question. The next lower estimate is also proved there:

**Theorem** Let \( \lambda_1, \ldots, \lambda_N \) be \( N \) real numbers, and suppose that \( a_n \geq 0 \) for all \( n \). Then

\[
\int_{-T}^{T} \left| \sum_{1 \leq n \leq N} a_n e(\lambda_n t) \right|^2 dt \geq T \sum_{n \leq N} a_n^2.
\]

We follow the idea of Corollary 3 of [H. Montgomery and R. Vaughan, 132] but rely on [E. Preissmann, 153] to get the following.

**Theorem (2013)** Let \((a_n)_{n \geq 1}\) be a series of complex numbers that are such that \( \sum n |a_n|^2 < \infty \) and \( \sum |a_n| < \infty \). We have, for \( T \geq 0 \),

\[
\int_{0}^{T} \left| \sum_{n \geq 1} a_n n^it \right|^2 dt = \sum_{n \leq N} |a_n|^2 (T + O^*(2\pi c_0(n + 1))'),
\]

where \( c_0 = \sqrt{1 + \frac{2}{3}\sqrt{6}} \). Moreover, when \( a_n \) is real-valued, the constant \( 2\pi c_0 \) may be reduced to \( \pi c_0 \).

This is Lemma 6.2 from [O. Ramaré, 156].

Corollary 6.3 and 6.4 of [O. Ramaré, 156] contain explicit versions of a Theorem of [P. Gallagher, 72]

**Theorem (2013)** Let \((a_n)_{n \geq 1}\) be a series of complex numbers that are such that \( \sum n |a_n|^2 < \infty \) and \( \sum |a_n| < \infty \). We have, for \( T \geq 0 \),

\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \text{ primitive} \mod q} \int_{-T}^{T} \left| \sum_{n} a_n \chi(n)n^it \right|^2 dt \leq 7 \sum_{n \geq 1} |a_n|^2 (n + Q^2 \max(T, 3)).
\]

**Theorem (2013)** Let \((a_n)_{n \geq 1}\) be a series of complex numbers that are such that \( \sum n |a_n|^2 < \infty \) and \( \sum |a_n| < \infty \). We have, for \( T \geq 0 \),

\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \text{ primitive} \mod q} \int_{-T}^{T} \left| \sum_{n} a_n \chi(n)n^it \right|^2 dt \leq \sum_{n \geq 1} |a_n|^2 (43n + \frac{33}{8}Q^2 \max(T, 70)).
\]

Last updated on July 14th, 2013, by Olivier Ramaré

[72] P. Gallagher, 1970, “A large sieve density estimate near \( \sigma = 1 \)”.

November 5, 2020
Part IV

Exponential sums / points close to curves
Chapter 9

Explicit results on exponential sums

Corresponding html file: ../Articles/Art05.html
Collecting references: [A. Granville and O. Ramaré, 75], [H. Daboussi and J. Rivat, 39].

Last updated on July 14th, 2012, by Olivier Ramaré

Chapter 10

Integer Points near Smooth Plane Curves

In what follows, $N \geq 1$ is an arbitrary large integer, $\delta \in (0, \frac{1}{2})$ and if $f : [N, 2N] \to \mathbb{R}$ is any positive function, then let $R(f, N, \delta)$ be the number of integers $n \in [N, 2N]$ such that $\|f(n)\| < \delta$, where as usual $\|x\|$ denotes the distance from $x \in \mathbb{R}$ to its nearest integer. Note that, since $\delta$ is very small, $R(f, N, \delta)$ roughly counts the number of integer points very close to the arc $y = f(x)$ with $N \leq x \leq 2N$. Hence the trivial estimate is given by $R(f, N, \delta) \leq N + 1$.

The number $R(f, N, \delta)$ arises fairly naturally in a large collection of problems in number theory, e.g. [M. Filaseta, 65], [M. Filaseta and O. Trifonov, 66], [M. Huxley, 84], [M. Huxley and P. Sargos, 85], [M. Huxley and P. Sargos, 86], [M. Huxley and O. Trifonov, 87] and bibref("Huxley*07"). We deal with either getting an asymptotic formula of the shape

$$R(f, N, \delta) = N\delta + \text{Error terms}$$

where the remainder terms depend on the derivatives of $f$ but not on $\delta$, or finding an upper bound for $R(f, N, \delta)$ as accurate as possible.

[85] M. Huxley and P. Sargos, 1995, “Integer points close to a plane curve of class $C^\alpha$. (Points entiers au voisinage d’une courbe plane de classe $C^\alpha$.)”
[86] M. Huxley and P. Sargos, 2006, “Integer points in the neighborhood of a plane curve of class $C^\alpha$. II. (Points entiers au voisinage d’une courbe plane de classe $C^\alpha$. II.)”.

10.1 Bounds using elementary methods

The basic result of the theory is well-known and may be found in [I. Vinogradov, 202]. The proof follows from a clever use of the mean-value theorem (see Theorem 5.6 of [O. Bordellès, 17] for instance).

**Theorem (First derivative test)** Let $f \in C^1[N,2N]$ such that there exist $\lambda_1 > 0$ and $c_1 \geq 1$ such that, for all $x \in [N,2N]$, we have

$$\lambda_1 \leq |f'(x)| \leq c_1 \lambda_1.$$

Then

$$R(f,N,\delta) \leq 2c_1 N \lambda_1 + 4c_1 N \delta + \frac{2\delta}{\lambda_1} + 1.$$

This result is useful when $\lambda_1$ is very small, so that the condition is in general too restrictive in the applications. Using a rather neat combinatorial trick, [M. Huxley, 84] succeeded in passing from the first derivative to the second derivative. This reduction step enables him to apply this theorem to a function being approximatively of the same order of magnitude as $f'$. This provides the following useful result.

**Theorem (Second derivative test)** Let $f \in C^2[N,2N]$ such that there exist $\lambda_2 > 0$ and $c_2 \geq 1$ such that, for all $x \in [N,2N]$, we have

$$\lambda_2 \leq |f''(x)| \leq c_2 \lambda_2 \quad \text{and} \quad N \lambda_2 \geq c_2^{-1}.$$

Then

$$R(f,N,\delta) \leq 6 \left\{ (3c_2)^{1/3} N \lambda_2^{1/3} + (12c_2)^{1/2} N \delta^{1/2} + 1 \right\}.$$

Both hypotheses above are often satisfied in practice, so that this result may be considered as the first useful tool of the theory. A proof of this Theorem may be found in Theorem 5.8 of [O. Bordellès, 17].

Using a $k$th version of Huxley’s reduction principle may allow us to generalize the above results. A better way is to split the integer points into two classes, namely the major arcs in which the points belong to a same algebraic curve of degree $\leq k - 1$, and the minor arcs. The points coming from the minor arcs are treated by divided differences techniques, generalizing the proof of both theorems above and, by a careful analysis of the points belonging to major arcs, [M. Huxley and P. Sargos, 85] and [M. Huxley and P. Sargos, 86] succeeded in proving the following fundamental result. A proof of an explicit version may be found in Theorem 5.12 of [O. Bordellès, 17].

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[85] M. Huxley and P. Sargos, 1995, “Integer points close to a plane curve of class $C^n$. (Points entiers au voisinage d’une courbe plane de classe $C^n$.)”
[86] M. Huxley and P. Sargos, 2006, “Integer points in the neighborhood of a plane curve of class $C^n$. II. (Points entiers au voisinage d’une courbe plane de classe $C^n$. II).”

