Linear Logic Pages

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Sequent calculus

Proofs

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Cut elimination (key cases) - Cut elimination (commutative cases) - Expansion of identities

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About provable formulas

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Miscellaneous

Invariants

About exponential rules

Phase semantics and Decision problems

MLL = *Multiplicative Linear Logic*
MALL = *Multiplicative Additive Linear Logic*
MELL = *Multiplicative Exponential Linear Logic*
LL = *Linear Logic*
WLL = *Affine Linear Logic*

NP = *non deterministic polynomial time*
PSPACE = *polynomial space*
NEXPTIME = *non deterministic exponential time*

Propositional case

Phase spaces - Phase models - Syntactical model

Theorem: If A is a propositional formula, the following properties are equivalent:

1. A is provable;
2. A holds in any phase model;
3. A holds in the syntactical model;
4. A is cut-free provable.

Corollary: *cut elimination* (propositional case)

Finite model property

Finite models - Semilinear models

	MLL	MALL	MELL	LL	WLL
finite model property	yes	yes	no	no	yes
semilinear model property	yes	yes	?	no	yes
decidability	yes	yes	?	no	yes

Provability problem

	MLL	MALL	MELL	LL
propositional case	NP-complete	PSPACE-complete	?	undecidable
first order case	NP-complete	NEXPTIME-complete	undecidable	undecidable
second order case	undecidable	undecidable	undecidable	undecidable

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Formulas

Formulas are built from atoms and units, using (binary) connectives, modalities, and quantifiers:

- An *atom* is a propositional (*i.e.* second order) variable α or its dual $\alpha^\perp$. More generally, it is an atomic predicate $\alpha(t_1, \dots, t_n)$ or its dual $\alpha^\perp(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are first order terms.
- The *multiplicative connectives* are $\otimes$ (*times*, or *multiplicative conjunction*) and its dual $⅋$ (*par*, or *multiplicative disjunction*). The corresponding units are $\mathbf{1}$ (*one*) and $\perp$ (*bottom*).
- The *additive connectives* are $\&$ (*with*, or *additive conjunction*) and its dual $\oplus$ (*plus*, or *additive disjunction*). The corresponding units are $\top$ (*top*) and $\mathbf{0}$ (*zero*).
- The *exponential* modalities are ! (*of course*) and its dual ? (*why not*).
- The *quantifiers* are $\forall$ (*for all*, or *universal quantifier*) and its dual $\exists$ (*exists*, or *existential quantifier*). A quantifier applies to a first order variable x or to a second order variable α.

If A is an atom, $A^\perp$ stands for its dual. In particular, $A^{\perp\perp} = A$. *Linear negation* is extended to all formulas by de Morgan equations:

$$(A \otimes B)^\perp = A^\perp ⅋ B^\perp, \qquad (A ⅋ B)^\perp = A^\perp \otimes B^\perp, \qquad \mathbf{1}^\perp = \perp, \qquad \perp^\perp = \mathbf{1},$$

$$(A \& B)^\perp = A^\perp \oplus B^\perp, \qquad (A \oplus B)^\perp = A^\perp \& B^\perp, \qquad \top^\perp = \mathbf{0}, \qquad \mathbf{0}^\perp = \top,$$

$$(!A)^\perp = ?A^\perp, \qquad (?A)^\perp = !A^\perp, \qquad (\forall\xi.A)^\perp = \exists\xi.A^\perp, \qquad (\exists\xi.A)^\perp = \forall\xi.A^\perp.$$

Linear implication is defined by $A \multimap B = A^\perp ⅋ B$.

If x is a first order variable and t is a first order term, $A[t/x]$ stands for the formula A where all free occurrences of x have been replaced by t (and bound variables have been renamed when necessary). Similarly, if α is a propositional variable and B is a formula, $A[B/\alpha]$ stands for the formula A where all free occurrences of α (respectively $\alpha^\perp$) have been replaced by B (respectively $B^\perp$).

Sequents and rules

Sequents are of the form $\vdash \Gamma$ where Γ is a (possibly empty) sequence of formulas $A_1, \ldots, A_n$. In practise, the sequent $\vdash \Gamma$ is identified with the sequence Γ, and a sequent consisting of a single formula is identified with this formula. $\Gamma^\perp$ stands for $A_1^\perp, \ldots, A_n^\perp$ (respectively $!\Gamma$ for $!A_1, \ldots, !A_n$ and $?\Gamma$ for $?A_1, \ldots, ?A_n$), and if Δ is another sequence of formulas, $\Gamma \vdash \Delta$ stands for $\vdash \Gamma^\perp, \Delta$.

