Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions

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Abstract

We prove that $\left| \sum_{d \leq x, \ (d,q)=1} \frac{\mu(d)}{d} \right| \leq 2.4 \left( \frac{q}{\varphi(q)} \right) / \log(\frac{x}{q})$ for every $x > q \geq 1$ and similar estimates for the Liouville functions. We give also better constants when $x/q$ is larger.

1 Introduction

In explicit analytic number theory, one needs very often to evaluate the average of a multiplicative function, say $f$. The usual strategy is to compare this function to a more usual model $f_0$, as in [12, Lemma 3.1]. This process is also well detailed in [2]. When the model is $f_0 = 1$, the situation is readily cleared out; it is also well studied when this model is the divisor function in [1, Corollary 2.2]. We signal here that the case of the characteristic function of the squarefree numbers is specifically handled in [4]. The next problem is to use the Moebius function as a model. In this case the necessary material can be found in [13], though of course the saving is much less and may be insufficient: when the model is $1$ or the divisor function, or the characteristic

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function of the squarefree integers, the saving is a power of the size of the variable, while now it is only a logarithm (or the square of one according to whether one says that the trivial estimate for \( \sum_{d \leq D} \mu(d)/d \) is 1 or log \( D \)). One of the consequences is that one has to be more careful, and thrifty, when it comes to small variations. The variations we consider here is the addition of a coprimality condition \((d, q) = 1\), for some fixed \( q \), on the ranging variable \( d \). Our first aim is thus to show how to get explicit estimates for the family of functions

\[
m_q(x) = \sum_{\substack{n \leq x, \\
(n,q) = 1}} \mu(n)/n,
m(x) = m_1(x)
\]

(1.1)

from explicit estimates concerning solely \( m(x) \). The definition of the Liouville function \( \lambda(n) \) is recalled below in (1.3), while the auxiliary function \( \ell_q \) is defined in (1.4).

**Theorem 1.1.** We have, when \( 1 \leq q < x \), where \( q \) is an integer and \( x \) a real number,

\[
\left| \sum_{\substack{n \leq x, \\
(n,q) = 1}} \frac{\mu(n)}{n} \right| \leq \frac{q}{\varphi(q)} \frac{2.4}{\log(x/q)},
\]

\[
\left| \sum_{\substack{n \leq x, \\
(n,q) = 1}} \frac{\lambda(n)}{n} \right| \leq \frac{q}{\varphi(q)} \frac{0.79}{\log(x/q)}.
\]

Moreover \( \log(x/q)|\ell_q(x)| \leq 0.155 \frac{q}{\varphi(q)} \) and \( \log(x/q)|m_q(x)| \leq \frac{3}{2} \frac{q}{\varphi(q)} \) when \( x/q \geq 3310 \). We also have \( \log(x/q)|m_q(x)| \leq \frac{7}{8} \frac{q}{\varphi(q)} \) when \( x/q \geq 9960 \).

The sole previous estimate on \( m_q(x) \) seems to be \cite[Lemma 10.2]{7} which bounds \( |m_q(x)| \) uniformly by 1. The estimate for \( m(x) \) that will provide the initial step comes from \cite{13}

\[
|m(x)| \leq 0.03/\log x \quad (x \geq X_0 = 11815).
\]

(1.2)

Let us first note that the simplest treatment of this condition via the Moebius function, i.e. writing

\[\mathbb{1}_{(d,q)=1} = \sum_{\delta | d, \delta | q} \mu(\delta),\]

does not work here. Indeed we get:

\[
\sum_{d \leq D, \ (d,q) = 1} \frac{\mu(d)}{d} = \sum_{\delta | q} \mu(\delta) \sum_{\delta | d \leq D} \frac{\mu(d)}{d} = \sum_{\delta | q} \frac{\mu(\delta)^2}{\delta} \sum_{d \leq D/\delta, \ (d,\delta) = 1} \frac{\mu(d)}{d}
\]
and we are back to the initial problem with different parameters. The classical workaround (used for instance in [10, near (7)] but already known by Landau) runs as follows: we determine a function $g_q$ such that $\mathbb{1}_{(n,q)=1}\mu(n) = g_q \ast \mu(n)$, where $\ast$ denotes the arithmetic convolution product. The drawback of this method is that the support of $g$ is not bounded (determining $g_q$ by comparing the Dirichlet series is a simple exercise). So if we write

