Properties of co-operations: diagrammatic proofs

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We propose an alternative approach of computations in bialgebras, based on diagram rewriting. We illustrate this graphical syntax by proving some properties of co-operations, for instance, coassociativity and cocommutativity. This amounts to check confluence of some rewriting systems.

1. Introduction

Traditionally, terms together with Sweedler notation are used to express computations in (generalized) bialgebras. Here, an algebra is a vector space equipped with an operation $\mu : A \otimes A \rightarrow A$ and a bialgebra is an algebra $A$ equipped with a co-operation $\delta : A \rightarrow A \otimes A$. The operation $\mu$ must be (bi)linear and satisfy some properties, for instance associativity and/or commutativity. Similarly, the co-operation $\delta$ must be linear and satisfy some co-properties, for instance co-associativity and/or co-commutativity. Furthermore, a compatibility relation between $\mu$ and $\delta$ must be satisfied for instance the Hopf identity. For more details, see (Loday 2008).

We shall consider an example from (Loday 2008): the definition of a Lie$^e$-Lie-bialgebra starting from a Ass$^e$-Ass-bialgebra (Section 4.4). For that purpose, we consider a vector space $V$, and we write $\overline{T}(V)$ for the (nonunital associative) algebra of noncommutative polynomials on $V$, and $\text{Lie}(V) \subseteq \overline{T}(V)$ for the algebra of Lie polynomials on $V$. We shall use the following equalities, expressed in Sweedler notation, where $X_1 \otimes X_2$ stands for comultiplication $\delta$ applied to $X$, as well as in diagrammatic notation, where co-operations, as well as operations, are represented by gates:

1. Nonunitary infinitesimal compatibility relation, or definition of deconcatenation (Section 4.2.1 of the same book):

   \[
   \delta(XY) = X \otimes Y + X_1 \otimes X_2 Y + XY_1 \otimes Y_2
   \]
2 Definition of Lie bracket \([\_, \_]\) : 
For \(X, Y \in \mathcal{T}(V)\), 
\[ [X, Y] := XY - YX \]

![Diagram of Lie bracket](image1)

3 Definition of Lie cobracket \(\delta_{(1)}\) : 
For \(X \in \mathcal{T}(V)\), 
\[ \delta_{(1)}(X) := X_1 \otimes X_2 - X_2 \otimes X_1 \]

![Diagram of Lie cobracket](image2)

4 Anti-cocommutativity of deconcatenation for Lie polynomials:
For \(X \in \text{Lie}(V)\), 
\[ X_1 \otimes X_2 = -X_2 \otimes X_1 \]

![Diagram of anti-cocommutativity](image3)

5 Corollary of the previous two equalities:
For \(X \in \text{Lie}(V)\), 
\[ 2X_1 \otimes X_2 = X_1 \otimes X_2 - X_2 \otimes X_1 = \delta_{(1)}(X) \]

![Diagram of corollary](image4)

Our example is a proof of the Lily compatibility relation:
For \(X, Y \in \text{Lie}(V)\), 
\[ \delta_{(1)}[X, Y] = 2(X \otimes Y - Y \otimes X) + \frac{1}{2}(X_1 \otimes [X_2, Y] + [X, Y_1] \otimes Y_2 + Y_1 \otimes [X, Y_2] + [X_1, Y] \otimes X_2) \]

Here, \(X_1 \otimes X_2\) stands for \(\delta_{(1)}\) applied to \(X\). This is Proposition 4.4.4 of (Loday 2008). In Figure 1, we translate the original proof of Proposition 4.4.4 into diagrammatic notation.

It should be noted that diagrams appear in (Loday 2008): see for instance pp. 105-106. But they are only used as convenient pictures. Here we consider diagrams as true mathematical objects to compute with.