A sequent is *provable* if it can be derived using the following rules:

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,\mathrm{x} \qquad \frac{}{\vdash A, A^\perp}\,\mathrm{id} \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\mathrm{cut}$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,\otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,\parr \qquad \frac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\,\bot$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\,\& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\,\oplus_2 \qquad \frac{}{\vdash \top, \Gamma}\,\top$$

$$\frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \qquad \frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,?\mathrm{d} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,?\mathrm{c} \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,?\mathrm{w}$$

$$\frac{\vdash A, \Gamma}{\vdash \forall \xi. A, \Gamma}\,\forall \qquad \frac{\vdash A[\tau/\xi], \Gamma}{\vdash \exists \xi. A, \Gamma}\,\exists$$

Note that *exchange* is the only structural rule. The rules for exponentials are respectively called *promotion*, *dereliction*, *contraction*, and *weakening*. In the $\forall$-rule, ξ must have no free occurrence in Γ, but it is well understood that a bound variable can always be renamed. In the $\exists$-rule, τ is a first order term (if ξ is a first order variable) or a formula (if ξ is a second order variable).

By exchange, any permutation of Γ can be derived from Γ, so that in practise, sequents are considered as finite multisets, and exchange is implicit. Similarly, the following rules are derivable:

$$\frac{\vdash \Gamma, \Delta}{\vdash ?\Gamma, \Delta}\,?\mathrm{D} \qquad \frac{\vdash ?\Gamma, ?\Gamma, \Delta}{\vdash ?\Gamma, \Delta}\,?\mathrm{C} \qquad \frac{\vdash \Delta}{\vdash ?\Gamma, \Delta}\,?\mathrm{W}$$

Basic equivalences and second order definability

We say that A and B are *equivalent* and we write $A \equiv B$ if both $A \vdash B$ and $B \vdash A$ are provable. This amounts to say that, for any Γ, the sequent $\vdash A, \Gamma$ is provable if and only if $\vdash B, \Gamma$ is provable, or more generally, that $\vdash \Gamma[A/\alpha]$ is provable if and only if $\vdash \Gamma[B/\alpha]$ is provable.

Here are some typical equivalences:

$$A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C, \qquad A \otimes B \equiv B \otimes A, \qquad A \otimes \mathbf{1} \equiv A$$

$$A \parr (B \parr C) \equiv (A \parr B) \parr C, \qquad A \parr B \equiv B \parr A, \qquad A \parr \bot \equiv A,$$

$$A \& (B \& C) \equiv (A \& B) \& C, \qquad A \& B \equiv B \& A, \qquad A \equiv A \& A, \qquad A \& \mathbf{1} \equiv A,$$

$$A \oplus (B \oplus C) \equiv (A \oplus B) \oplus C, \qquad A \oplus B \equiv B \oplus A, \qquad A \equiv A \oplus A, \qquad A \oplus \mathbf{0} \equiv A,$$

$$A \otimes (B \oplus C) \equiv (A \otimes B) \oplus (A \otimes C), \qquad A \otimes \mathbf{0} \equiv \mathbf{0},$$

$$A \parr (B \& C) \equiv (A \parr B) \& (A \parr C), \qquad A \parr \top \equiv \top,$$

$$!!A \equiv !A, \qquad !A \equiv !A \otimes !A, \qquad !\mathbf{1} \equiv \mathbf{1}, \qquad !(A \& B) \equiv !A \otimes !B, \qquad !\top \equiv \mathbf{1},$$

$$??A \equiv ?A, \qquad ?A \equiv ?A \parr ?A, \qquad ?\bot \equiv \bot, \qquad ?(A \oplus B) \equiv ?A \parr ?B, \qquad ?\mathbf{0} \equiv \bot,$$

$$\forall\xi.\forall\zeta.A \equiv \forall\zeta.\forall\xi.A, \qquad \forall\xi.(A \& B) \equiv \forall\xi.A \& \forall\xi.B, \qquad A \equiv \forall\xi.A, \qquad A \parr \forall\xi.B \equiv \forall\xi.(A \parr B),$$

$$\exists\xi.\exists\zeta.A \equiv \exists\zeta.\exists\xi.A, \qquad \exists\xi.(A \oplus B) \equiv \exists\xi.A \oplus \exists\xi.B, \qquad A \equiv \exists\xi.A, \qquad A \otimes \exists\xi.B \equiv \exists\xi.(A \otimes B).$$

In the last two equivalences of the last two lines, the variable ξ must have no free occurrence in A.

By definition of linear implication, we also get:

$$A \multimap (B \multimap C) \equiv (A \otimes B) \multimap C, \qquad A \multimap (B \parr C) \equiv (A \multimap B) \parr C,$$

$$A \multimap B \equiv B^\perp \multimap A^\perp, \qquad \mathbf{1} \multimap A \equiv A, \qquad A \multimap \bot \equiv A^\perp,$$

$$A \multimap (B \& C) \equiv (A \multimap B) \& (A \multimap C), \qquad (A \oplus B) \multimap C \equiv (A \multimap C) \& (B \multimap C),$$

$$A \multimap \top \equiv \top, \qquad \mathbf{0} \multimap A \equiv \top,$$

$$A \multimap \forall\xi.B \equiv \forall\xi.(A \multimap B), \qquad \exists\xi.B \multimap A \equiv \forall\xi.(B \multimap A).$$

In the last two equivalences, the variable ξ must have no free occurrence in A.

