FUN WITH OPERATOR ALGEBRAS

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Abstract. In this note, we give two little amusing results and some questions on operator algebras. First, we expose an elementary sufficient condition on index of subfactor to have no nontrivial intermediate subfactor (called maximal subfactor). Then, we build a non-hyperfinite type III factor from an action of the free group on the circle. Finally we give some questions and remarks on the conditional expectation.

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1. Maximal subfactors.

Reminder 1.1. Let \( M \subset I \subset N \) be an intermediate subfactor of finite index, then \([N : M] = [N : I][I : M]\).

Reminder 1.2. Let \( R \) be the hyperfinite II\(_1\) factor, then each finite group \( G \) acts faithfully on only one manner on \( R \) (modulo \( \text{Int}(R) \)).

Reminder 1.3. Let \( R^G \) be the fixed point subalgebra for the action of \( G \), then \( R^G \subset R \) is a subfactor and \([R : R^G] = |G|\).

Remark 1.4. Let \( H \subset G \) be a subgroup, then we obtain the intermediate subfactor: \( R^G \subset R^H \subset R \).

Reminder 1.5. (Galois correspondence) Let the intermediate subfactor \( R^G \subset N \subset R \) with \( G \) a finite group, then \( N = R^H \) with \( H \) a subgroup of \( G \) ([7], [8]).

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Lemma 1.6. Let \( G \) be a group. It has no non-trivial subgroup if and only if it is equal to \( \mathbb{Z}/p\mathbb{Z} \) with \( p \) a prime number.

Proof. Let \( g \in G \), then \( \langle g \rangle \) is a subgroup of \( G \), but \( \langle g \rangle = \{ e \} \), \( \mathbb{Z}/n\mathbb{Z} \) or \( \mathbb{Z}/d\mathbb{Z} \), a subgroup of \( \mathbb{Z}/n\mathbb{Z} \) if \( d/n \).

Corollary 1.7. Let the subfactor \( R^G \subset R \). It has no non-trivial intermediate subfactor if and only if \( G = \mathbb{Z}/p\mathbb{Z} \) with \( p \) a prime number.

Definition 1.8. A subfactor \( M \subset N \) is called maximal if it admits no nontrivial intermediate subfactor.

Remark 1.9. Let \( \tilde{n} = 4\cos^2\left(\frac{\pi}{m}\right) \), then \( \tilde{3} = 1, \tilde{4} = 2, \tilde{5} = 2, 618..., \tilde{6} = 3, ... \)

Between 4 and 8, the product of such numbers are ordered as:

\( (\tilde{4}; \tilde{4.5}; \tilde{4.6}; \tilde{4.7}; \tilde{4.8}; \tilde{5}; \tilde{4.9}; \tilde{4.10}; \tilde{5.6}; \tilde{4.3}) \simeq (4; 5.2; 6; 6.4; 6.82; 6.85; 7.0; 7.2; 7.8; 8) \).

Theorem 1.10. (Sufficient condition on index for maximal subfactor).
If a subfactor admits an index in the set:
\( \{3, 4, 5, ...\} \cup \{4, 8\} \setminus \{4^2, 4.5, 4.6, 4.7, 4.8, 5, 5^2, 4.9, 4.10, 5.6, 4^3\} \),
then it’s a maximal subfactor.

Proof. The set of possible index is \( \{4\cos^2\left(\frac{\pi}{m}\right)\}|m = 3, 4, ...\} \cup \{4, \infty\} \), the result follows by reminder 1.1 and a sieve of Eratosthenes.

Remark 1.11. There is a maximal subfactor for each possible finite index: for index \( < 4 \) it’s obvious the previous argument, and corollary 2.2.5 of the Jones’ paper [6] gives subfactors realizing the continuum of index \( \geq 4 \), they are also maximal.


Irreducible versus maximal.

Definition 1.13. A subfactor \( M \subset N \) is called irreducible if \( M' \cap N = \mathbb{C} \).

Question 1.14. Is there a link between maximal and irreducible subfactor ?

