Homotopy of computation: The Minneapolis Program

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Introduction

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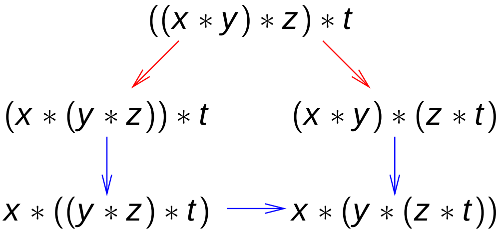

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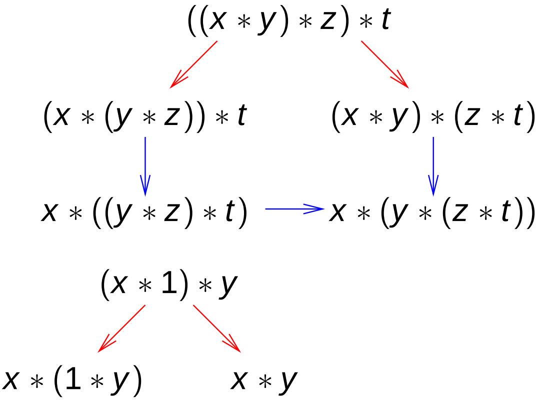

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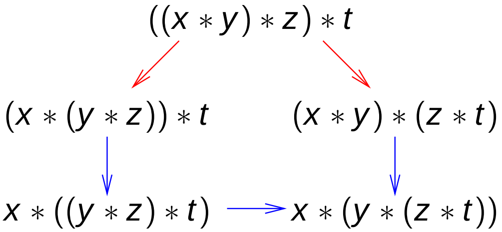

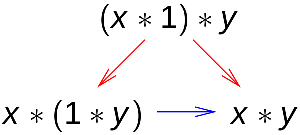

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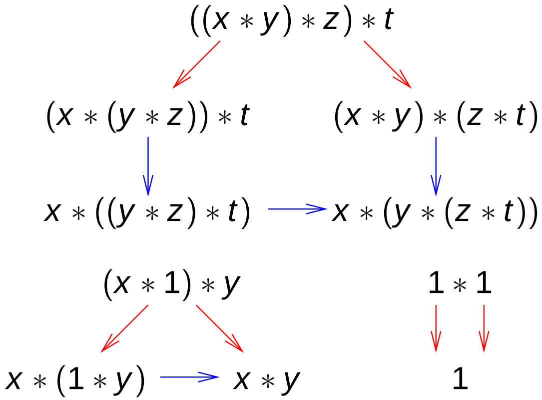

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Theorem (Mac Lane): All conditions follow from those.

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Theorem (Kobayashi): In that case, all $H_n(M)$ are of finite type.

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Directed homotopy for parallel computation

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- or a preorder (*rewriting*)

- category ($\rightarrow$ *directed homotopy*)

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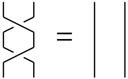

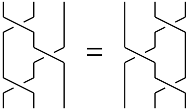

- free 2-monoid (special case of a 2-category)

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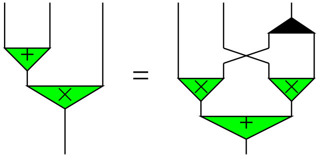

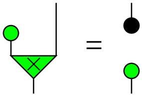

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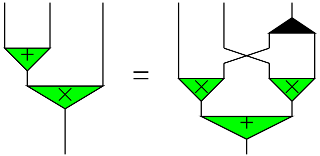
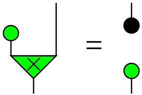
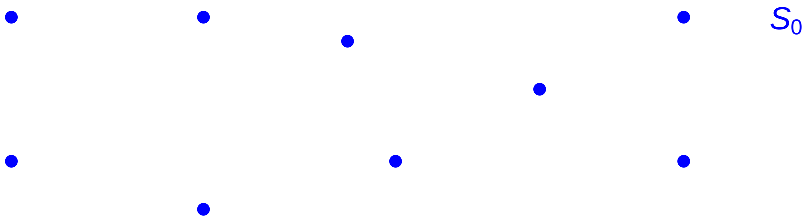



In general:

