Approximate formulae for $L(1, \chi)$, II

by

Olivier Ramaré (Lille)

1. Introduction and results. Upper bounds of $|L(1, \chi)|$ are mainly useful in number theory to study class numbers of algebraic extensions. In [1]–[3] Louboutin establishes bounds for $|L(1, \chi)|$ that take into account the behavior of $\chi$ at small primes. His method uses special representations of $L(1, \chi)$ and does not extend to odd characters. For instance in [2] he uses $L(1, \chi) = 2\sum_n \sum_{l \leq n} \chi(l)/(n(n+1)(n+2))$ which comes from an integration by parts; such a formula fails in the odd case. But the effect of this integration by parts is in fact similar to the introduction of a smoothing, something we did in [5], the only difficulty being to handle properly the Fourier transform of functions behaving like $1/t$ near $\infty$. This method gives good numerical results in a uniform way.

In this note we improve on the results given in [2] and [3] and extend them to the odd character case. Let us mention that we take this opportunity to correct several typos occurring in [5].

We first state a general formula.

**Theorem.** Let $\chi$ be a primitive Dirichlet character modulo $q$ and let $h$ be an integer prime to $q$. Let $F: \mathbb{R} \to \mathbb{R}$ be such that $f(t) = F(t)/t$ is in $C^2(\mathbb{R})$ (also at 0), vanishes at $\pm \infty$ and $f'$ and $f''$ are in $\mathcal{L}^1(\mathbb{R})$. Assume also that $F$ is even if $\chi$ is odd, and odd if $\chi$ is even. Then, for every $\delta > 0$, we have

$$
\prod_{p|\chi} \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi) = \sum_{n \geq 1} \chi(n) \frac{1 - F(\delta n)}{n} \\
+ \frac{\chi(-h) \tau(\chi)}{qh} \sum_{m \geq 1} c_h(m) \chi(m) \int_{-\infty}^{\infty} F(t) \frac{1}{t} e(mt/(\delta qh)) dt.
$$

2000 Mathematics Subject Classification: Primary 11M20.
Here the Gauss sum \( \tau(\chi) \) is defined by

\[
\tau(\chi) = \sum_{a \mod q} \chi(a)e(a/q)
\]

and the Ramanujan sums \( c_h(m) \) by

\[
c_h(m) = \sum_{a \mod * h} e(ma/q).
\]

Of course \( e(\cdot) = e^{2i\pi \cdot} \), and \( a \mod * h \) denotes summation over all invertible residue classes modulo \( h \). We further restrict our attention to square-free \( h \).

Here are two interesting choices for \( F \) which we take directly from Proposition 2 of [5]. Set

\[
F_3(t) = \left( \frac{\sin \pi t}{\pi} \right)^2 \left( \frac{2}{t} + \sum_{m \in \mathbb{Z}} \frac{\text{sgn}(m)}{(t-m)^2} \right),
\]

\[
j(u) = \int_{-\infty}^{\infty} F_3(t) e(ut) dt = \mathbf{1}_{[-1,1]}(u) \int_{|u|}^{1} (\pi(1-t) \cot \pi t + 1) dt,
\]

\[
F_4(t) = 1 - \left( \frac{\sin \pi t}{\pi t} \right)^2
\]

which satisfies

\[
\int_{-\infty}^{\infty} F_4(t) e(ut) dt = -i\pi (1-|u|)^2 \mathbf{1}_{[-1,1]}(u).
\]

Notice furthermore that \( F_3 \) and \( F_4 \) take their values in \([0,1]\).

In order to compute efficiently the resulting sums we select several levels of hypotheses, starting by the most general ones. We use the Euler \( \phi \)-function and the number \( \omega(t) \) of distinct prime factors of \( t \).

**COROLLARY 1.** Let \( \chi \) be a primitive Dirichlet character modulo \( q \) and \( h \) an integer prime to \( q \). Assume \( q \) is divisible by a square-free \( k \) and set \( \kappa_\chi = 0 \) if \( \chi \) is even, and \( \kappa_\chi = 5 - 2\log 6 = 1.41648 \ldots \) if \( \chi \) is odd. Then

\[
\prod_{p|h} \left( 1 - \frac{\chi(p)}{p} \right) L(1,\chi) = \phi(hk) \left[ \frac{\log q + 2 \sum_{p|h} \log p}{p-1} + \omega(h) \log 4 + \kappa_\chi \right]
\]

is bounded from above if \( \chi \) is even and \( q \geq k^2 4^{\omega(h)} \) by