Reminder 1.15. An irreducible type II_1 subfactor of finite index admits finitely many intermediate subfactors, with many non-trivial examples, ie, with more than zero intermediate subfactor (see [12]).

So irreducibility no implies maximality.

Reminder 1.16. The subfactors of Jones realizing the continuum of index \( \geq 4 \) are maximal and non-irreducible.

So maximality no imples irreducibility.
Reminder 1.17. The index of a finite depth type II$_1$ subfactor is a cyclotomic integer (see [4]).

Now by theorem 1.10 every subfactor of index in $[4, 5]$ is maximal and it exists subfactor for each such index, but $[4, 5]$ admits non cyclotomic integers. So, maximality no implies finite depth.

2. A non-hyperfinite type III factor.

Definition 2.1. Let $\mathcal{M}$ be a factor, then, the Connes spectrum $S(\mathcal{M})$ is the intersection of the spectrum of all the modular operators $\Delta$ (without 0).

Reminder 2.2. Let $\mathcal{M}$ be a type III factor, then $S(\mathcal{M}) = \{1\}, \lambda^2$, or $\mathbb{R}_+^*$, which permits to distinguish type III$_0$, III$_\lambda$ ($0 < \lambda < 1$), or III$_1$ (see [3]).

Reminder 2.3. A von Neumann algebra $\mathcal{M} \neq \mathbb{C}$ is a type III$_1$ factor if the modular action $\sigma$ is ergodic (i.e. fixing only the scalar operator), or iff the cross-product $\mathcal{M} \rtimes_\sigma \mathbb{R}$ is equal to $\mathcal{N} \otimes B(H)$, with $\mathcal{N}$ a II$_1$ factor, called the core of $\mathcal{M}$.

Definition 2.4. Let $\mathcal{N}$ be a II$_1$ factor, then its fundamental group $\mathcal{F}(\mathcal{N})$ is the set of real numbers $\lambda$ such that there is an automorphism rescaling the trace of $\mathcal{N} \otimes B(H)$ by a factor $\lambda$.

Reminder 2.5. If $\mathcal{N}$ is the core of a III$_1$ factor then $\mathcal{F}(\mathcal{N}) = \mathbb{R}_+^*$ (see [11]).

Remark 2.6. The converse is false, see [9].

Definition 2.7. Let $s, r_\theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, defined by $s(x) = x^2$ (choosing representatives in $[0, 1[$) and $r_\theta(x) = x + \theta$. Now, identifying $\mathbb{R}/\mathbb{Z}$ and $\mathbb{S}^1$, we define the action $\alpha$ of $\mathbb{F}_2 = \langle a, b \rangle$, generated by $\alpha(a) = s$ and $\alpha(b) = r_\theta$ in $\text{Homeo}(\mathbb{S}^1)$.

Lemma 2.8. If $\theta$ is transcendantal, the action $\alpha$ is faithful.

Proof. A relation $s^{n_1}r_\theta^{m_1} \ldots s^{n_k}r_\theta^{m_k} = e$ can be translated into an algebraic equation in $x$ and $\theta$, which $\theta$ has to be a root $\forall x$. Then, if $\theta$ is transcendantal, we are sure that there is no relation.

Remark 2.9. For a fixed transcendantal $\theta$, each non-trivial relations can be realized for at most finitely many $x \in \mathbb{R}/\mathbb{Z}$, ie roots of the related algebraic equation.

Theorem 2.10. $\mathcal{M} = L^\infty(\mathbb{S}^1, \text{Leb}) \rtimes_\alpha \mathbb{F}_2$ is a non-hyperfinite type III factor.