In fact, a diagram represents a morphism in a \(\text{PRO}\). Recall that a \(\text{PRO}\) is a strict monoidal...
Fig. 1. Proof of the Lily compatibility relation

\[
\delta_{\mathcal{L}} [X, Y] = (\delta - \tau \delta) (XY - YX)
\]
\[
= \delta (XY) - \delta (YX) - \tau \delta (XY) + \tau \delta (YX)
\]
\[
= X \otimes Y + X_1 \otimes X_2 Y + XY_1 \otimes Y_2 - Y \otimes X - Y_1 \otimes Y_2 X - YX_1 \otimes X_2
\]
\[
- Y \otimes X - X_2 Y \otimes X_1 - Y_2 \otimes XY_2 + X \otimes Y + Y_2 \otimes Y_1 + X_2 \otimes YX_1
\]
\[
= X \otimes Y + X_1 \otimes X_2 Y + XY_1 \otimes Y_2 - Y \otimes X - Y_1 \otimes Y_2 X - YX_1 \otimes X_2
\]
\[
- Y \otimes X + X_1 Y \otimes X_2 + Y_1 \otimes XY_2 + X \otimes Y - Y_2 X \otimes Y_1 + YX_1 \otimes Y_2
\]
\[
= 2(X \otimes Y - Y \otimes X)
\]
\[
+ X_1 \otimes [X_2, Y] + [X, Y_1] \otimes Y_2 + Y_1 \otimes [X, Y_2] + [X_1, Y] \otimes X_2
\]
\[
= 2(X \otimes Y - Y \otimes X)
\]
\[
+ \frac{1}{2} [X_1, [X_2, Y]] + [X, Y_1] \otimes Y_2 + Y_1 \otimes [X, Y_2] + [X_1, Y] \otimes X_2 + [X_1, Y] \otimes [X_1, Y]
\]
category, whose objects are natural numbers and where the monoidal product of two objects \( p, q \) is \( p + q \). Such a \( \text{PRO} \) defines an \( \text{operad} \), which is a monadic Schur functor \( \mathcal{P} : \text{Vect} \rightarrow \text{Vect} \), where \( \text{Vect} \) is the category of vector spaces over some field: see (Loday 2008). In \( \text{Vect} \), sum and coefficients play a crucial role. Hence, both should appear explicitly in diagrams.

Therefore, we use \( \Sigma \)-diagrams, which are formal sums of diagrams. Those \( \Sigma \)-diagrams may not be familiar to mathematicians or computer scientists working in proof theory or in category theory, but they appear, for instance, in (Ehrhard Regnier 2006), where sum stands for nondeterminism.

Our paper is organised as follows:
- In Section 2, we present basic algebraic notions, and we define deconcatenation;
- In Section 3, we give a precise definition of diagrams and \( \Sigma \)-diagrams;
- In Section 4, we prove a well-known result, using the diagrammatic notation: coassociativity of deconcatenation for semi-groups;
- In Section 5, we prove the same result for monoids, using the previous one;
- In Section 6, we study \( \text{shuffle} \) for monoids, and we prove its coassociativity and its cocommutativity, using a similar method.

2. Deconcatenation for semi-groups

Recall that a \textit{semi-group} is a set with an associative operation, and a \textit{monoid} is a semi-group with a \textit{unit}.

Let \( \mathcal{A} \) be an alphabet. The elements of \( \mathcal{A} \) are called \textit{letters}.

**Definition 1.** We write \( \mathcal{A}^+ \) for the \textit{free semi-group} generated by \( \mathcal{A} \). Its elements are nonempty lists of letters. They are called \textit{nonempty words}.

For instance, if our alphabet is \( \mathcal{A} = \{a, b\} \), then \( aabba \) is a nonempty word in \( \mathcal{A}^+ \). \textit{Concatenation} is the operation \( \cdot \) which, to each pair \((u, v) \in (\mathcal{A}^+)^2\), associates the word \( uv \in \mathcal{A}^+ \). For instance, \( abba \cdot bbab = ababba \).

**Remark 1.** Concatenation is associative. For instance, \((ab \cdot b) \cdot a = abb \cdot a = abba = ab \cdot ba = ab \cdot (b \cdot a)\).

**Definition 2.** The \textit{free \( \mathbb{Q} \)-vector space} generated by a set \( X \) is the vector space \( \mathbb{Q}X \) whose elements are formal sums of elements of \( X \), with coefficients in \( \mathbb{Q} \).