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Theorem (kth derivative test) Let \( k \geq 3 \) be an integer and \( f \in C^k[N, 2N] \) such that there exist \( \lambda_k > 0 \) and \( c_k \geq 1 \) such that, for all \( x \in [N, 2N] \), we have

\[
\lambda_k \leq |f^{(k)}(x)| \leq c_k \lambda_k.
\]

Let \( \delta \in (0, \frac{1}{4}) \). Then

\[
\mathcal{R}(f, N, \delta) \leq \alpha_k N \lambda_k^{\frac{2}{k(k+1)}} + \beta_k N \delta^{\frac{2}{k(k-1)}} + 8k^3 \left( \frac{\delta}{\lambda_k} \right)^{1/k} + 2k^2 (5e^3 + 1)
\]

where

\[
\alpha_k = 2k^2 c_k^{\frac{2}{k(k+1)}} \quad \text{and} \quad \beta_k = 4k^2 \left( 5e^3 c_k^{\frac{2}{k(k-1)}} + 1 \right).
\]

10.2 Bounds using exponential sums techniques

The next result leads us to estimate \( \mathcal{R}(f, N, \delta) \) with the help of exponential sums (see [S. W. Graham and G. Kolesnik, 1991, Van der Corput’s Method of Exponential Sums] for instance), which have been extensively studied in the 20th century by many specialists, such as van der Corput, Weyl or Vinogradov. Nevertheless, even using the finest exponent pairs to date, the result generally does not significantly improve on the previous estimates seen above. A simple proof of the following inequality may be found in [M. Filaseta, 1990, “Short interval results for squarefree numbers”].

Theorem (kth derivative test) Let \( f : [N, 2N] \rightarrow \mathbb{R} \) be any function and \( \delta \in (0, \frac{1}{4}) \). Set \( K = \left\lfloor (8\delta)^{-1} \right\rfloor + 1 \). Then, for any positive integer \( H \leq K \), we have

\[
\mathcal{R}(f, N, \delta) \leq 4N \frac{H}{H} + 4 \sum_{h=1}^{H} \left| \sum_{N \leq n \leq 2N} e(hf(n)) \right|.
\]

10.3 Integer points on curves

This last part is somewhat out of the scope of the TME-EMT project, but may help the reader in orienting him/herself in the litterature.

When \( \delta \rightarrow 0 \), we are led to counting the number of integer points lying on curves, and we denote this number by \( \mathcal{R}(f, N, 0) \). This problem goes back to Jarník [V. Jarník, 1925, “Über die Gitterpunkte auf konvexen Kurven”] who proved that a strictly convex arc \( y = f(x) \) with length \( L \) has at most

\[
\leq \frac{3}{(2\pi)^{1/3}} L^{2/3} + O \left( L^{1/3} \right)
\]

integer points and this is a nearly best possible result under the sole hypothesis of convexity. However, [H. Swinnerton-Dyer, 1974, “The number of lattice points on a convex curve”] proved that if \( f \in C^3[0, N] \)
is such that $|f(x)| \leq N$ and $f'''(x) \neq 0$ for all $x \in [0, N]$, then the number of integer points on the arc $y = f(x)$ with $0 \leq x \leq N$ is $\ll N^{3/5+\varepsilon}$. This result was later generalized by [E. Bombieri and J. Pila, 14] who showed among other things the following estimate.

**Theorem (1989)** Let $N \geq 1$, $k \geq 4$ be integers and define $K = \frac{(k+2)}{2}$. Let $I$ be an interval with length $N$ and $f \in C^K(I)$ satisfying $|f'(x)| \leq 1$, $f'''(x) > 0$ and such that the number of solutions of the equation $f^{(K)}(x) = 0$ is $\leq m$. Then there exists a constant $c_0 = c_0(k) > 0$ such that

$$R(f, N, 0) \leq c_0(m + 1)N^{1/2+3/(k+3)}.$$
Part V

Size of $L(1, \chi)$ and character sums
Chapter 11

Size of $L(1, \chi)$

Corresponding html file: ../Articles/Art07.html
Collecting references: [S. Louboutin, 117],

11.1 Upper bounds for $|L(1, \chi)|$

[S. Louboutin, 118], [A. Granville and K. Soundararajan, 78], [A. Granville and K. Soundararajan, 76]. [O. Ramaré, 157], [O. Ramaré, 158], [S. Louboutin, 119],

11.2 Lower bounds for $|L(1, \chi)|$

[S. Louboutin, 113] announces the following lower bound proved in [S. R. Louboutin, 121].

Theorem (2013) For any non-quadratic primitive Dirichlet character $\chi$ of conductor $f$, we have $|L(1, \chi)| \geq 1/(10 \log(f/\pi))$.

Last updated on September 11th, 2014, by Olivier Ramaré
Chapter 12

Character sums

Corresponding html file: ../Articles/Art15.html

12.1 Explicit Polya-Vinogradov inequalities

The main Theorem of [Z. M. Qiu, 154] implies the following result.

**Theorem (1991)** For \( \chi \) a primitive character to the modulus \( q > 1 \), we have

\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq \frac{4}{\pi^2} \sqrt{q} \log q + 0.38 \sqrt{q} + \frac{0.637}{\sqrt{q}}.
\]

When \( \chi \) is not especially primitive, but is still non-principal, we have

\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq \frac{8}{3\pi^2} \sqrt{q} \log q + 0.63 \sqrt{q} + \frac{1.05}{\sqrt{q}}.
\]

This was improved later by [G. Bachman and L. Rachakonda, 4] into the following.

**Theorem (2001)** For \( \chi \) a non-principal character to the modulus \( q > 1 \), we have

\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq \frac{1}{3\log 3} \sqrt{q} \log q + 6.5 \sqrt{q}.
\]

These results are superseded by [D. Frolenkov, 69] and more recently by [D. A. Frolenkov and K. Soundararajan, 70] into the following.

**Theorem (2013)** For \( \chi \) a non-principal character to the modulus \( q \geq 1000 \), we have

\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq \frac{1}{\pi \sqrt{2}} \sqrt{q} (\log q + 6) + \sqrt{q}.
\]

In the same paper they improve upon estimates of [C. Pomerance, 152] and get the following.

**Theorem (2013)** For \( \chi \) a primitive character to the modulus \( q \geq 1200 \), we have
\[
\max_{M,N} \left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq \begin{cases} 
\frac{2}{\pi} \sqrt{q} \log q + \sqrt{q}, & \chi \text{ even}, \\
\frac{1}{2\pi} \sqrt{q} \log q + \sqrt{q}, & \chi \text{ odd}. 
\end{cases}
\]
This latter estimates holds as soon as \( q \geq 40 \).

In case \( \chi \) odd, the constant \( 1/(2\pi) \) has already been asymptotically obtained in [E. Landau, 103] and is still unsurpassed. When \( \chi \) is odd and \( M = 1 \), the best asymptotical constant up to now is \( 1/(3\pi) \) from Theorem 7 of [A. Granville and K. Soundararajan, 77],

In case \( \chi \) even, we have
\[
\max_{M,N} \left| \sum_{a=M+1}^{N} \chi(a) \right| = 2 \max_{N} \left| \sum_{a=1}^{N} \chi(a) \right|.
\]
(The LHS is always less than the RHS. Equality is then easily proved). The asymptotical best constant is \( 23/(35\pi\sqrt{3}) \) from Theorem 7 of [A. Granville and K. Soundararajan, 77].

### 12.2 Burgess type estimates

The following from [E. Treviño, 192] is an explicit version of Burgess with the only restriction being \( p \geq 10^7 \).

**Theorem (2015)** Let \( p \) be a prime such that \( p \geq 10^7 \). Let \( \chi \) be a non-principal character mod \( p \). Let \( r \) be a positive integer, and let \( M \) and \( N \) be non-negative integers with \( N \geq 1 \). Then
\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq 2.74 N^{1-\frac{1}{r}} p^{\frac{r+1}{r^2}} (\log p)^{\frac{1}{r}}.
\]
From the same paper, we get the following more specific result.