Proof. The action $\alpha$ of $\mathbb{F}_2$ on $\mathbb{S}^1$ is:

(a) Measure class preserving: the set of null measure subspaces is invariant.
(b) Free: a fixed point set for $\gamma \neq e$ is at most finite, so with null measure.
(c) Properly ergodic: ergodicity comes from irrational rotation, next, every $F_2$-orbit have null measure.
(d) Non-amenable: by Connes-Feldman-Weiss [2], if such an action is amenable, there exist a transformation $T$ of $\mathbb{S}^1$, such that $\forall x \in \mathbb{S}^1$ up to a null set, $F_2.x = Tz.x$. Then, it exists $n \in \mathbb{Z}$ and $\gamma \in F_2$, such that $a.x = T^n.x$ and $T.x = \gamma.x$. So, $a.x = \gamma^n.x$ and $x$ is in the null set of algebraics with $\theta$.
(e) Non equivalent measure preserving: by ergodicity, an equivalent invariant measure $m$ is proportionnal to Leb. Then $m([1/4, 1/2]) = 2m([1/16, 1/4])$, and by $\alpha(a)$ invariance, $m([1/4, 1/2]) = m([1/16, 1/4])$. In fact, the only invariant measure are 0 or $\infty$.

(a), (b), (c) give a factor, (d) gives non-hyperfinite, (e) gives a type III. □

Remark 2.11. See [1] for groups acting on $\mathbb{S}^1$ without preserving finite measure.

Remark that if there is a conditional expectation in $\mathcal{M}$ with respect to $\mathcal{N}(F_2)$ (ie $\sigma$ leaves $\mathcal{N}(F_2)$ invariant, see Golodec [5] or Takesaki [10]), then we show $\mathcal{M}$ non-hyperfinite, with another way.

3. On the conditional expectation

Let $H$ be a separable Hilbert space and $X$ the borelian space of all the factors included in $B(H)$. Let $\simeq$ be the isomorphism of von Neumann algebra. Let $M \in X$ be a factor, $\Omega \in H$ be a cyclic, separating vector (ie $M\Omega$, $M'\Omega$ are dense in $H$). Let $\sigma^\Omega_1$ be the modular action on $M$.

Reminder 3.1. Let $N$ be a subfactor of $M$, then there is a conditional expectation in $\mathcal{M}$ with respect to $\mathcal{N}(F_2)$ iff $\sigma^\Omega_1$ is the modular action on $M$.

We note $N \subset_E M$

Question 3.2. Does $M \subset_E M \rtimes_{\sigma^\Omega_1} \mathbb{R}$ ?

Definition 3.3. Let $C_M = \{N \in X \mid \exists R \in X$ with $R \subset_E M$ and $N \simeq R\}$

Definition 3.4. Let $C^N = \{M \in X \mid \exists R \in X$ with $N \subset_E R$ and $M \simeq R\}$

Remark 3.5. $C^C = X$, $C_C = \{\mathbb{C}\}$, $C_{B(H)}$ is the class of hyperfinite factors.

Lemma 3.6. Let $M_1, M_2 \in X$ then $M_1, M_2 \subset_E M = M_1 \bar{\otimes} M_2$

Proof. By KMS uniqueness (see [14] p 493) $\sigma^\Omega_1 = \sigma^\Omega_1 \otimes \sigma^\Omega_2$. □

Remark 3.7. $N \subset_E M$ iff $C_M \subset C_N$. Then $C_{M_1}, C_{M_2} \subset C_{M_1 \bar{\otimes} M_2}$. 

Definition 3.8. Let $S = \{ C_M \mid M \in X \}$. It’s a partially ordered set such that $\forall a, b \in S, \exists c$ such that $a, b \subseteq c$.

Question 3.9. Is there $R \in X$ such that $C_R = X$?

Definition 3.10. A factor $M$ is said to be prime if $M = M_1 \bar{\otimes} M_2$ imply that either $M_1$ or $M_2$ are finite dimensional.

Definition 3.11. A class $C_M$ is said to be minimal if $C_N \subset C_M$ implies $C_N = C_M$ or $C_N = \{ C \}$.

Definition 3.12. A von Neumann algebra is said to be diffuse if it contains no minimal projection (a factor is diffuse iff of type II or III).

Question 3.13. For a diffuse factor $M$, $C_M$ is minimal iff $M$ is prime?

Question 3.14. What’s about $C_{vN(F_2)}$ or $C_{vN(F_2) \otimes B(H)}$?

REFERENCES


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