For instance, if \( X = \{x, y\} \), we have \( x + y = x + y + y = y + y + y = 3y \in \mathbb{Q}X \).

**Remark 2.** If \( X \) is a finite set, \( \mathbb{Q}X \) is isomorphic to \( \mathbb{Q}^n \), where \( n \) is the cardinality of \( X \). For instance, \( \mathbb{Q}X \) is isomorphic to \( \mathbb{Q}^2 \) in the above example.

**Definition 3.** The \textit{nonunital algebra} generated by a semi-group \( S \) is the free \( \mathbb{Q} \)-vector space \( \mathbb{Q}S \) generated by the set \( S \), equipped with the multiplication extending the one of the semi-group \( S \), and distributive over sum.
Definition 4. If $U$ and $V$ are $\mathbb{Q}$-vector spaces, the tensor product $U \otimes V$ is the free $\mathbb{Q}$-vector space generated by elements of the form $u \otimes v$ with $u \in U$ and $v \in V$, quotiented by the following equalities:
- $(u + u') \otimes v = u \otimes v + u' \otimes v$;
- $u \otimes (v + v') = u \otimes v + u \otimes v'$;
- $(\lambda u) \otimes v = \lambda (u \otimes v) = u \otimes (\lambda v)$ for all $\lambda \in \mathbb{Q}$.

We write $U^{\otimes n}$ for the $\mathbb{Q}$-vector space $U \otimes \cdots \otimes U$ ($n$ times).

Remark 3. By the universal property of tensor (a bilinear map from $U \times V$ to $Z$ induces a unique linear map from $U \otimes V$ to $Z$) we have $(\mathbb{Q}X)^{\otimes n} = \mathbb{Q}X^n$. Hence, we get $u_1 \otimes \cdots \otimes u_n \in \mathbb{Q}X^n$ for any $u_1, \cdots, u_n \in \mathbb{Q}X$.

Definition 5. The right and the left actions of $\mathbb{Q}S$ on $\mathbb{Q}S^2$ are given as follows:

$$(u \otimes v) \cdot w = u \otimes (v \cdot w)$$

and

$$u \cdot (v \otimes w) = (u \cdot v) \otimes w$$

for any $u, v, w \in S$.

For instance, $(ab \otimes a) \cdot a = (ab \otimes aa)$.

Definition 6. Let $\mathcal{A}$ be an alphabet and let $S = \mathcal{A}^*$. Deconcatenation is the co-operation $\delta : \mathbb{Q}S \rightarrow \mathbb{Q}S \otimes \mathbb{Q}S = \mathbb{Q}S^2$ recursively defined by:

- $\delta(a) := 0$ for any $a \in \mathcal{A}$;
- $\delta(u \cdot v) := u \cdot \delta(v) + \delta(u) \cdot v + u \otimes v$ for any $u, v \in S$.

Remark 4. In fact, $\delta(u) \cdot v$ consists of all terms of $\delta(u \cdot v)$ whose first component is a prefix of $u$ and similarly, $u \cdot \delta(v)$ consists of all terms of $\delta(u \cdot v)$ whose second component is a suffix of $v$.

Remark 5. The co-operation $\delta$ is described by the following equality:

$$\delta(w) := \sum_{u \otimes v} u \otimes v$$

for any $w \in S$.

For instance, $\delta(abaa) = a \otimes baa + ab \otimes aa + aba \otimes a$.

Theorem 1. Deconcatenation is coassociative:

$$(\text{id}_{\mathbb{Q}S} \otimes \delta) \circ \delta = (\delta \otimes \text{id}_{\mathbb{Q}S}) \circ \delta$$

or in other words:

for all $w \in S$, if $\delta(w) = \sum u_i \otimes v_i$, then $\sum u_i \otimes \delta(v_i) = \sum \delta(u_i) \otimes v_i$.

A diagrammatic proof of this classical result is given in section 4.

3. Diagrams and $\Sigma$-diagrams

For any $m, n \in \mathbb{N}$, a diagram $\phi : m \rightarrow n$, with $m$ inputs and $n$ outputs is pictured as follows:
It is interpreted as a map $f : X^m \to X^n$, where $X$ is some fixed set.