**Theorem (2015)** Let \( p \) be a prime. Let \( \chi \) be a non-principal character mod \( p \). Let \( M \) and \( N \) be non-negative integers with \( N \geq 1 \), let \( 2 \leq r \leq 10 \) be a positive
integer, and let \( p_0 \) be a positive real number. Then for \( p \geq p_0 \), there exists \( c_1(r) \), a constant depending on \( r \) and \( p_0 \) such that
\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq c_1(r)N^{1-\frac{1}{r}}p^{\frac{r+1}{4r}}(\log p)^{\frac{1}{2}}
\]
where \( c_1(r) \) is given by
\[
\begin{align*}
& 1 \quad 2.7381 \quad 2.5173 \quad 2.3549 \\
& 2 \quad 2.0197 \quad 1.7385 \quad 1.3695 \\
& 3 \quad 1.7308 \quad 1.5151 \quad 1.3104 \\
& 4 \quad 1.6107 \quad 1.4572 \quad 1.2987 \\
& 5 \quad 1.5482 \quad 1.4274 \quad 1.2901 \\
& 6 \quad 1.5052 \quad 1.4042 \quad 1.2813 \\
& 7 \quad 1.4703 \quad 1.3846 \quad 1.2729 \\
& 8 \quad 1.4411 \quad 1.3662 \quad 1.2641 \\
& 9 \quad 1.4160 \quad 1.3495 \quad 1.2562 \\
& 10 \quad 1.3921 \quad 1.3378 \quad 1.2486
\end{align*}
\]

We can get a smaller exponent on \( \log \) if we restrict the range of \( N \) or if we have \( r \geq 3 \).

**Theorem (2015)** Let \( p \) be a prime. Let \( \chi \) be a non-principal character mod \( p \). Let \( M \) and \( N \) be non-negative integers with \( 1 \leq N \leq 2p^{\frac{r}{2}+\frac{1}{r}} \) or \( r \geq 3 \). Let \( r \leq 10 \) be a positive integer, and let \( p_0 \) be a positive real number. Then for \( p \geq p_0 \), there exists \( c_2(r) \), a constant depending on \( r \) and \( p_0 \) such that
\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq c_2(r)N^{1-\frac{1}{r}}p^{\frac{r+1}{4r}}(\log p)^{\frac{1}{2}},
\]
where \( c_2(r) \) is given by
\[
\begin{align*}
& 1 \quad 3.7451 \quad 3.5700 \quad 3.5341 \\
& 2 \quad 2.3200 \quad 2.1901 \quad 2.1071 \\
& 3 \quad 2.0881 \quad 1.9831 \quad 1.9037 \\
& 4 \quad 1.9373 \quad 1.8504 \quad 1.7748 \\
& 5 \quad 1.8293 \quad 1.7559 \quad 1.6843 \\
& 6 \quad 1.7461 \quad 1.6836 \quad 1.6167 \\
& 7 \quad 1.6802 \quad 1.6262 \quad 1.5638 \\
& 8 \quad 1.6260 \quad 1.5786 \quad 1.5210 \\
\end{align*}
\]

Kevin McGown in [K. J. McGown, 130] has slightly worse constants in a slightly larger range of \( N \) for smaller values of \( p \).


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**Theorem (2012)** Let \( p \geq 2 \cdot 10^4 \) be a prime number. Let \( M \) and \( N \) be non-negative integers with \( 1 \leq N \leq 4p^{\frac{1}{2} + \frac{1}{r}} \). Suppose \( \chi \) is a non-principal character mod \( p \). Then there exists a computable constant \( C(r) \) such that

\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq C(r)N^{1-\frac{1}{r}}p^{\frac{r+1}{2r}}(\log p)^{\frac{1}{2}},
\]

where \( C(r) \) is given by

\[
\begin{array}{cccc}
r & C(r) & C(r) & C(r) \\
2 & 10.0366 & 9 & 2.1467 \\
3 & 4.9539 & 10 & 2.0492 \\
4 & 3.6493 & 11 & 1.9712 \\
5 & 3.0356 & 12 & 1.9073 \\
6 & 2.6765 & 13 & 1.8540 \\
7 & 2.4400 & 14 & 1.8088 \\
8 & 2.2721 & 15 & 1.7700 \\
\end{array}
\]

Finally, if the character is quadratic (and with a more restrictive range), we have slightly stronger results due to Booker in [A. Booker, 15].

**Theorem (2006)** Let \( p > 10^{20} \) be a prime number with \( p \equiv 1 \pmod{4} \). Let \( r \in \{2, 3, 4, \ldots, 15\} \). Let \( M \) and \( N \) be real numbers such that \( 0 < M, N \leq 2\sqrt{p} \). Let \( \chi \) be a non-principal quadratic character mod \( p \). Then

\[
\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq \alpha(r)N^{1-\frac{1}{r}}p^{\frac{r+1}{2r}}(\log p + \beta(r))^{\frac{1}{2}},
\]

where \( \alpha(r) \) and \( \beta(r) \) are given by

\[
\begin{array}{cccc}
r & \alpha(r) & \beta(r) & \beta(r) \\
2 & 1.8221 & 8.9077 & 9.1.4548 \\
3 & 1.8000 & 5.3948 & 10.1.4231 \\
4 & 1.7263 & 3.6658 & 11.1.3958 \\
5 & 1.6526 & 2.5405 & 12.1.3721 \\
6 & 1.5892 & 1.7059 & 13.1.3512 \\
7 & 1.5363 & 1.0405 & 14.1.3328 \\
8 & 1.4921 & 0.4856 & 15.1.3164 \\
\end{array}
\]

Last updated on April 28th, 2017, by Olivier Ramaré

[A. Booker, 2006, “Quadratic class numbers and character sums”.

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Part VI

Zeros and zero-free regions
Chapter 13

Bounds for $|\zeta(s)|$, $|L(s, \chi)|$
and related questions

Corresponding html file: ../Articles/Art06.html
Collecting references: [T. Trudgian, 197], [H. Kadiri and N. Ng, 98],

13.1 Size of $|\zeta(s)|$ and of $L$-series

Theorem 4 of [H. Rademacher, 155] gives the convexity bound. See also section 4.1 of [T. S. Trudgian, 200].

Theorem (1959) In the strip $-\eta \leq \sigma \leq 1 + \eta$, $0 < \eta \leq 1/2$, the Dedekind zeta function $\zeta_K(s)$ belonging to the algebraic number field $K$ of degree $n$ and discriminant $d$ satisfies the inequality

$$|\zeta_K(s)| \leq 3 \left| \frac{1 + s}{1 - s} \right| \left( \frac{|d||1 + s|}{2\pi} \right)^{\frac{1+\eta-\sigma}{2}} \zeta(1 + \eta)^n.$$ 

On the line $\Re s = 1/2$, Lemma 2 of [R. Lehman, 107] gives a better result, namely

Theorem (1970) If $t \geq 1/5$, we have $|\zeta(\frac{1}{2} + it)| \leq 4(t/(2\pi))^{1/4}$.

In fact, Lehman states this Theorem for $t \geq 64/(2\pi)$, but modern means of computations makes it easy to check that it holds as soon as $t \geq 0.2$. See also equation (56) of [R. J. Backlund, 5] reproduced below.

[98] H. Kadiri and N. Ng, 2012, “Explicit zero density theorems for Dedekind zeta functions”.
CHAPTER 13. BOUNDS FOR $|\zeta(S)|$, $|L(S, \chi)|$ AND RELATED QUESTIONS

For Dirichlet $L$-series, Theorem 3 of [H. Rademacher, 155] gives the corresponding convexity bound.

**Theorem (1959)** In the strip $-\eta \leq \sigma \leq 1 + \eta$, $0 < \eta \leq 1/2$, for all moduli $q > 1$ and all primitive characters $\chi$ modulo $q$, the inequality

$$|L(s, \chi)| \leq \left( q \frac{|1 + s|}{2\pi} \right)^{\frac{1 + \eta - \sigma}{2}} \zeta(1 + \eta)$$

holds.

This paper contains other similar convexity bounds.

**Corollary to Theorem 3 of [Y. Cheng and S. Graham, 29]** goes beyond convexity.

**Theorem (2001)** For $0 \leq t \leq e$, we have $|\zeta(\frac{1}{2} + it)| \leq 2.657$. For $t \geq e$, we have $|\zeta(\frac{1}{2} + it)| \leq 3t^{1/6} \log t$. Section 5 of [T. S. Trudgian, 200] shows that one can replace the constant 3 by 2.38.

This is improved in [G. A. Hiary, 82].