- objects are n-dimensional words
- computations are $n+1$-dimensional words

Burroni's polygraphs



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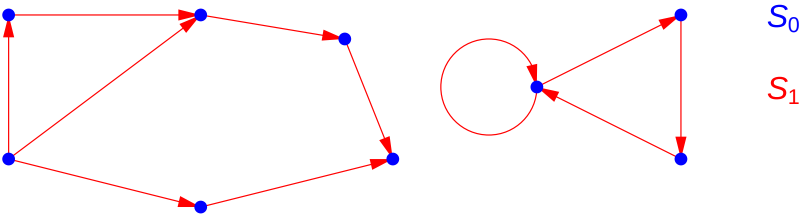

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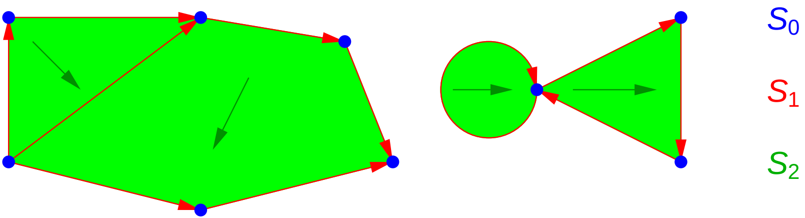

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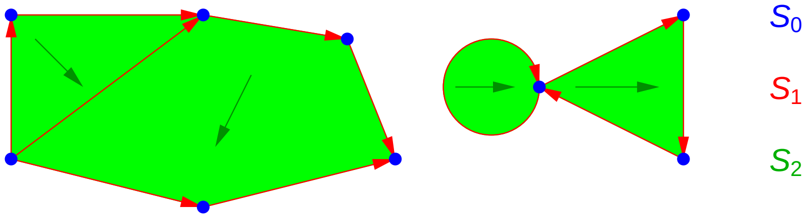

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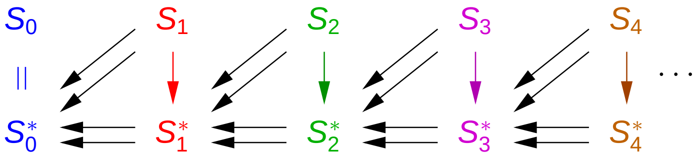

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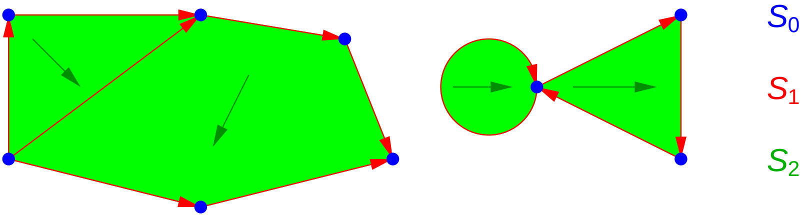

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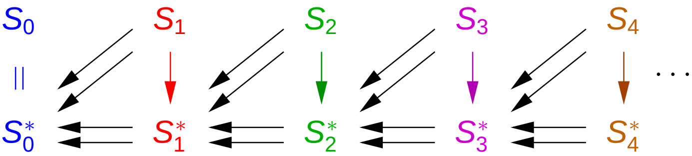

We get a free ω-category ($\longrightarrow$ *directed cellular complex*).

Polygraphic resolutions

Definition: a *polygraphic resolution* is an ω-functor

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C_0 & \leftleftarrows & C_1 & \leftleftarrows & C_2 & \leftleftarrows & C_3 & \leftleftarrows & C_4 & &
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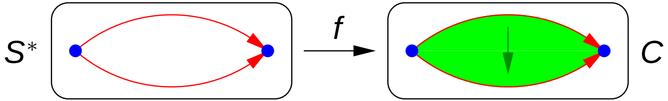

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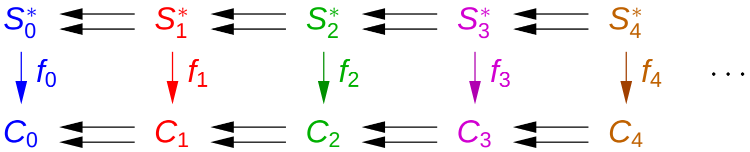