\[
\frac{\phi(h) 2^{\omega(h)-1}}{h \sqrt{q}} \times \begin{cases} 
\log(q 4^{-\omega(h)+1}) & \text{if } q \geq k^2 4^{\omega(h)}, \\
1.81 + \omega(h) \log 4 - \log q & \text{if } k = 1,
\end{cases}
\]
and if $\chi$ is odd by

$$\frac{3\pi\phi(hk)}{2hkq} \prod_{p|hk} p^2 - 1 + \left\{ \begin{array}{ll} \frac{\pi\phi(h)2\omega(k)}{2h \sqrt{q}} & \text{if } k^2 \max \left( \frac{11}{10} \cdot 4\omega(h), h^2 4^{-\omega(h)+1} \right), \\
0 & \text{if } k = 1. \end{array} \right.$$  

This improves on Theorems 1, 4 and 5 of [3] in the quality of the bounds and in their range, and also by the fact that it covers the case of odd characters. For instance in Theorem 5 of [3], where Louboutin studies separately the cases $h = 3$ and $k = 2$, he gets the upper bound $\frac{1}{6}(\log q + 4.83 \ldots + o(1))$ for even characters, while we get $\frac{1}{6}(\log q + 3.87 \ldots + 3(\log q)/\sqrt{q})$. Recently in [4], by generalizing his method introduced in [2], Louboutin has reached a similar result for the case of even characters, albeit with a slightly larger constant $\kappa_\chi = 2 + \gamma - \log(4\pi) = 0.046 \ldots$ instead of $\kappa_\chi = 0$. This enabled him to replace $\frac{1}{6}(\log q + 4.83 \ldots + o(1))$ by $\frac{1}{6}(\log q + 3.91 \ldots)$.

Notice that the upper bound in the case of even characters is non-positive when $k = 1$ as soon as $q \geq 6.2 \cdot 4^{\omega(h)}$.

When $h = 2$ we can get slightly more precise results:

**Corollary 2.** Let $\chi$ be a primitive Dirichlet character modulo odd $q$. Then

$$|1 - \chi(2)/2|L(1, \chi)| \leq \frac{1}{4}(\log q + \kappa(\chi))$$

where $\kappa(\chi) = 4 \log 2$ if $\chi$ is even, and $\kappa(\chi) = 5 - 2 \log(3/2)$ otherwise.

In [2], the value $\kappa(\chi) \simeq 2.818 \ldots$ is proved to hold true for even characters while $4 \log 2 = 2.772 \ldots$.

We introduce the character $\psi$ induced by $\chi$ modulo $qh$. Furthermore $(m, t)$ denotes the gcd of $m$ and $t$.

As for the typos in [5], first, Proposition 2 gives a wrong formula for $L(1, \chi)$ if $\chi$ is even: the sign preceding $\tau(\chi)$ should be $+$ and not $-$. Then Lemma 8 gives a fancy value for $\varphi_4$. In fact $\varphi_4(t) = -i\pi(1 - |t|)^21_{[-1,1]}(t)$, which is what is proved and used throughout the paper! Finally, in the 6th line of page 264, it is written, “and this last summand is non-negative”, while this summand is without any doubt non-positive.

We thank the referee for his careful reading and for improving Lemma 11.

**2. Lemmas.** We essentially combine Louboutin’s proof [2] and ours [5], while generalizing both situations.

First here is a generalization of the new part in Louboutin’s paper [2]:

**Lemma 1.** For every $m$ in $\mathbb{Z}$, we have

$$\sum_{a \mod qh} \psi(a)e(am/(qh)) = c_h(m)\chi(h)\overline{\chi}(m)\tau(\chi).$$
Proof. By the Chinese remainder theorem,
\[
\sum_{a \mod hq} \psi(a)e(am/(hq)) = \sum_x \sum_y \psi(xq + yh)e((xq + yh)m/(hq))
\]
\[
= \sum_{x \mod h} e(xm/h) \sum_{y \mod q} \chi(yh)e(ym/q)
\]
\[
= c_h(m)\chi(h)\overline{\chi}(m)\tau(\chi),
\]
where \(c_h(m)\) is the Ramanujan sum defined by (2).