**Theorem (2016)** When $t \geq 3$, we have $|\zeta(\frac{1}{2} + it)| \leq 0.63t^{1/6} \log t$.

Concerning $L$-series, the situation is more difficult but [G. A. Hiary, 81] manages, among other and more precise results, to prove the following.

**Theorem (2016)** Assume $\chi$ is a primitive Dirichlet character modulo $q > 1$. Assume further that $q$ is a sixth power. Then, when $|t| \geq 200$, we have

$$|L(\frac{1}{2} + it, \chi)| \leq 9.05d(q)(q|t|)^{1/6}(\log q|t|)^{3/2}$$

where $d(q)$ is the number of divisors of $q$.

It is often useful to have a representation of the Riemann zeta function or of $L$-series inside the critical strip. In the case of $L$-series, [R. Spira, 187] and [R. Rumely, 180] proceed via decomposition in Hurwitz zeta function which they compute through an Euler-MacLaurin development. We have a more efficient approximation of the Riemann zeta function provided by the Riemann Siegel

[82] G. A. Hiary, 2016, “An explicit van der Corput estimate for $\zeta(1/2 + it)$”.

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13.2 On the total number of zeroes

formula, see for instance equations (3-2)–(3.3) of [A. Odlyzko, 143]. This expression is due to [W. Gabcke, 71]. See also equations (2.4)-(2.5) of [R. Lehman, 108], a corrected version of Theorem 2 of [E. Titchmarsh, 191].

In general, we have the following estimate taken from equations (53)-(54), (56) and (76) of [R. J. Backlund, 5] (see also [R. Backlund, 6]).

Theorem (1918)  
- When $t \geq 50$ and $\sigma \geq 1$, we have $|\zeta(\sigma + it)| \leq \log t - 0.048$.
- When $t \geq 50$ and $0 \leq \sigma \leq 1$, we have $|\zeta(\sigma + it)| \leq \frac{e^2}{t^{2/4}} \left( \frac{1}{\pi^2} \right)^{1/2} \log t$.
- When $t \geq 50$ and $-1/2 \leq \sigma \leq 0$, we have $|\zeta(\sigma + it)| \leq \left( \frac{e^2}{t^{2/4}} \right)^{1/2} \log t$.

On the line $\Re s = 1$, one can rely on [T. Trudgian, 195].

Theorem (2012) When $t \geq 3$, we have $|\zeta(1 + it)| \leq \frac{4}{3} \log t$.

Asymptotically better bounds are available since the huge work of [K. Ford, 67].

Theorem (2002) When $t \geq 3$ and $1/2 \leq \sigma \leq 1$, we have $|\zeta(\sigma + it)| \leq 76.2 t^{4.45(1-\sigma)^{3/2}} (\log t)^{2/3}$.

The constants are still too large for this result to be of use in any decent region. See [M. Kulas, 102] for an earlier estimate.

13.2 On the total number of zeroes

The first explicit estimate for the number of zeros of the Riemann $\zeta$-function goes back to [R. Backlund, 6]. An elegant consequence of the result of Backlund is the following easy estimate taken from Lemma 1 of [R. S. Lehman, 106].

Theorem (1966) If $\varphi$ is a continuous function which is positive and monotone decreasing for $2\pi e \leq T_1 \leq t \leq T_2$, then

$$\sum_{T_1 < \gamma \leq T_2} \varphi(\gamma) = \frac{1}{2\pi} \int_{T_1}^{T_2} \varphi(t) \log \frac{t}{2\pi} dt + O^* \left( 4\varphi(T_1) \log T_1 + 2 \int_{T_1}^{T_2} \varphi(t) \log t \ dt \right)$$

where the summation is over all zeros of the Riemann $\zeta$-function of imaginary part between $T_1$ and $T_2$, with multiplicity.

[71] W. Gabcke, 1979, “Neue Herleitung und explizite Restabschatzung der Riemann-Siegel-Formel”.

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Theorem 19 of [J. Rosser, 174] gives a bound for the total number of zeroes.

**Theorem (1941)** For $T \geq 2$, we have

$$N(T) = \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O^\ast \left( 0.137 \log T + 0.443 \log \log T + 1.588 \right)$$

where the summation is over all zeros of the Riemann $\zeta$-function of imaginary part between 0 and $T$, with multiplicity.

It is noted in Lemma 1 of [O. Ramaré and Y. Saouter, 171] that the $O$-term can be replaced by the simpler $O^\ast (0.67 \log \frac{T}{2\pi})$ when $T \geq 10^3$.

This is improved in Corollary 1 of [T. S. Trudgian, 200] into

**Theorem (1941)** For $T \geq e$, we have

$$N(T) = \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O^\ast \left( 0.112 \log T + 0.278 \log \log T + 2.510 + \frac{1}{5T} \right)$$

where the summation is over all zeros of the Riemann $\zeta$-function of imaginary part between 0 and $T$, with multiplicity.

### 13.3 $L^2$-averages

We can find in [H. Helfgott, 80] the proof of the following estimate. Though it is unpublished yet, the full proof is available.

**Theorem (1941)** Let $0 < \sigma \leq 1$ and $T \geq 3$. Then

$$\frac{1}{2\pi} \left( \int_{\sigma-iT}^{\sigma+iT} |\zeta(s)|^2 \frac{ds}{s^2} \right)^{1/2} \leq \kappa_{\sigma,T} \begin{cases} \frac{c_{1,\sigma}}{\log T} + \frac{c_{1,\sigma}}{T^3} & \text{when } \sigma > 1/2, \\ \frac{c_{1,\sigma}}{\log T} + \frac{c_{3,\sigma}}{T^2} & \text{when } \sigma = 1/2, \\ \frac{c_{3,\sigma}}{T^{2\sigma}} & \text{when } \sigma < 1/2. \end{cases}$$

where

$$c_{1,\sigma} = \zeta(2\sigma)/2, c_{1,\sigma}^\flat = c^2 \frac{3^{2\sigma}}{2^{2\sigma}}, c_{2,\sigma} = 3c^2 + \frac{1 - \log 3}{2}, c = 9/16,$$

$$c_{3,\sigma} = \left( \frac{c^2}{2\sigma} + \frac{1/6}{1 - 2\sigma} \right) \left( 1 + \frac{1}{\sigma} \right)^{2\sigma}, \kappa_{\sigma,T} = \begin{cases} \frac{9/4}{\left(1 - \frac{9/2}{1 - (1+\sigma)^2}\right)} & \text{when } 1/2 \leq \sigma \leq 1, \\ \frac{9/4}{\left(1 - \frac{9/2}{1 - (1+\sigma)^2}\right)} & \text{when } 0 < \sigma < 1/2. \end{cases}$$

---

[80] H. Helfgott, 2017, “$L^2$ bounds for tails of $\zeta(s)$ on a vertical line’’.

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13.4 Bounds on the real line

After some estimates from [G. Bastien and M. Rogalski, 9], Lemma 5.1 of [O. Ramaré, 156] shows the following.

**Theorem (2013)** When $\sigma > 1$ and $t$ is any real number, we have $|\zeta(\sigma + it)| \leq e^{\gamma(\sigma-1)/(\sigma - 1)}$.

Here is the Theorem of [H. Delange, 42]. See also Lemma 2.3 of [K. Ford, 68] for a slightly weaker version.

**Theorem (1987)** When $\sigma > 1$ and $t$ is any real number, we have

$$-\Re \zeta'(\sigma + it) \leq \frac{1}{\sigma - 1} - \frac{1}{2\sigma^2}.$$
Chapter 14

Explicit zero-free regions for the $\zeta$ and $L$ functions

Corresponding html file: ../Articles/Art08.html

14.1 Numerical verifications of the Generalized Riemann Hypothesis

Numerical verifications of the Riemann hypothesis for the Riemann $\zeta$-function have been pushed extremely far. B. Riemann himself computed the first zeros. Concerning more recent published papers, we mention [J. Van de Lune, H. te Riele, and D.T.Winter, 201] who proved that

Theorem (1986) Every zero $\rho$ of $\zeta$ that have a real part between 0 and 1 and an imaginary part not more, in absolute value, than $\leq T_0 = 545\,439\,823$ are in fact on the critical line, i.e. satisfy $\Re \rho = 1/2$.