$$\sum_{\substack{d \leq D, \\ (d,q)=1}} \mu(d)/d = \sum_{\delta \leq D} \frac{g_q(\delta)}{\delta} \sum_{d \leq D/\delta} \frac{\mu(d)}{d},$$

we are forced to two things:

1. using estimates for $\sum_{d \leq D/\delta} \mu(d)/d$ when $D/\delta$ can be small,
2. completing the sum over $\delta$ to get a decent result.

Both steps introduce quite a loss when $q$ is not specified. We propose here a different approach by introducing the Liouville function as an intermediary. This function $\lambda(n)$ is the completely multiplicative function that is $1$ on integers that have an even number of prime factors – counted with multiplicity – and $-1$ otherwise. It satisfies

$$\sum_{n \geq 1} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}. \tag{1.3}$$

We introduce the family of auxiliary functions

$$\ell_q(x) = \sum_{\substack{n \leq x, \\ (n,q)=1}} \lambda(n)/n, \quad \ell(x) = \ell_1(x). \tag{1.4}$$

Our process runs as follows: we derive bounds for $\ell(x)$ from bounds on $m(x)$ and some computations, derive bounds on $\ell_q(x)$ from bounds on $\ell(x)$, and finally derive bounds on $\mu_q(x)$ from bounds on $\ell_q(x)$. The theoretical steps are contained in the three Lemmas 2.3, 2.5 and 3.2.

We complete this introduction by signalling that [14] contains explicit estimates with a large range of uniformity for sums of the shape

$$\sum_{\substack{d \leq x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}$$

and for a similar sum but with the summand $\mu(d) \log(x/d)/d^{1+\varepsilon}$. The path we followed there is essentially elementary and the saving is less.
I thank Harald Helfgott for interesting discussions that pushed me into pulling this note out of its drawer. Special thanks are also due to the referee for his/her very careful reading; several numerical errors have been corrected in the process, and the arguments are also now better exposed.

2 From the Moebius function to the Liouville function

Lemma 2.1. For $2 \leq x \leq 906\,000\,000$, we have $|\ell(x)| \leq 1.347/\sqrt{x}$.

For $2 \leq x \leq 1.1 \cdot 10^{10}$, we have $|\ell(x)| \leq 1.41/\sqrt{x}$.

For $1 \leq x \leq 1.1 \cdot 10^{10}$, we have $|\ell(x)| \leq \sqrt{2}/x$.

The computations have been run with PARI/GP (see [11]), speeded by using gp2c as described for instance in [1]. We mention here that [6] proposes an algorithm to compute isolated values of $M(x)$. This can most probably be adapted to compute isolated values of $\ell(x)$, but does not seem to offer any improvement for bounding $|\ell(x)|$ on a large range. In [3], the authors show that

$$\ell(x) \geq 0, \quad (x < 72\,185\,376\,951\,205)$$

and that

$$\ell(x) \geq -2.0757642 \times 10^{-9}, \quad (x \leq 75\,000\,000\,000\,000)$$

This takes care of the lower bound for $\ell(x)$. The computations we ran are much less demanding in time and algorithm, but however rely on a large enough sieve-kind of table to compute values of $\lambda(n)$ on some very large range. Harald Helfgott (indirectly) pointed out to me that the RAM-memory can be very large nowadays, allowing to precompute large quantities to which one has an almost immediate access. Here is a simplified version of the main loop:

```plaintext
{getbounds(zmin:small, valini:real, zmax:small) =
   my(maxi:real, valuesliouville:vecsmall, gotit:vecsmall, valuel:real, bound:small, pa:small);
   /* Precomputing lambda(n): */
   valuesliouville = vectorsmall(zmax-zmin+1, m, 1);
   gotit = vectorsmall(zmax-zmin+1, m, 1);
   forprime (p:small = 2, floor(sqrt(zmax+0.0))},
```