Now, Lemma 3 of [5] can be extended to

**Lemma 2.** The sum \(\sum_{n \in \mathbb{Z}}^w f(\delta n)\chi(n)\) exists in the restricted sense given in [5] and
\[
\sum_{n \in \mathbb{Z}}^w f(\delta n)\psi(n) = \frac{\chi(-h)\tau(\chi)}{qh} \sum_{m \in \mathbb{Z}\setminus\{0\}} c_h(m)\chi(m) \int_{-\infty}^{\infty} f(\delta t)e(mt/(qh))\,dt.
\]

**Note:** \(\int_{-\infty}^{\infty} g(t)e(ut)\,dt = \lim_{T \to \infty} \int_{-T}^{T} g(t)e(ut)\,dt\) for \(u \neq 0\).

Now we state and prove lemmas that give approximations of the relevant quantities.

**Lemma 3.** For \(\delta > 0\) and \(hk \geq 2\) we have
\[
\frac{hk}{\phi(hk)} \sum_{n \geq 1}^{(n,hk)=1} \frac{1 - F_3(\delta n)}{n} = -\log \delta - 1 + \sum_{p|hk} \frac{\log p}{p - 1}.
\]

**Proof.** We have
\[
\sum_{n \geq 1}^{(n,hk)=1} \frac{1 - F_3(\delta n)}{n} = \sum_{d|hk} \mu(d) \sum_{n \geq 1}^{d|n} \frac{1 - F_3(\delta n)}{n}
\]
\[
= \sum_{d|hk} \mu(d) \sum_{n \geq 1}^{d|n} \frac{1 - F_3(d\delta n)}{n}.
\]

Lemma 16 of [5] gives the value of the above if \(hk = 1\), which is \(-\log \delta - 1 + \delta\).
This equality is stated only for \(\delta \leq 1\) but since only analytic functions are involved, it naturally extends to \(\delta > 0\). We infer that
\[
\sum_{n \geq 1}^{(n,hk)=1} \frac{1 - F_3(\delta n)}{n} = \sum_{d|hk} \frac{\mu(d)}{d} (-\log(d\delta) - 1 + d\delta)
\]
\[
= -\frac{\phi(hk)}{hk} \log \delta - \frac{\phi(hk)}{hk} + \frac{\phi(hk)}{hk} \sum_{p|hk} \frac{\log p}{p - 1}
\]
provided \(hk \geq 2\).
**Lemma 4.** For \( \delta uq \geq 1 \) we have
\[
\delta uq - 2 \log(e\delta uq) \leq \sum_{1 \leq m \leq \delta uq} j(m/(\delta uq)) \leq \delta uq - \log(2\pi \delta uq/e).
\]

The upper bound is proved between (6.3) and (6.4) in [5]. There also the restriction \( \delta \leq 1 \) can be dispensed with. The lower bound comes simply from a comparison to an integral since \( j \) is non-increasing and since \( j(t) \leq -2\log|t| \) for \( t \leq 1 \) (shown to be true in Lemma 7 of [5]),
\[
(7) \quad \int_0^r j(t) \, dt \leq -2(r \log r - r) \quad (r \in [0, 1]).
\]

**Lemma 5.** For \( \delta > 0 \) and \( h' = h/(2, h) \) we have
\[
\sum_{1 \leq m \leq \delta q} \frac{\phi((m, h))}{\phi(h)} j(m/(\delta hq)) \leq 2^{\omega(h)} \delta q + 1 - \log(2\pi \delta q) + H(h') \sum_{p|h} \frac{\log p}{p}.
\]

**Proof.** Let us introduce the non-negative multiplicative function \( H = \mu \ast \phi \). We have \( H(p) = p - 2 \). We get
\[
\sum_{1 \leq m \leq \delta q} \frac{\phi((m, h))}{\phi(h)} j(m/(\delta q)) = \sum_{d|h} H(d) \sum_{1 \leq m \leq \delta q/d} j(dm/(\delta q)) 
\leq \sum_{d|h} \frac{hH(d)}{d} \delta q + \phi(h)(1 - \log(2\pi \delta hq)) + \sum_{d|h} H(d) \log d.
\]
Now and since \( h \) is square-free we see that \( \sum_{d|h} hH(d)/d = 2^{\omega(h)} \phi(h) \).