The bound 545 439 823 is increased to 1 000 000 000 in [D. Platt, 147]. In [D. J. Platt, 150], this bound is further increased to 30 610 046 000. Between these results, [S. Wedeniwski, 181] announced that, he and many collaborators proved, using a network method:

Theorem (2002) $T_0 = 29\,538\,618\,432$ is admissible in the theorem above.

And [X. Gourdon and P. Demichel, 73] went one step further

[73] X. Gourdon and P. Demichel, 2004, “The $10^{13}$ first zeros of the Riemann Zeta Function and zeros computations at very large height”.
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Theorem (2004) $T_0 = 2.445 \cdot 10^{12}$ is admissible in the theorem above.

These two last announcements have not been subject to any academic papers.

One of the key ingredient is an explicit Riemann-Siegel formula due to [W. Gabcke, 71] (the preprint of Gourdon mentioned above gives a version of Gabcke’s result) and such a formula is missing in the case of Dirichlet $L$-function.

Let us introduce some terminology. We say that a modulus $q \geq 1$ (i.e. an integer!) satisfies $GRH(H)$ for some numerical value $H$ when every zero $\rho$ of the $L$-function associated to a primitive Dirichlet character of conductor $q$ and whose real part lies within the critical line (i.e. has a real part lying inside the open interval $(0, 1)$) and whose imaginary part is below, in absolute value, $H$, in fact satisfies $\Re \rho = 1/2$.

By employing an Euler-McLaurin formula, [R. Rumely, 180] has proved that

Theorem (1993) • Every $q \leq 13$ satisfies $GRH(10000)$.

• Every $q$ belonging to one of the sets
  - $\{k \leq 72\}$
  - $\{k \leq 112, k$ non premier$\}$
  - $\{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163,$
    $169, 180, 216, 243, 256, 360, 420, 432\}$

satisfies $GRH(2500)$.

These computations have been extended by [M. Bennett, 10] by using Rumely’s program. All these computations have been superseded by the work of D. Platt. [D. Platt, 147] and [D. Platt, 148] use two fast Fourier transforms, one in the $t$-aspect and one in the $q$-aspect, as well as an approximate functional equation to prove via extremely rigorous computations that

Theorem (2011-2013) Every modulus $q \leq 400000$ satisfies $GRH(100000000/q)$.

We mention here the algorithm of [S. Omar, 144] that enables one to prove efficiently that some $L$-functions have no zero within the rectangle $1/2 \leq \sigma \leq 1$ et $2\sigma - |t| = 1$ though this algorithm has not been put in practice.

There are much better results concerning real zeros of Dirichlet $L$-functions associated to real characters.

[71] W. Gabcke, 1979, “Neue Herleitung und explizite Restabschaetzung der Riemann-Siegel-Formel”.

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14.2 Asymptotical zero-free regions

The first fully explicit zero free region for the Riemann zeta-function is due to [J. Rosser, 177] in Lemma 19 (essentially with $R_0 = 19$ in the notations below). This is next improved upon in Theorem 1 of [J. Rosser and L. Schoenfeld, 179] by using a device of [S. Stechkin, 189] (getting essentially $R_0 = 9.646$). The next step is in [O. Ramaré and R. Rumely, 170] where the second order term is improved upon, relying on [S. Stechkin, 188].

Next, in [H. Kadiri, 97] and later in [H. Kadiri, 96], the following result is proven.

**Theorem (2002)** The Riemann $\zeta$-function has no zeros in the region

$$\Re s \geq 1 - \frac{1}{R_0 \log(|\Im s| + 2)}$$

with $R_0 = 5.70175$.

[W.-J. Jang and S.-H. Kwon, 90] improved the value of $R_0$ by showing that $R_0 = 5.68371$ is admissible. By plugging a better trigonometric polynomial in the same method, it is proved in [M. J. Mossinghoff and T. S. Trudgian, 138] that

**Theorem (2015)** The Riemann $\zeta$-function has no zeros in the region

$$\Re s \geq 1 - \frac{1}{R_0 \log(|\Im s| + 2)}$$

with $R_0 = 5.573412$.

Concerning Dirichlet $L$-function, the first explicit zero-free region has been obtained in [K. McCurley, 129] by adapting [J. Rosser and L. Schoenfeld, 179]. [H. Kadiri, 97] (cf also [H. Kadiri, 93]) improves that into:

**Theorem (2002)** The Dirichlet $L$-functions associated to a character of conductor $q$ has no zero in the region:

$$\Re s \geq 1 - \frac{1}{R_1 \log(q \max(1, |\Im s|))}$$

with $R_1 = 6.4355$, to the exception of at most one of them which would hence be attached to a real-valued character. This exceptional one would have at most one zero inside the forbidden region (and which is loosely called a "Siegel zero").

[177] J. Rosser, 1938, “The $n$-th prime is greater than $n \log n$”.
[179] J. Rosser and L. Schoenfeld, 1975, “Sharper bounds for the Chebyshev Functions $\theta(X)$ and $\psi(X)$”.
[96] H. Kadiri, 2005, “Une région explicite sans zéros pour la fonction $\zeta$ de Riemann”.

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In [Kadiri*18], the next theorem is proved.

**Theorem (2016)** The Dirichlet $L$-functions associated to a character of conductor $q \in [3, 400,000]$ has no zero in the region:

$$\Re s \geq 1 - \frac{1}{R_2 \log(q \max(1, |\Im s|))} \quad \text{with } R_1 = 5.60.$$ 

Concerning the Vinogradov-Korobov zero-free region, [K. Ford, 68] shows that

**Theorem (2001)** The Riemann $\zeta$-function has no zeros in the region

$$\Re s \geq 1 - \frac{1}{58(\log |\Im s|)^{2/3}(\log \log |\Im s|)^{1/3}} \quad (|\Im s| \geq 3).$$

Concerning the Dedekind $\zeta$-function, see [H. Kadiri, 94].

14.3 Real zeros

[J. Rosser, 175], [J. Rosser, 176], [K. S. Chua, 30], [M. Watkins, 205],

14.4 Density estimates

After initial work of [J. Chen and T. Wang, 28] and [M. Liu and T. Wang, 111], here are the latest two best results. We first define

$$N(\sigma, T, \chi) = \sum_{\substack{\rho = \beta + i\gamma, \\
L(\rho, \chi) = 0, \\
\sigma \leq \beta, |\gamma| \leq T}} 1$$

which thus counts the number of zeroes $\rho$ of $L(s, \chi)$, zeroes whose real part is denoted by $\beta$ (and assumed to be larger than $\sigma$), and whose imaginary part in absolute value $\gamma$ is assumed to be not more than $T$. For the Riemann $\zeta$-function (i.e. when $\chi = \chi_0$ the principal character modulo 1), it is customary to count only the zeroes with positive imaginary part. The relevant number is usually denoted by $N(\sigma, T)$. We have $2N(\sigma, T) = N(\sigma, T, \chi_0)$.

For low values, we start with the main Theorem of [H. Kadiri and N. Ng, 98]. We reproduce only a special case.

[98] H. Kadiri and N. Ng, 2012, “Explicit zero density theorems for Dedekind zeta functions”.

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Theorem (2013) Let $T \geq 3.061 \cdot 10^{10}$. We have $2N(17/20, T, \chi_0) \leq 0.5561T + 0.7586 \log T - 268658$ where $\chi_0$ is the principal character modulo 1.

Otherwise, here is the result of [O. Ramaré, 156].