4
bound = floor(log(zmax+0.0)/log(p+0.0));
pa = 1;
for(a:small = 1, bound,
    pa *= p;
    for(k:small = 1, floor((zmax+0.0)/pa),
        if(k*pa >= zmin,
            valuesliouville[k*pa-zmin+1] *= -1;
            gotit[k*pa-zmin+1] *= p,)))));

/* Correction in case of a large prime factor: */
for(n:small = zmin, zmax,
    if(gotit[n-zmin+1] < n,
        valuesliouville[n-zmin+1] *= -1,));

valuel = (valini + 0.0) + valuesliouville[1]/zmin;
maxi = max( valini*sqrt(zmin+0.0), abs(valuesl*sqrt(zmin+1.0)));

/* Main loop: */
for(n:small = zmin+1, zmax,
    valuel += valuesliouville[n-zmin+1]/n;
    maxi = max(maxi, abs(valuel)*sqrt(n+1.0)));

return([maxi, valuel]);
}

We used this loop to compute our maximum on intervals of length $2 \cdot 10^7$. The main function aggregates these results by making the interval vary. The computations took about half a day on a 64 bits fast desktop equipped of 8G of RAM. In the actual script, we also checked that the computed value of $\ell(x)$ is non-negative in this range. Going farther would improve on the final constants, but only when $x/q$ is large. We compared $|\ell(x)|$ with $1/\sqrt{x}$, and this seems correct for small values, but the works [9] and [8] suggest that the maximal order is larger than that.

**Lemma 2.2.** The function

$$T(y) : y \rightarrow \frac{\log y}{y} \int_{\sqrt{X_0}}^{y} \frac{dv}{\log v}$$

satisfies $T(y) \leq 1.119$ for $y \geq 10^5$. 

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Proof. A repeated integration by parts shows that

\[ T(y) = \log \frac{y}{\log y} \left( \frac{y}{\log y} - \frac{\sqrt{X_0}}{\log \sqrt{X_0}} + \frac{y}{(\log y)^2} - \frac{\sqrt{X_0}}{(\log \sqrt{X_0})^2} + 2 \int_{\sqrt{X_0}}^{y} \frac{dv}{(\log v)^3} \right) \]

\[ \leq \log \frac{y}{\log y} \left( \frac{y}{\log y} - \frac{\sqrt{X_0}}{\log \sqrt{X_0}} + \frac{y}{(\log y)^2} - \frac{\sqrt{X_0}}{(\log \sqrt{X_0})^2} \right) + \frac{2T(y)}{(\log \sqrt{X_0})^2} \]

from which we deduce that

\[ T(y) \leq 1.1001 \cdot \left( 1 + \frac{1}{\log y} \right). \]

This shows that \( T(y) \leq 1.113 \) when \( y \geq 10^{40} \). We then check numerically that the function \( T \) is increasing and then decreasing, with a maximum around 12478.8 with value 1.118 598 + \( O(10^{-6}) \). But this is only an observation, since a computer computes only a sample of values. Since the derivative of \( T \) can easily be bounded, we obtain the claimed upper bound. The reader may also consult [5] where a similar process is fully detailed.

The following lemma is a simple exercise:

**Lemma 2.3.** We have

\[ \ell_q(x) = \sum_{u^2 \leq x, (u,q)=1} m_q(x/u^2)/u^2. \]  

(2.1)

We shall use it only when \( q = 1 \), but it is equally easy to state it in general.

**Lemma 2.4.** For \( x > 1 \), we have \(|\ell(x)| \leq 0.79/\log x \).

For \( x \geq 3310 \), we have \(|\ell(x)| \leq 0.155/\log x \).