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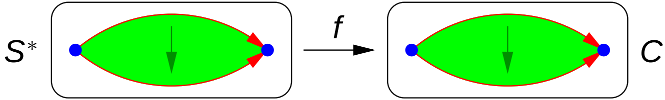

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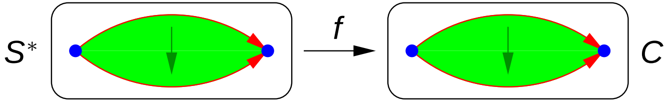

Theorem (Métayer): Any ω-category C has a polygraphic resolution, which is unique up to *homotopical equivalence*.

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Theorem (Lafont & Métayer): In the case of a monoid M, the polygraphic homology coincides with the classical one.

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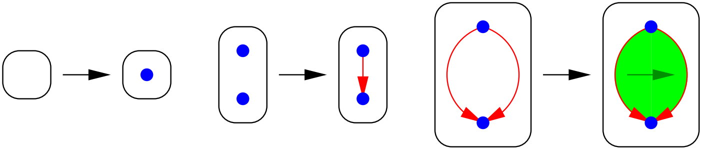

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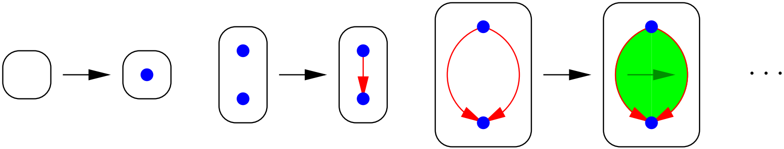

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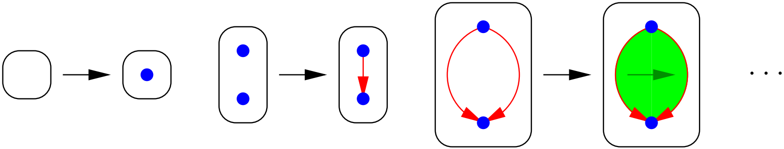

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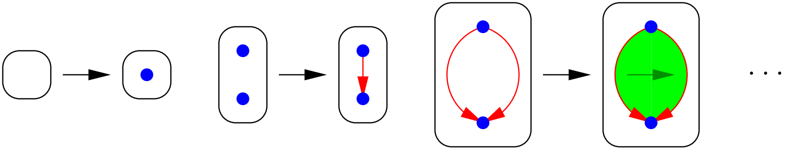

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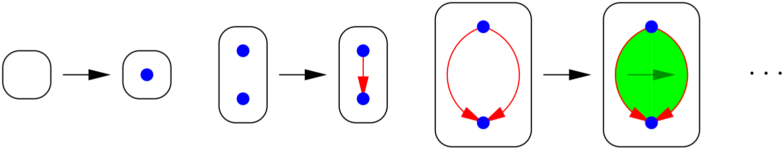

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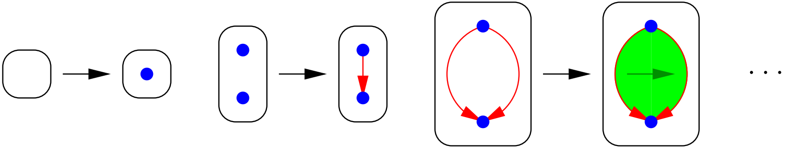

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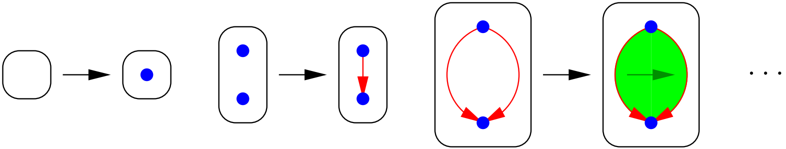

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- the notion of *n-connection* (reversible cylinders)
- division, transport, coherence . . .

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Homotopy of rewriting revisited

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Homotopy and Type theory

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Homotopy and Type theory

Modeling Martin Löf's Intentional Type Theory with fibrations in some model structure (Awodey).

Internal language for homotopy theory?