Theorem (2016) For $T \geq 2000$ and $T \geq Q \geq 10$, as well as $\sigma \geq 0.52$, we have

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \mod q} N(\sigma, T, \chi) \leq 20(56 Q^5 T^3)^{1-\sigma} \log^{5-2\sigma}(Q^2 T) + 32 Q^2 \log^2(Q^2 T)$$

where $\chi \mod q$ denotes a sum over all primitive Dirichlet character $\chi$ to the modulus $q$. Furthermore, we have

$$N(\sigma, T, \chi_0) \leq 6T \log T \log \left(1 + \frac{6.87}{2T}(3T)^{8(1-\sigma)/3} \log^{4-2\sigma}(T)\right) + 103(\log T)^2$$

where $\chi_0$ is the principal character modulo 1.

In [H. Kadiri, A. Lumley, and N. Ng, 99] this result is improved upon, we refer to their paper for their result by quote a corollary.

For $T \geq 1$, we have $N(0.9, T) \leq 11.5 T^{4/14} \log^{16/9}(T) + 3.2 \log^2(T)$ where $N(\sigma, T) = N(\sigma, T, \chi_0)$ and $\chi_0$ is the principal character modulo 1.

14.5 Miscellanea

The LMFDB\(^1\) database contains the first zeros of many $L$-functions. A part of Andrew Odlyzko’s website\(^2\) contains extensive tables concerning zeroes of the Riemann zeta function.

Last updated on June 18th, 2019, by Olivier Ramaré

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[99] H. Kadiri, A. Lumley, and N. Ng, 2018, “Explicit zero density for the Riemann zeta function”.

\(^1\)http://www.lmfdb.org
\(^2\)http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html
Part VII

Sieve and short interval results
Chapter 15

Short intervals containing primes

Corresponding html file: ../Articles/Art09.html

15.1 Interval with primes, without any congruence condition

The story seems to start in 1845 when Bertrand conjectured after numerical trials that the interval \([n, 2n - 3]\) contains a prime as soon as \(n \geq 4\). This was proved by Čebišev in 1852 in a famous work where he got the first good quantitative estimates for the number of primes less than a given bound, say \(x\). By now, analytical means combined with sieve methods (see [R. Baker, G. Harman, and J. Pintz, 7]) ensures us that each of the intervals \([x, x + 0.525 x^{0.525}]\) for \(x \geq x_0\) contains at least one prime. This statement concerns only for the (very) large integers.

It falls very close to what we can get under the assumption of the Riemann Hypothesis: the interval \([x - K \sqrt{x} \log x, x]\) contains a prime, where \(K\) is an effective large constant and \(x\) is sufficiently large (cf [D. Wolke, 206] for an account on this subject). A theorem of Schoenfeld [L. Schoenfeld, 184] also tells us that the interval

\[
[x - \sqrt{x} \log^2 x / (4\pi), x]
\]

contains a prime for \(x \geq 599\) under the Riemann Hypothesis. These results are still far from the conjecture in [H. Cramer, 38] on probabilistic grounds: the


[184] L. Schoenfeld, 1976, “Sharper bounds for the Chebyshev Functions \(\theta(x)\) and \(\psi(x)\)”.

[38] H. Cramer, 1936, “On the order of magnitude of the difference between consecutive prime numbers”.
interval \([x - K \log^2 x, x]\) contains a prime for any \(K > 1\) and \(x \geq x_0(K)\). Note that this statement has been proved for almost all intervals in a quadratic average sense in [A. Selberg, 185] assuming the Riemann Hypothesis and replacing \(K\) by a function \(K(x)\) tending arbitrarily slowly to infinity. [L. Schoenfeld, 184] proved the following.

**Theorem (1976)** Let \(x\) be a real number larger than 2010760. Then the interval

\[
\left(x\left(1 - \frac{1}{16597}\right), x\right)
\]

contains at least one prime.

The main ingredient is the explicit formula and a numerical verification of the Riemann hypothesis.

From a numerical point of view, the Riemann Hypothesis is known to hold up to a very large height (and larger than in 1976). [S. Wedeniwski, 181] and the Zeta grid project verified this hypothesis till height \(T_0 = 2.41 \cdot 10^{11}\) and [X. Gourdon and P. Demichel, 73] till height \(T_0 = 2.44 \cdot 10^{12}\) thus extending the work [J. Van de Lune, H. te Riele, and D.T.Winter, 201] who had conducted such a verification in 1986 till height \(5.45 \times 10^8\). This latter computations has appeared in a refereed journal, but this is not the case so far concerning the other computations; section 4 of the paper [Y. Saouter and P. Demichel, 182] casts some doubts on whether all the zeros where checked. Discussions in 2012 with Dave Platt from the university of Bristol led me to believe that the results of [S. Wedeniwski, 181] can be replicated in a very rigorous setting, but that it may be difficult to do so with the results of [X. Gourdon and P. Demichel, 73] with the hardware at our disposal.

In [O. Ramaré and Y. Saouter, 171], we used the value \(T_0 = 3.3 \cdot 10^9\) and obtained the following.

**Theorem (2002)** Let \(x\) be a real number larger than 10726905041. Then the interval

\[
\left(x\left(1 - \frac{1}{28314000}\right), x\right)
\]

contains at least one prime.

If one is interested in somewhat larger value, the paper [O. Ramaré and Y. Saouter, 171] also contains the following.

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[184] L. Schoenfeld, 1976, “Sharper bounds for the Chebyshev Functions \(\theta(x)\) and \(\psi(x)\) II”.


[73] X. Gourdon and P. Demichel, 2004, “The \(10^{13}\) first zeros of the Riemann Zeta Function and zeros computations at very large height”.


[182] Y. Saouter and P. Demichel, 2010, “A sharp region where \(\pi(x) - \text{li}(x)\) is positive”.


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15.1 **Interval with primes, without any congruence condition**

**Theorem (2002)** Let $x$ be a real number larger than $\exp(53)$. Then the interval

$$[x \left(1 - \frac{1}{204879661}\right), x]$$

contains at least one prime.

Increasing the lower bound in $x$ only improves the constant by less than 5 percent. If we rely on [X. Gourdon and P. Demichel, 73], we can prove that

**Theorem (2004, conditional)** Let $x$ be a real number larger than $\exp(60)$. Then the interval

$$[x \left(1 - \frac{1}{14500755538}\right), x]$$

contains at least one prime.

Note that all prime gaps have been computed up to $10^{15}$ in [T. Nicely, 140], extending a result of [A. Young and J. Potler, 207].

In [T. Trudgian, 199], we find

**Theorem (2016)** Let $x$ be a real number larger than $2898242$. The interval

$$[x, x \left(1 + \frac{1}{111 \log x^2}\right)]$$

contains at least one prime.

In [P. Dusart, 58], we find

**Theorem (2016)** Let $x$ be a real number larger than $468991632$. The interval

$$[x, x \left(1 + \frac{1}{5000 \log x^2}\right), x]$$

contains at least one prime.

Let $x$ be a real number larger than $89693$. The interval

$$[x, x \left(1 + \frac{1}{\log^3 x}\right)]$$

contains at least one prime.

The proof of these latter results has an asymptotical part, for $x \geq 10^{20}$ where we used the numerical verification of the Riemann hypothesis together with two other arguments: a (very strong) smoothing argument and a use of the Brun-Titchmarsh inequality.

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[199] T. Trudgian, 2016, “Updating the error term in the prime number theorem”.

[58] P. Dusart, 2018, “Estimates of some functions over primes”.

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The second part is of algorithmic nature and covers the range $10^{19} \leq x \leq 10^{20}$ and uses prime generation techniques [U. Maurer, 125]: we only look at families of numbers whose primality can be established with one or two Fermat-like or Pocklington’s congruences. This kind of technique has been already used in a quite similar problem in [J.-M. Deshouillers, H. te Riele, and Y. Saouter, 47]. The generation technique we use relies on a theorem proven in [J. Brillhart, D. Lehmer, and J. Selfridge, 22] and enables us to generate dense enough families for the upper part of the range to be investigated. For the remaining (smaller) range, we use theorems of [G. Jaeschke, 89] that yield a fast primality test (for this limited range).

**Theorem (2002)** Under the Riemann Hypothesis, the interval $]x - \frac{8}{5} \sqrt{x} \log x, x]$ contains a prime for $x \geq 2$.