For \( x \geq 8918 \), we have \(|\ell(x)| \leq 0.099/\log x \).

Proof. We appeal to Lemma 2.3 (with \( q = 1 \)) and separate the sum according to \( u \leq U \) or \( u > U \) where \( x/U^2 \geq X_0 \). When \( u \leq U \) we apply (1.2), in the other case we use that \( |m(x)| \leq 1 \)

\[ |\ell(x)| \leq 0.03 \sum_{u \leq U} \frac{1}{u^2 \log(x/u^2)} + \frac{1+U^{-1}}{U} \]

With \( f(t) = 1/(t^2 \log(x/t^2)) \), we check that

\[ f'(t) = -\frac{2}{t^3 \log(x/t^2)} + \frac{2}{t^3 \log^2(x/t^2)}. \]
This quantity is negative for \(1 \leq t \leq U\), since then \(x/t^2 \geq x/U^2 \geq X_0 > e\). We thus have

\[
\sum_{u \leq U} \frac{1}{u^2 \log(x/u^2)} \leq f(1) + \int_1^U f(t) dt = \frac{1}{\log x} + \int_1^U \frac{dt}{t^2 \log(x/t^2)}.
\]

Changing variables we get

\[
\sum_{u \leq U} \frac{1}{u^2 \log(x/u^2)} \leq \frac{1}{\log x} + \frac{1}{\sqrt{x}} \int_{\sqrt{x}/U^2}^{\sqrt{x}} \frac{dv}{2 \log v}.
\]

It follows that

\[
|\ell(x)| \leq \frac{0.03}{\log x} + \frac{0.03}{\sqrt{x}} \int_{\sqrt{x}}^{\sqrt{x}/X_0} \frac{dv}{2 \log v} + \frac{1 + \sqrt{X_0/X}}{\sqrt{x}/X_0}.
\]

We apply Lemma 2.2 at this level. Hence, when \(x \geq 10^{10}\),

\[
|\ell(x)| \leq \frac{0.03}{\log x} + \frac{0.03 \cdot 1.119}{\log x} + \frac{1 + \sqrt{X_0/x}}{\sqrt{x}/X_0} \leq \frac{0.06357}{\log x} + \frac{(1 + \sqrt{X_0/x}) \log x}{\sqrt{x}/X_0} \frac{1}{\log x} \leq \frac{0.089}{\log x} \leq \frac{0.099}{\log x}.
\]

We extend it to \(x \geq 17715\) via Lemma 2.1, part one and two, and to \(x \geq 8918\) by direct inspection. This inequality extends to \(x \geq 1\) by weakening the constant 0.099 to 0.79. It is straightforward to use some mild computations to check the validity of the bound 0.155 when \(x \geq 3310\).

**Adding coprimality conditions**

Our tool is provided by the simple elementary lemma.

**Lemma 2.5.** We have

\[
\ell_q(x) = \sum_{d|q} \mu^2(d) \ell(x/d).
\]

The second part of Theorem 1.1 follows immediately by combining Lemma 2.5 together with Lemma 2.4. Actually, what comes out is the bound

\[
|\ell_q(x)| \leq \frac{0.79}{\log(x/q)} \sum_{d|q} \frac{\mu^2(d)}{d} = \frac{0.79}{\log(x/q)} \prod_{p|q} \frac{p + 1}{p}.
\]
As the function \( q/\varphi(q) \) is easier to remember and \( \prod_{p|q} \frac{p+1}{p} \leq q/\varphi(q) \), we simplify the above into

\[
|\ell_q(x)| \leq \frac{0.79}{\log(x/q)} \frac{q}{\varphi(q)}.
\]

When \( x/q \geq 3310 \), one can replace 0.79 by 0.155, and when \( x/q \geq 8918 \), by 1/10.

### 3 Back to the Moebius function with coprimality coditions

Let us start with a wide ranging estimate:

**Lemma 3.1.** We have, for every integer \( q \geq 1 \) and every real number \( x \geq 1 \),

\[
|\ell_q(x)| \leq \frac{\pi^2}{6}.
\]

**Proof.** This a direct consequence of Lemma 2.3 and [7, Lemma 10.2].

The following lemma is again a simple exercise.