There are two operations on diagrams:

- The parallel composition of $\phi : m \to n$ and $\phi' : m' \to n'$ is $\phi \ast \phi' : m + m' \to n + n'$:

- The sequential composition of $\phi : m \to n$ and $\psi : n \to p$ is $\psi \circ \phi : m \to p$:

They are interpreted as follows:

- if $f : X^m \to X^n$ is the interpretation of $\phi : m \to n$ and $f' : X^{m'} \to X^{n'}$ is the interpretation of $\phi' : m' \to n'$, then $f \times f' : X^{m + m'} \to X^{n + n'}$ is the interpretation of the parallel composition $\phi \ast \phi' : m + m' \to n + n'$;

- if $f : X^m \to X^n$ is the interpretation of $\phi : m \to n$ and $g : X^n \to X^p$ is the interpretation of $\psi : n \to p$, then $g \circ f : X^m \to X^p$ is the interpretation of the sequential composition $\psi \circ \phi : m \to p$.

The identity diagram $\text{Id}_n : n \to n$ is pictured as follows:

Atomic diagrams are called gates (or generators).

**Definition 7.** An elementary diagram is a formal composition $\text{Id}_p \otimes \alpha \otimes \text{Id}_q : p + m + q \to p + n + q$ where $\alpha : m \to n$ is a gate.
Definition 8. A diagram is a sequential composition \( \phi_1 \circ \cdots \circ \phi_n \) of elementary diagrams \( \phi_1, \cdots, \phi_n \).

In fact, diagrams are defined modulo interchange:

\[
\begin{array}{c}
\phi \\
\vdots \\
\phi' \\
\vdots \\
\phi \\
\vdots \\
\phi' \\
\vdots \\
\phi \\
\end{array}
\]

\[
\begin{array}{c}
\phi \\
\vdots \\
\phi \\
\vdots \\
\phi \\
\vdots \\
\phi \\
\end{array}
\]

\[
\begin{array}{c}
\phi \\
\vdots \\
\phi \\
\vdots \\
\phi \\
\vdots \\
\phi \\
\end{array}
\]

For more details on diagrams, see (Lafont 2003) and (Lafont 2010).

Remark 6. Diagrams are the morphisms of a free PRO. Moreover, any PRO is the quotient of a free PRO by some relations. Hence, diagrams are the syntax of PROs.

Definition 9. A \( \Sigma \)-diagram \( \Phi : m \to n \) is a (finite) formal sum \( \Sigma \lambda_i \phi_i \) where the coefficients \( \lambda_i \) are in \( \mathbb{Q} \) and the \( \phi_i : m \to n \) are diagrams with the same number \( m \) of inputs and the same number \( n \) of outputs.

Parallel and sequential composition are extended to \( \Sigma \)-diagrams by distributivity over sum:

\[
(\Sigma \lambda_i \phi_i) \ast \Psi = \Sigma \lambda_i (\phi_i \ast \Psi);
\]

\[
\Phi \ast (\Sigma \lambda_i \psi_i) = \Sigma \lambda_i (\Phi \ast \psi_i);
\]

\[
(\Sigma \lambda_i \phi_i) \circ \Psi = \Sigma \lambda_i (\phi_i \circ \Psi);
\]

\[
\Phi \circ (\Sigma \lambda_i \psi_i) = \Sigma \lambda_i (\Phi \circ \psi_i).
\]

Remark 7. The field \( \mathbb{Q} \) can be replaced by any field of characteristic 0, for instance \( \mathbb{R} \) or \( \mathbb{C} \).

Note that we use uppercase greek letters \( \Phi, \Psi \) for \( \Sigma \)-diagrams.

On \( \Sigma \)-diagrams, there is a binary sum, which is pictured as follows:

\[
\begin{array}{c}
\Phi \\
\vdots \\
\phi \\
\vdots \\
\phi \\
\vdots \\
\phi \\
\end{array}
+ 
\begin{array}{c}
\Psi \\
\vdots \\
\psi \\
\vdots \\
\psi \\
\vdots \\
\psi \\
\end{array}
\]

Note that the \( \Sigma \)-diagrams \( \Phi \) and \( \Psi \) have the same numbers of inputs \( m \) and the same number of outputs \( n \). Similarly we define the \( \Sigma \)-diagram \( \lambda \Phi : m \to n \) for any scalar \( \lambda \), and the null \( \Sigma \)-diagram \( 0 : m \to n \).