**Lemma 6.** For \( \delta \geq k/q \) we have
\[
\sum_{1 \leq m \leq \delta q \atop (m, k) = 1} \frac{\phi((m, h))}{\phi(h)} j(m/(\delta hq)) \leq 2^{\omega(h)} \frac{\phi(k)}{k} \delta q + 2^{\omega(k)} \log(e\delta q/2).
\]

**Proof.** Following the proof of Lemma 5, our sum equals
\[
\sum_{d|h} H(d) \sum_{l|k} \mu(l) \sum_{1 \leq m \leq \delta q/(dl)} j(dm/(\delta hq)) 
\leq \delta q 2^{\omega(h)} \phi(h) \frac{\phi(k)}{k} + \sum_{d|h} H(d) \sum_{l|k \atop \mu(l) = -1} 2 \log(e\delta q/(dl)) 
\leq \delta q 2^{\omega(h)} \phi(h) \frac{\phi(k)}{k} + \phi(h) 2^{\omega(k)} \log(e\delta q/2)
\]
provided that \( \delta q/k \geq 1 \).
LEMMA 7. For $\delta > 0$ and $hk \geq 2$ we have

$$\frac{hk}{\phi(hk)} \sum_{n \geq 1 \atop (n, hk) = 1} \frac{1 - F_4(\delta n)}{n} = \log \delta + \frac{3}{2} - \log(2\pi) + \sum_{p \mid hk} \log p \frac{1}{p - 1}$$

$$+ \frac{2\phi(hk)}{hk} \sum_{d \mid hk} \mu(d) \int_0^1 (1 - t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| \frac{dt}{d}.$$

When $hk = 2$ the last summand is non-positive, and in general if $\delta \leq 1/(2hk)$, it is not more than $\frac{\pi^3}{6} \delta^2 \prod_{p \mid hk} (p^2 - 1)/p^2$.


$$\sum_{n \geq 1 \atop (n, hk) = 1} \frac{1 - F_4(\delta n)}{n} = -\log \delta + \frac{3}{2} - \log(2\pi) + 2 \int_0^1 (1 - t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| \frac{dt}{d}$$

and we use the same technique as in the previous lemma. The error term is non-positive if $hk = 2$ as shown in [5] between (7.2) and (7.3). Furthermore the integral is shown there (in Lemma 18) to be not more than $\pi^3 \delta^2 / 12$ as soon as $\delta \leq 1/2$.

A simple comparison to an integral yields:

LEMMA 8. For $\delta uq \geq 1$ we have

$$\frac{\delta uq}{3} - 1 \leq \sum_{1 \leq m \leq \delta uq} \left( 1 - \frac{m}{\delta uq} \right)^2 \leq \frac{\delta uq}{3}.$$

LEMMA 9. For $\delta \geq k/q$ we have

$$\sum_{1 \leq m \leq \delta uq \atop (m, k) = 1} \frac{\phi((m, h))}{\phi(h)} \left( 1 - \frac{m}{\delta hq} \right)^2 \leq \frac{\phi(k)}{k} \frac{\delta q}{3} 2^{\omega(k)} + 2^{\omega(k) - 1}$$

where the last summand can be omitted if $k = 1$.

Proof. We proceed as in Lemma 6 to deduce that our sum is

$$\sum_{d \mid h} H(d) \sum_{l \mid k} \mu(l) \sum_{1 \leq m \leq \delta q/(dl) \atop 1 \leq m \leq \delta uq/(dl)} \left( 1 - \frac{dlm}{\delta q} \right)^2$$

and the conclusion follows readily.

From [6, (3.22), (2.11) and (3.26)], we get...
Lemma 10. We have
\[
\sum_{1 < p \leq X} \frac{\log p}{p} \leq \log X - 1.332 + \frac{1}{2 \log X} \quad (X \geq 319),
\]
\[
\prod_{2 < p \leq X} \frac{p - 1}{p} \leq \frac{2e^{-\gamma}}{\log X} \left(1 + \frac{1}{2 \log^2 X}\right) \quad (X > 1),
\]
where $\gamma$ is Euler’s constant.

Lemma 11. For $h > 1$, we have
\[
\prod_{2 < p | h} \frac{p - 2}{p - 1} \sum_{2 < p | h} \frac{\log p}{p - 2} \leq 0.7414.
\]