**Lemma 3.2.** We have

\[
m_q(x) = \sum_{\substack{u^2 \leq x, \ (u,q) = 1}} \frac{\mu(u)}{u^2} \ell_q(x/u^2).
\]

**Proof of Theorem 1.1.** We have to prove several estimates of type

\[
\varphi(q) \log(x/q)|m_q(x)| \leq c, \quad x/q \geq N.
\]

We put \( x^* = x/q \) and \( y = \log x^* = \log(x/q) \) and separate the proof in two parts. First we consider the case \( 1 \leq y \leq 8 \), and later the case \( y > 8 \).

**Case (A) :** \( 1 \leq y \leq 8 \). We appeal to Lemma 3.2. We have for a real parameter \( U \) such that \( U^2 \leq x^* \)

\[
|m_q(x)| \leq \sum_{u^2 \leq x} \frac{\mu^2(u)}{u^2} |\ell_q(x/u^2)| \leq \sum_{u \leq U} \frac{q}{\varphi(q)} \frac{0.79 \mu^2(u)}{u^2} \log(x/u^2) + \frac{\pi^2}{6} \sum_{u > U} \frac{\mu^2(u)}{u^2} \log(x/q).
\]

\[
\leq \frac{q/\varphi(q)}{\log(x/q)} \left( \sum_{u \leq U} \frac{0.79 \mu^2(u)}{u^2 (1 - \frac{2 \log u}{\log(x/q)})} + \frac{\pi^2}{6} \sum_{u > U} \frac{\mu^2(u)}{u^2} \log(x/q) \right).
\]

\[^1\text{If we were to adapt the proof presented in [7] to the case of } \lambda \text{ instead of } \mu, \text{ we would reach the bound 2 and not } \pi^2/6.\]
This is our starting inequality. We define

$$\rho(U, y) = 0.79 \sum_{u \leq U} \frac{\mu^2(u)}{u^2 (1 - \frac{2 \log u}{y})} + \frac{\pi^2}{6} \sum_{u > U} \frac{\mu^2(u)}{u^2} y.$$  \hspace{1cm} (3.2)

Note that $\rho(U, y) = \rho([U], y)$ where $[U]$ is the integer part of $U$. For each $y$ we need to select one $U$ such that $\rho(U, y) \leq 2.4$. We choose $U = 1$ for $y \in [1, a_1]$; $U = 2$ for $y \in [a_1, a_2]$; $U = 3$ for $y \in [a_2, a_3]$; and $U = 7$ for $y \in [a_3, 8]$.

Where $a_1 = 1.8665 \cdots$ is a solution of $\rho(1, y) = \rho(2, y)$; $a_2 = 2.6774 \cdots$ is a solution of $\rho(2, y) = \rho(3, y)$; $a_3 = 4.1237 \cdots$ is a solution of $\rho(3, y) = \rho(7, y)$.

Each of these three functions is a sum of a linear term $ay$ and terms of type $\frac{A_y}{(y - 2 \log n)}$ with $A > 0$. These are convex for $y > 2 \log n$. In this way it is very easy to show that $\rho(1, y)$ is convex in $[1, a_1]$, $\rho(2, y)$ is convex in $[a_1, a_2]$, $\rho(3, y)$ is convex in $[a_2, a_3]$ and finally that $\rho(7, y)$ is convex in $[a_3, 8]$. So, for example, to show the inequality $\rho(3, y) \leq 2.4$ in the interval $[a_2, a_3]$ we only have to show that $\rho(3, a_2)$, and $\rho(3, a_3) \leq 2.4$. This presents no difficulty.