A \( \Sigma \)-diagram \( \Phi : m \to n \) is interpreted as a \( \mathbb{Q} \)-linear map \( f : \mathbb{Q}^m \to \mathbb{Q}^n \). The interpretation of the operations is similar to the case of diagrams, except for parallel composition, which is interpreted by \( \otimes \) instead of \( \times \). The interpretation of + is straightforward.

Definition 10. A rewrite rule is of the form \( \phi \to \Psi \) where \( \phi : m \to n \) is a diagram and \( \Psi : m \to n \) is a \( \Sigma \)-diagram.

Note that the left member \( \phi \) must be a diagram, not a \( \Sigma \)-diagram.

Elementary reduction, written \( \to \), is defined as usual by applying a rule \( \phi \to \Psi \) in a context given by two diagrams \( \xi : r \to p + m + q \) and \( \omega : p + n + q \to s \):
Reduction is the linear reflexive transitive closure of elementary reduction, that is the smallest relation $\rightarrow^*$ containing $\rightarrow$ such that:

- $\Phi \rightarrow^* \Phi$ for any $\Sigma$-diagram $\Phi$;
- $\Phi \rightarrow^* \Phi''$ whenever $\Phi \rightarrow^* \Phi'$ and $\Phi' \rightarrow^* \Phi''$;
- $\sum \lambda_i \Phi_i \rightarrow^* \sum \lambda_i \Psi_i$ whenever $\Phi_i \rightarrow^* \Psi_i$ for all $i$.

4. Diagrammatic proof of Theorem 1

Now we assume that $\mathcal{X}$ is the free semi-group $\mathcal{A}^*$. The gates are:

- concatenation $\cdot$
- deconcatenation $\delta$

Hence, we consider $\Sigma$-diagrams built on those gates. In other words, those two gates are the generators of our free $PRO$. From Definition 6 (of deconcatenation), we deduce the following interaction rule:

$$\delta(u \cdot v) = u \cdot \delta(v) + \delta(u) \cdot v + u \otimes v$$

We recall that $u \cdot \delta(v)$ and $\delta(u) \cdot v$ are given by Definition 5. Similar kinds of rules are introduced in (Lafont 1997) and (Ehrhard Regnier 2006).

We prove the coassociativity of deconcatenation, which corresponds to the following rewrite rule:
The key argument of our proof is described by a confluence diagram:

\[
\begin{array}{c}
\Psi \\
\phi \\
\phi' \\
\end{array}
\rightarrow
\begin{array}{c}
\Psi' \\
\phi'' \\
\end{array}
\]

where \( \phi = \) and \( \phi' = \)

Note that there are two kinds of arrow:

- broken arrow for coassociativity;
- solid arrow for interaction.

**Proof.** Formally, Theorem 1 is proved by induction on the length of the input word:

- Coassociativity holds obviously for letters, since \( \delta(a) = 0 \) for any \( a \in A \). In fact, this equality is expressed by the following rule (using an extra gate for each letter \( a \)):

\[
\begin{array}{c}
\Rightarrow 0 \\
\end{array}
\]

Using that rule, we get:

\[
\begin{array}{c}
\Rightarrow 0 \\
\end{array}
\]

- Now, let \( u \) and \( v \) be words in \( A^+ \) for which deconcatenation is coassociative. Let us to prove that deconcatenation is coassociative for \( w = u \cdot v \). In other words, the following reduction holds:

\[
\begin{array}{c}
\Rightarrow \\
\end{array}
\]

We apply interaction to the left and right members:
The two results differ only on two terms:

By induction hypothesis, we can apply coassociativity to the left $\Sigma$-diagram, and we get the right one.