This is improved upon in [A. W. Dudek, 53] into:

**Theorem (2015)** Under the Riemann Hypothesis, the interval $]x - \frac{4}{5} \sqrt{x} \log x, x]$ contains a prime for $x \geq 2$.

In [E. Carneiro, M. Milinovich, and K. Soundararajan, 26], the authors go one step further and prove the next result.

**Theorem (2019)** Under the Riemann Hypothesis, the interval $]x - \frac{22}{25} \sqrt{x} \log x, x]$ contains a prime for $x \geq 4$.

Let us recall here that a second line of approach following the original work of Čebyshev is still under examination though it does not give results as good as analytical means (see [N. Costa Pereira, 37] for the latest result).

### 15.2 Interval with primes, with congruence condition

Collecting references: [K. McCurley, 128], [K. McCurley, 127], [H. Kadiri, 95].

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[22] J. Brillhart, D. Lehmer, and J. Selfridge, 1975, “New primality criteria and factorizations for $2^m \pm 1$.”
[37] N. Costa Pereira, 1989, “Elementary estimates for the Chebyshev function $\psi(X)$ and for the Möbius function $M(X)$”.

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Chapter 16

Sieve bounds

Corresponding html file: ../Articles/Art14.html

16.1 Some upper bounds

Theorem 2 of [H. Montgomery and R. Vaughan, 133] contains the following explicit version of the Brun-Titchmarsh Theorem.

**Theorem (1973)** Let $x$ and $y$ be positive real numbers, and let $k$ and $\ell$ be relatively prime positive integers. Then

$$\pi(x + y; k, \ell) - \pi(x; k, \ell) < \frac{2y}{\phi(k) \log(y/k)}$$

provided only that $y > k$.

Here as usual, we have used the notation

$$\pi(z; k, \ell) = \sum_{\substack{p \leq z, \\ p \equiv \ell \ [k]}} 1,$$

i.e. the number of primes up to $z$ that are coprime to $\ell$ modulo $k$. See [J. B"uthe, 23] for a generic weighted version of this inequality.

Here is a bound concerning a sieve of dimension 2 proved by [H. Siebert, 186].

**Theorem (1976)** Let $a$ and $b$ be coprime integers with $2|ab$. Then we have, for $x > 1$,

$$\sum_{\substack{p \leq x, \\ ap + b \text{ prime}}} 1 \leq 16\omega \frac{x}{(\log x)^2} \prod_{\substack{p|ab, \\ p > 2}} \frac{p - 1}{p - 2} \omega = \prod_{p > 2} (1 - (p - 1)^{-2}).$$

16.2 Combinatorial sieve estimates

The combinatorial sieve is known to be difficult from an explicit viewpoint. For the linear sieve, the reader may look at Chapter 9, Theorem 9.7 and 9.8 from [M. B. Nathanson, 139].

Last updated on July 18th, 2013, by Olivier Ramaré
Part VIII

Analytic Number Theory in Number Fields
Chapter 17

Bounds on the Dedekind zeta-function

Corresponding html file: ../Articles/Art18.html

17.1 Size

The knowledge on the general Dedekind zeta is less accomplished than the one of the Riemann zeta-function, but we still have interesting results. Theorem 4 of [H. Rademacher, 155] gives the convexity bound. See also section 4.1 of [T. S. Trudgian, 200].

Theorem (1959) In the strip $-\eta \leq \sigma \leq 1 + \eta$, $0 < \eta \leq 1/2$, the Dedekind zeta function $\zeta_K(s)$ belonging to the algebraic number field $K$ of degree $n$ and discriminant $d$ satisfies the inequality

$$|\zeta_K(s)| \leq 3 \left| \frac{1 + s}{1 - s} \right| \left( \frac{|d| |1 + s|}{2 \pi} \right)^{\frac{1 + \eta - \sigma}{2}} \zeta(1 + \eta)^n.$$

17.2 Zeroes and zero-free regions

We denote by $N_K(T)$ the number of zeros $\rho$, of the Dedekind zeta-function of the number field $K$ of degree $n$ and discriminant $d_K$, zeros that lie in the critical strip $0 < \Re \rho = \sigma < 1$ and which verify $|\Im \rho| \leq T$. After a first result in [H. Kadiri and N. Ng, 98], we find in [T. Trudgian, 196] The following result.

[98] H. Kadiri and N. Ng, 2012, “Explicit zero density theorems for Dedekind zeta functions”.
Theorem (1959) When $T \geq 1$, we have $N_K(T) = \frac{T}{\pi} \log \left( \left| d_K \right| \left( \frac{T}{2 \pi e} \right)^n \right) + O^* (0.316 (\log |d_K| + n \log T) + 5.872 n + 3.655)$. 

In [H. Kadiri, 94], a zero-free region is proved.

Theorem (1959) Let $K$ be a number field of degree $n$ over $\mathbb{Q}$ and of discriminant $d \geq 2$. The associated Dedekind zeta-function $\zeta_K$ has no zeros in the region

$$\sigma \geq 1 - \frac{1}{12.55 \log |d_K| + n (9.69 \log |t| + 3.03) + 58.63}, |t| \geq 1$$

and at most one zero in the region

$$\sigma \geq 1 - \frac{1}{12.74 \log |d_K|}, |t| \leq 1.$$

The exceptional zero, if it exists, is simple and real.


Last updated on February 14th, 2017, by Olivier Ramaré.


November 5, 2020
Part IX

Applications
Chapter 18

Explicit bounds for class numbers

Let $K$ be a number field of degree $n \geq 2$, signature $(r_1, r_2)$, absolute value of discriminant $d_K$, class number $h_K$, regulator $R_K$ and $w_K$ the number of roots of unity in $K$. We further denote by $\kappa_K$ the residue at $s = 1$ of the Dedekind zeta-function $\zeta_K(s)$ attached to $K$.

Estimating $h_K$ is a long-standing problem in algebraic number theory.

18.1 Majorising $h_K R_K$

One of the classic way is the use of the so-called analytic class number formula stating that

$$h_K R_K = w_K \sqrt{d_K} \frac{\kappa_K}{2^{r_1} (2\pi)^{r_2}}$$

and to use Hecke’s integral representation of the Dedekind zeta function to bound $\kappa_K$. This is done in [S. Louboutin, 114] and in [S. Louboutin, 116] with additional properties of log-convexity of some functions related to $\zeta_K$ and enabled Louboutin to reach the following bound:

$$h_K R_K \leq \frac{w_K}{2} \left( \frac{2}{\pi} \right)^{r_2} \left( \frac{e \log d_K}{4n - 4} \right)^{n-1} \sqrt{d_K}.$$ 

Furthermore, if $\zeta_K(\beta) = 0$ for some $\frac{1}{2} \leq \beta < 1$, then we have

$$h_K R_K \leq (1 - \beta) w_K \left( \frac{2}{\pi} \right)^{r_2} \left( \frac{e \log d_K}{4n} \right)^n \sqrt{d_K}.$$ 

[114] S. Louboutin, 2000, “Explicit bounds for residues of Dedekind zeta functions, values of $L$-functions at $s = 1$, and relative class numbers”.

[116] S. Louboutin, 2001, “Explicit upper bounds for residues of Dedekind zeta functions and values of $L$-functions at $s = 1$, and explicit lower bounds for relative class numbers of CM-fields”.
When \( K \) is abelian, then the residue \( \kappa_K \) may be expressed as a product of values at \( s = 1 \) of \( L \)-functions associated to primitive Dirichlet characters attached to \( K \). On using estimates for such \( L \)-functions from [O. Ramaré, 157], we get for instance

\[
h_K R_K \leq \frac{w_K}{2} \left( \frac{2}{\pi} \right)^{r_2} \left( \frac{\log d_K}{4n-4} + \frac{5 - \log 36}{4} \right)^{n-1} \sqrt{d_K}.
\]

Note that the constant \( \frac{1}{4}(5 - \log 36) = 0.354 \cdots \) can be improved upon in many cases. For instance, when \( K \) is abelian and totally real (i.e. \( r_2 = 0 \)), a result from [O. Ramaré, 157] implies that the constant may be replaced by 0, so that

\[
h_K R_K \leq \left( \frac{\log d_K}{4n-4} \right)^{n-1} \sqrt{d_K}.
\]

### 18.2 Majorising \( h_K \)

One may also estimate \( h_K \) alone, without any contamination by the regulator since this contamination is often difficult to control, see [M. Pohst and H. Zassenhaus, 151].