The maximum value obtained is $\rho(2, a_2) = 2.38790 \cdots$ with

$$a_2 = \frac{237 + 100\pi^2 \log 3}{50\pi^2},$$

$$\rho(2, a_2) = \frac{237}{20\pi^2} + \pi^2 \left(\frac{79 \log 2}{948 + 400\pi^2 \log(3/2)} - \frac{5 \log 3}{12}\right) + \log 243.$$

Case (B) : $y > 8$.

We start from Lemma 3.2, from which we deduce the simpler bound:

$$|m_q(x)| \leq \sum_{u^2 \leq x} \left| e_q(x/u^2) \right|/u^2$$

which we then exploit in the same way as what is done in the proof of Lemma 2.4, replacing the bound $|m(x)| \leq 1$ by Lemma 3.1. With $x = eU^2 q$ and $x^* = x/q$, we thus get

$$|m_q(x)| \leq \frac{q}{\varphi(q) \log x^*} \left(0.79 + \frac{0.79 \varphi(q)}{\varphi(q)} \int_1^{\sqrt{x^*/e}} \frac{du}{u^2 \log(x^*/u^2)} + \frac{\pi^2 \sqrt{e}}{6} \frac{\log 1 + \sqrt{e/x^*}}{\sqrt{x^*}} \right)$$

$$\leq \frac{q}{\varphi(q) \log x^*} \left(0.79 + \frac{0.79 q}{\varphi(q) \sqrt{x^*}} \int_1^{\sqrt{x^*}} \frac{dv}{2 \log v} + \frac{\pi^2 \sqrt{e}}{6} \frac{\log 1 + \sqrt{e/x^*}}{\sqrt{x^*}} \right)$$

$$\leq c(x^*) \frac{q}{\varphi(q) \log x^*}$$

with

$$c(x^*) = 0.79 + 0.79 \frac{\log x^*}{\sqrt{x^*}} \int_1^{\sqrt{x^*}} \frac{dv}{2 \log v} + \frac{\pi^2 \sqrt{e}}{6} \frac{\log 1 + \sqrt{e/x^*}}{\sqrt{x^*}} \log x^*.$$
Some numerical work shows that $c(x^*) \leq 2.4$ when $x^* \geq 1862$, so our inequality is proved for $y > \log 1862 = 7.52941 \cdots$. This with part (A) proves that $\varphi(q) \log(x/q)|m_q(x)| \leq 2.4$ for $1 \leq q < x$.

When $x^* \geq 3310$, we can single out the term $u = 1$ in (3.1) and modify the coefficient of the bound on this term from 0.79 into 0.155, then we treat the rest of the sum in the same way as before. We get a similar bound with $c(x^*)$ substituted by:

$$c_1(x^*) = 0.155 + 0.79 \frac{\log x^*}{4 \log(x^*/4)} + 0.79 \frac{\log x^*}{\sqrt{x^*/4}} \int_{\sqrt{e}}^{\sqrt{x^*/4}} \frac{dv}{2 \log v}$$

$$+ \frac{\pi^2}{6} \sqrt{e} \frac{1 + \sqrt{e/x^*}}{\sqrt{x^*}} \log x^* .$$

This yields a maximum not more than $1.466 < 3/2$. When $x^* \geq 3 \times 3310$, we single out the terms of index 1, 2, and 3 similarly. This means substituting $c_2(x^*)$ to $c_1(x^*)$ where the $c_2(x^*)$ is defined by

$$c_2(x^*) = 0.155 + 0.79 \frac{\log x^*}{4 \log(x^*/4)} + 0.155 \frac{\log x^*}{9 \log(x^*/9)} + 0.79 \frac{\log x^*}{25 \log(x^*/25)}$$

$$+ 0.79 \frac{\log x^*}{\sqrt{x^*/25}} \int_{\sqrt{e}}^{\sqrt{x^*/25}} \frac{dv}{2 \log v} + \frac{\pi^2}{6} \sqrt{e} \frac{1 + \sqrt{e/x^*}^{-1/2}}{\sqrt{x^*}} \log x^* .$$

This yields a maximum not more than $0.871 < 7/8$. The proof of Theorem 1.1 is complete.

References