**Remark 8.** This proof expresses the confluence of the conflict between coassociativity and interaction. In fact, to get a complete rewrite system for (noncommutative and noncocommutative) bialgebras, we need an associativity rule for concatenation:

The resulting system has two *critical peaks*:
- the conflict between coassociativity and interaction,
- the conflict between associativity and interaction.
We have checked the confluence of the first one. For the second one, we simply reverse diagrams. Hence we get a confluent rewrite system. Termination is straightforward. Note that the latter argument is diagrammatic, and we do not need to consider inputs of diagrams.

5. Concatenation and deconcatenation for monoids

Let \( \mathcal{A} \) be an alphabet.

**Definition 11.** We write \( \mathcal{A}' \) for the free monoid generated by \( \mathcal{A} \). Its elements are those of \( \mathcal{A}' \) and the empty word \( \varepsilon \).

**Remark 9.** The unit for concatenation is \( \varepsilon \).

We write \( M \) for \( \mathcal{A}' \), and \( S \) for \( \mathcal{A}^{\circ} \).

**Definition 12.** The (unital) \( \mathbb{Q} \)-algebra \( \mathbb{Q}M \), is the free \( \mathbb{Q} \)-vector space generated by the set \( M \), equipped with a multiplication \( \cdot \) extending the one of the monoid \( M \) and distributive over sum.

**Definition 13.** Full deconcatenation \( \Delta : \mathbb{Q}M \to \mathbb{Q}M^2 \), is defined as follows:

\[
\Delta(w) := \sum_{u \cdot v = w} u \otimes v.
\]

**Remark 10.** In particular, we get \( \Delta(\varepsilon) = \varepsilon \otimes \varepsilon \).

**Definition 14.** Primitive deconcatenation \( \delta : \mathbb{Q}M \to \mathbb{Q}M^2 \) extending \( \delta : \mathbb{Q}S \to \mathbb{Q}S^2 \), is defined as follows:

\[
delta(w) = \sum_{u \cdot v = w, u \neq \varepsilon} u \otimes v, \text{ for } w \neq \varepsilon
\]

\[
delta(\varepsilon) = -\varepsilon \otimes \varepsilon
\]

**Remark 11.** The relation between the two deconcatenations is:

\[
\Delta(u) = \delta(u) + u \otimes \varepsilon + \varepsilon \otimes u.
\]

This explains why we defined \( \delta(\varepsilon) := -\varepsilon \otimes \varepsilon \):

\[
\Delta(\varepsilon) = \delta(\varepsilon) + \varepsilon \otimes \varepsilon + \varepsilon \otimes \varepsilon = -\varepsilon \otimes \varepsilon + 2\varepsilon \otimes \varepsilon = \varepsilon \otimes \varepsilon.
\]

**Theorem 2.** Full deconcatenation is coassociative.

**Proof.** We introduce two new gates:

\[
\text{full deconcatenation } \Delta \quad \text{unit } \varepsilon
\]

We also introduce two new rules:
\[ \Delta(u) = \delta(u) + u \otimes \varepsilon + \varepsilon \otimes u \]
\[ \delta(\varepsilon) = -\varepsilon \otimes \varepsilon \]

Coassociativity of full deconcatenation is pictured as follows:

Reducing those two diagrams by the new rules gives:

Hence, it remains to show the following equality for \( u \in \mathcal{A}^* \):

We have two cases:
- if \( u = \varepsilon \), we get \( \varepsilon \otimes \varepsilon \otimes \varepsilon \) in both cases;
- if \( u \in \mathcal{A}^* \), we apply Theorem 1.

**Remark 12.** From \( \Delta(u) = \delta(u) + u \otimes \varepsilon + \varepsilon \otimes u \) and \( \delta(u \cdot v) = u \cdot \delta(v) + \delta(u) \cdot v + u \otimes v \) we deduce:
\[ \Delta(u \cdot v) = u \cdot \Delta(v) + \Delta(u) \cdot v - u \otimes v. \] This equality corresponds to the following interaction rule:
Using this rule, we get an alternative proof of Theorem 2, which is very similar to the proof of Theorem 1.

6. Concatenation and shuffle for monoids

In this section, we also consider the monoid $M = \mathcal{A}$. Here, the syntax is also interpreted in $\mathbb{Q}$-vector spaces but we only need diagrams (not $\Sigma$-diagrams).