In this case, one rather uses explicit bounds for the Piltz-Dirichlet divisor functions \( \tau_n \) (see [O. Bordellès, 18] and [O. Bordellès, 16]) and get

\[
h_K \leq \frac{M_K}{(n-1)!} \left( \frac{\log (M_K^2 d_K)}{2} + n - 2 \right)^{n-1} \sqrt{d_K}
\]

as soon as

\[n \geq 3, \quad d_K \geq 139M_K^2 \]

The constant \( M_K \) is known as the Minkowski constant of \( K \).

### 18.3 Using the influence of small primes

It is explained in [S. Louboutin, 120] how the behavior of certain small primes may substantially improve on the previous bounds. To make things more significant, define, for a rational prime \( p \),

\[
\Pi_K(p) = \prod_{p \mid \mathfrak{p}} \left( 1 - \frac{1}{N_K(p)} \right)^{-1}.
\]

[157] O. Ramaré, 2001, “Approximate Formulae for \( L(1, \chi) \)”.


[18] O. Bordellès, 2002, “Explicit upper bounds for the average order of \( d_n(m) \) and application to class number”.


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From [S. Louboutin, 120], we have among other things

\[ h_K R_K \leq w_K \left( \frac{2}{\pi} \right)^{r_2} \frac{\Pi K(2)}{\Pi Q(2)^n} \left( \frac{e \log d_K}{4n - 4} \times e^{n \log 4 / \log d_K} \right)^{n-1} \sqrt{d_K} \]

where \( K \) is any number field of degree \( n \geq 3 \). In particular, when 2 is inert in \( K \), then

\[ h_K R_K \leq w_K \left( \frac{2}{2(2^n - 1)} \right)^{r_2} \left( e^{ \log \left( \frac{d_K}{d_K + \kappa_K} \right) 4 \pi n} \times e^{n \log 4 / \log d_K} \right)^{n-1} \sqrt{d_K}. \]

18.4 The \( h_K^- \) of CM-fields

Let \( K \) be here a CM-field of degree \( 2n > 2 \), i.e. a totally complex quadratic extension \( K \) of its maximal totally real subfield \( K^+ \). It is well known that \( h_K^+ \) divides \( h_K \). The quotient is denoted by \( h_K^- \) and is called the relative class number of \( K \). The analytic class number formula yields

\[ h_K^- = Q_K w_K \left( \frac{d_K}{d_K + \kappa_K} \right)^{1/2} \frac{\kappa_K}{\kappa_K^+} = Q_K w_K \left( \frac{d_K}{d_K + \kappa_K} \right)^{1/2} L(1, \chi) \]

where \( \chi \) is the quadratic character of degree 1 attached to the extension \( K/K^+ \) and \( Q_K \in \{1, 2\} \) is the Hasse unit index of \( K \). Here are three results originating in this formula.

From [S. Louboutin, 114]:

**Theorem (2000)** We have

\[ h_K^- \leq 2Q_K w_K \left( \frac{d_K}{d_K^+} \right)^{1/2} \left( \frac{e \log(d_K/d_K^+)}{4\pi n} \right)^n. \]

From [S. Louboutin, 115]:

**Theorem (2003)** Assume that \((\zeta_K/\zeta_{K^+})(\sigma) \geq 0 \) whenever \( 0 < \sigma < 1 \). Then we have

\[ h_K^- \geq \frac{Q_K w_K}{\pi e \log d_K} \left( \frac{d_K}{d_K^+} \right)^{1/2} \left( \frac{n - 1}{\pi e \log d_K} \right)^{n-1}. \]

Again from [S. Louboutin, 115]:

**Theorem (2003)** Let \( c = 6 - 4\sqrt{2} = 0.3431 \cdots \). Assume that \( d_K \geq 2800^n \) and that either \( K \) does not contain any imaginary quadratic subfield, or that the real zeros in the range \( 1 - \frac{e}{\log d_N} \leq \sigma < 1 \) of the Dedekind zeta-functions of

\[ [114] \text{S. Louboutin, 2000, “Explicit bounds for residues of Dedekind zeta functions, values of } L\text{-functions at } s=1, \text{and relative class numbers”}. \]

\[ [115] \text{S. Louboutin, 2003, “Explicit lower bounds for residues at } s=1 \text{ of Dedekind zeta functions and relative class numbers of CM-fields”}. \]
the imaginary quadratic subfields of $K$ are nor zeros of $\zeta_K(s)$, where $N$ is the normal closure of $K$. Then we have

$$h_K^{-} \geq \frac{cQKw_K}{4ne^{c/2}[N:Q]} \left( \frac{d_K}{d_{K+}} \right)^{1/2} \left( \frac{n}{\pi e \log d_K} \right)^n.$$ 

And a third result from [S. Louboutin, 115]:

**Theorem (2003)** Assume $n > 2$, $d_K > 2800^n$ and that $K$ contains an imaginary quadratic subfield $F$ such that $\zeta_F(\beta) = \zeta_K(\beta) = 0$ for some $\beta$ satisfying $1 - \frac{2}{\log d_K} \leq \beta < 1$. Then we have

$$h_K^{-} \geq \frac{6}{(\pi e)^2} \left( \frac{d_K}{d_{K+}} \right)^{1/2-1/n} \left( \frac{n}{\pi e \log d_K} \right)^{n-1}.$$ 

Last updated on August 23rd, 2012, by Olivier Bordellès

Chapter 19

Primitive Roots

Corresponding html file: ../Articles/Art19.html

Last updated on July 14th, 2012, by Olivier Ramaré
Part X

Development
Chapter 20

README

20.1 How to write

This part is technical and destined at the development team only.

1. Sections are coded via `<div class="section">1. Title</div>`. The numbering is done by hand.

2. Theorems, Lemmas, Propositions are coded via

   `<span class="THM">Theorem (1996) </span`
   `<blockquote class="outer-thm">
   `<div class="thm">
   Every zero $\rho$ of $\zeta$ that have a real part between 0 and 1 and an imaginary part not more, in absolute value, than $T_0=545439823$ are in fact on the critical line, i.e. satisfy $\Re \rho=1/2$.
   </div>
   </blockquote>`

Note that the part `<div></div>` should be on a single line.

3. Mathematics are entered latex style and processed via MathJax. Macros are to be avoided, of course.

4. A reference is introduced on one line in the form

   `<script language="javascript">bibref("Ramare*12")</script>`

   where Ramare*12 is the key of the bibtex entry, which has to be introduced in the Local-TME-EMT.bib file.

The file `Latex/booklet.tex` is the master file of the PDF booklet and has to be edited by hand. The Perl script `Biblio/UpDateBiblio.pl` will create tex-files that are to be used as chapter for each html-file found in `Articles/`, save the file `Template_Article.html` of course.
20.2 How to contribute

Everyone is most welcome to help us keep track of the results. You can do so by simply sending to the development team a mail with the proper information (in bibtex format for the relevant part). You may also propose a new annotated bibliography, for instance for "Explicit results in the combinatorial sieve", or any other missing entry. There are other ways to contribute, like modifying the CSS so that this site would be readable under windows, a fact we do not guarantee. Or by proposing a rewrite of some already present bibliography. Or anything else we did not think about.

20.3 Adding a part or a chapter

Adding a chapter requires several steps.

- Modify the file accueil.html.

- Modify in the file ../MiseEnPage.js the variable Architecture_TME_EMT. This modification changes the numbers and each file Articles/Art**.html has to be subsequently modified at the level of the command BandeauGeneral(2, "../", [0, 2]).

- Modify accordingly the file Latex/booklet.tex.
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