Proof. First writing $h = h_1 p_1$ where $p_1$ is a prime factor, the reader readily checks that our quantity is a non-increasing function of $p_1$. We thus find that its maximum is obtained when $h = \prod_{2 < p \leq X} p$. As a function of $X$, it numerically seems increasing and GP/PARI needs at most 10 seconds to prove it is $\leq 0.72$ if the product is taken over primes $\leq 10^6$. Using Lemma 10, we get
\[
S(X) = \sum_{2 < p \leq X} \frac{\log p}{p - 2} = \sum_{2 < p \leq X} \frac{2 \log p}{p(p - 2)} + \sum_{1 < p \leq X} \frac{\log p}{p} - \frac{\log 2}{2}
\leq 1.27 + \log X - 1.332 + \frac{1}{2 \log X} - 0.346
\leq \log X - 0.4
\]
for $X \geq 10^6$. Furthermore, still invoking Lemma 10, we have
\[
\Pi(X) = \prod_{2 < p \leq X} \frac{p - 2}{p - 1}
\leq \prod_{2 < p \leq X} \left(1 - \frac{1}{(p - 1)^2}\right) \prod_{2 < p \leq X} \frac{p - 1}{p}
\leq \prod_{2 < p \leq 10^6} \left(1 - \frac{1}{(p - 1)^2}\right) \frac{2e^{-\gamma}}{\log X} \left(1 + \frac{1}{2 \log^2 X}\right)
\]
also for $X \geq 10^6$. Since $(1 - 0.4y)(1 + 0.5y^2) \leq 1$ if $0 \leq y \leq 0.4$, our function is not more than
\[
2e^{-\gamma} \prod_{2 < p \leq 10^6} \left(1 - \frac{1}{(p - 1)^2}\right) \leq 0.7414.
\]
3. Proof of the Theorem. Let us start with
\[ L(1, \psi) = \sum_{n \geq 1} \psi(n) \frac{1 - F(\delta n)}{n} + \sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n}. \]
Thanks to the hypothesis concerning the respective parities of \( F \) and \( \chi \), we get
\[ \sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) \delta f(\delta n), \]
to which we apply Lemma 2, and the Theorem follows readily.

4. Proofs of the corollaries. For even characters we take \( F = F_3 \).
Combining the Theorem with Lemmas 3 and 6, and noticing that \( |c_\chi(m)| \leq \phi((h, m)) \), we get
\[ \left| \prod_{p|h} \left( 1 - \frac{\chi(p)}{p} \right) L(1, \chi) \right| \frac{hk}{\phi(hk)} \leq -\log \delta - 1 + \sum_{p|h} \frac{\log p}{p - 1} + \frac{1}{\sqrt{q}} \left( 2^{\omega(h)} \delta q + \frac{k2^{\omega(k)}}{\phi(k)} \log(e\delta q/2) \right) \]
provided \( \delta \geq k/q \). We simply have to choose \( \delta = 1/(2^{\omega(h)} \sqrt{q}) \) and the claimed formula follows readily.

For odd characters we use \( F = F_4 \) and Lemmas 7 and 9 to get
\[ \left| \prod_{p|h} \left( 1 - \frac{\chi(p)}{p} \right) L(1, \chi) \right| \frac{hk}{\phi(hk)} \leq -\log \delta + \frac{3}{2} - \log(2\pi) \]
\[ + \sum_{p|h} \frac{\log p}{p - 1} + \frac{\pi^3}{6} \delta^2 \prod_{p|h} \frac{p^2 - 1}{p} + \frac{\pi}{\sqrt{q}} \left( \frac{\delta 2^{\omega(h)} q}{3} + \frac{2^{\omega(k)-1}}{k} \frac{k}{\phi(k)} \right) \]
provided \( \delta \in [k/q, 1/(2hk)] \). We take \( \delta = 3/(2^{\omega(h)} \pi \sqrt{q}) \) and the claimed formula follows readily.

To prove the second corollary (i.e. with \( k = 1 \)), we simply adapt the above proof, but we can simplify the bound in the even case. We first obtain
\[ \frac{1}{\sqrt{q}} \left( 1 - \log((2\pi/e)\sqrt{q} 2^{-\omega(h)}) + \prod_{2 < p|h} \frac{p - 2}{p - 1} \sum_{2 < p|h} \frac{\log p}{p - 2} \right). \]
The last factor is bounded in Lemma 11 by 0.7414, so the above term is not more than \((1.81 + \omega(h) \log 4 - \log q)/(2\sqrt{q})\) as announced.

When \( h = 2 \), the claimed upper bounds are proved if \( q \geq 39 \), in part because the term in \( \delta^2 \) appearing in (12) disappears by Lemma 7. We complete the verification by appealing to GP/PARI as indicated in [5]. The maximum of \( \kappa(\chi) \) for even characters of module \( \leq 1000 \) is \( \leq 1.705 \), attained
for $q = 109$, while the maximum of $\kappa(\chi)$ for odd characters of module $\leq 1000$ is $\leq 3.360$, attained for $q = 131$.

References


UMR 8524
Université Lille I
59 655 Villeneuve d’Ascq Cedex, France
E-mail: ramare@agat.univ-lille1.fr

Received on 20.9.2002
and in revised form on 26.6.2003