**Definition 15.** Shuffle $\sigma : \mathcal{Q}M \rightarrow \mathcal{Q}M^2$ is defined as follows on a word $w = a_1 \cdots a_n$ of length $n$:

$$\sigma(w) := \sum_{(u,v) \in I_w} u \otimes v,$$

where $I_w$ is the set of pairs $(u,v)$ of words of the form:

- $u = a_{i_1} \cdots a_{i_p}$ with $1 \leq i_1 < i_2 < \cdots < i_p \leq n$,
- $v = a_{j_1} \cdots a_{j_q}$ with $1 \leq j_1 < j_2 < \cdots < j_q \leq n$,

where $\{i_1, \cdots, i_p\} \cup \{j_1, \cdots, j_q\} = \{1, \cdots, n\}$ and $\{i_1, \cdots, i_p\} \cap \{j_1, \cdots, j_q\} = \emptyset$.

For instance,

$$\sigma(abaa) = ab \otimes aa + 2aba \otimes a + abaa \otimes \varepsilon + 2aa \otimes ba + aaa \otimes b$$

$$+ aa \otimes ab + 2a \otimes aba + \varepsilon \otimes abaa + 2ba \otimes aa + aaa \otimes b$$

Shuffle is pictured as follows:

**Remark 13.** We have $\sigma(\varepsilon) = \varepsilon \otimes \varepsilon$ and $\sigma(a) = \varepsilon \otimes a + a \otimes \varepsilon$ for all $a \in \mathcal{A}$. Furthermore, we have:

$$\sigma(w \cdot w') = \sum_{(u,v) \in I_{w \cdot w'}} u \otimes u' \otimes v \cdot v'$$

for all $w, w' \in \mathcal{A}$.

The latter equality is expressed by the following Hopf interaction rule:
Theorem 3. Shuffle is coassociative.

Proof. Coassociativity of shuffle corresponds to the following rule:

We prove this by induction on the length of the input word:

— For the empty word we have:

— For any \( a \in \mathcal{A} \) we introduce a new gate and we get:

— Now, let \( u \) and \( v \) be two words in \( M \) for which shuffle is coassociative. We want to show that shuffle is also coassociative for \( w = u \cdot v \). In other words, the following reduction holds:

We apply interaction to the left and right members:
By induction hypothesis, we can apply coassociativity to the upper diagram, and we get the lower one.

Theorem 4. Shuffle is co-commutative:

\[ \sigma(w) = \sum_{(u,v) \in I} v \otimes u \] for all \( w \in M \).

Proof. Cocommutativity of shuffle corresponds to the following rule:

- For the empty word we have:

- For any letters \( a \in \mathcal{A} \) we get:

- Now, let \( u \) and \( v \) be two words in \( M \) for which shuffle is cocommutative. We want to show that shuffle is also cocommutative for \( w = u \cdot v \). In other words, the following reduction holds:
We apply interaction to the left and right members:

By induction hypothesis, we can apply cocommutativity to the upper diagram, and we get the lower one.

In fact, in this section, we have used a new gate:

We should introduce the following new rules:
Crossing satisfies an extra equation (Yang-Baxter) which is not needed here. For more details on this kind of rewriting, see (Lafont 2003).

7. Conclusion

$\Sigma$-diagrams are used by mathematicians working on bialgebras, in order to describe precisely the relations between operations and co-operations. But usually, the diagrammatic syntax is not formally defined, and it is not used inside computations or proofs.

In this paper, we have given a precise definition of these $\Sigma$-diagrams and some examples of computations using them. We notice that:

— computation with $\Sigma$-diagrams is well handled by rewriting: see (Lafont 2003) for the case of diagrams;
— $\Sigma$-diagrams are very similar to differential interaction nets: see (Ehrhard Regnier 2006).

Hence, in future work, we shall develop a general theory of rewriting for $\Sigma$-diagrams as well as programs implementing those rewriting techniques.

References

Lafont, Y. (2010), Diagram rewriting and operads. To appear in Séminaires et Congrès de la SMF.