Note also the following equivalences:

$$\mathbf{1} \equiv \forall\alpha.\alpha \multimap \alpha, \qquad A \oplus B \equiv \forall\alpha.!(A \multimap \alpha) \multimap !(B \multimap \alpha) \multimap \alpha, \qquad \mathbf{0} \equiv \forall\alpha.\alpha,$$

$$\bot \equiv \exists\alpha.\alpha \otimes \alpha^\perp, \qquad A \& B \equiv \exists\alpha.!(\alpha \multimap A) \otimes !(\alpha \multimap B) \otimes \alpha, \qquad \top \equiv \exists\alpha.\alpha.$$

In other words:

- Multiplicative units are definable in terms of second order quantifiers and multiplicative connectives.
- Additive connectives are definable in terms of second order quantifiers, multiplicative connectives, and exponentials.
- Additive units are definable in terms of second order quantifiers.

Two-sided sequent calculus

Formulas are built in the same way as in the one-sided calculus, except that *linear negation* and *linear implication* are primitive connectives. Sequents are of the form $\Gamma \vdash \Delta$ where Γ and Δ are finite sequences of formulas. The rules are the following:

$$\frac{\Gamma \vdash \Delta, A, B, \Theta}{\Gamma \vdash \Delta, B, A, \Theta} \vdash\mathrm{x} \qquad \frac{\Gamma, A, B, \Delta \vdash \Theta}{\Gamma, B, A, \Delta \vdash \Theta} \mathrm{x}\vdash \qquad \frac{}{A \vdash A} \mathrm{id} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Delta, \Lambda} \mathrm{cut}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta} \vdash .^{\perp} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta} .^{\perp}\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \vdash\multimap \qquad \frac{\Gamma \vdash A, \Delta \qquad \Theta, B \vdash \Lambda}{\Gamma, \Theta, A \multimap B \vdash \Delta, \Lambda} \multimap\vdash$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Theta \vdash B, \Lambda}{\Gamma, \Theta \vdash A \otimes B, \Delta, \Lambda} \vdash\otimes \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes\vdash \qquad \frac{}{\vdash \mathbf{1}} \vdash\mathbf{1} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \mathbf{1} \vdash \Delta} \mathbf{1}\vdash$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Delta, B \vdash \Lambda}{\Gamma, \Theta, A \parr B \vdash \Delta, \Lambda} \parr\vdash \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta} \vdash\parr \qquad \frac{}{\perp \vdash} \perp\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta} \vdash\perp$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta} \vdash\& \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&_1\vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \,\&\, B \vdash \Delta} \&_2\vdash \qquad \frac{}{\Gamma \vdash \top, \Delta} \vdash\top$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \oplus\vdash \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \vdash\oplus_1 \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \vdash\oplus_2 \qquad \frac{}{\Gamma, \mathbf{0} \vdash \Delta} \mathbf{0}\vdash$$

$$\frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} \vdash! \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} !\mathrm{d}\vdash \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} !\mathrm{c}\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} !\mathrm{w}\vdash$$

$$\frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta} ?\vdash \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta} \vdash?\mathrm{d} \qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta} \vdash?\mathrm{c} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta} \vdash?\mathrm{w}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall\xi.A, \Delta} \vdash\forall \qquad \frac{\Gamma, A[\tau/\xi] \vdash \Delta}{\Gamma, \forall\xi.A \vdash \Delta} \forall\vdash \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, \exists\xi.A \vdash \Delta} \exists\vdash \qquad \frac{\Gamma \vdash A[\tau/\xi], \Delta}{\Gamma \vdash \exists\xi.A, \Delta} \vdash\exists$$

In the $\vdash\forall$-rule and in the $\exists\vdash$-rule, ξ must have no free occurrence in Γ, Δ.

De Morgan equations become equivalences, so that any formula A of the two-sided calculus is equivalent to a formula $\widehat{A}$ of the one-sided calculus. A sequent $\Gamma \vdash \Delta$ is provable in the two-sided calculus if and only if the sequent $\vdash \widehat{\Gamma}^{\perp}, \widehat{\Delta}$ is provable in the one-sided calculus. In particular, a sequent $\vdash \Gamma$ is provable in the one-sided calculus if and only if it is provable in the two-sided calculus.

Hybrid sequent calculus

Formulas are the same as in the one-sided calculus. Sequents are of the form $\vdash \Theta \;;\; \Gamma$ where Θ and Γ are finite sequences of formulas. We write $\vdash \Gamma$ for $\vdash \;;\; \Gamma$. The rules are the following (J.-M. Andreoli):

$$\frac{\vdash \Theta \;;\; \Gamma, A, B, \Delta}{\vdash \Theta \;;\; \Gamma, B, A, \Delta} \mathrm{x} \qquad \frac{\vdash \Theta, A, \Theta' \;;\; A, \Gamma}{\vdash \Theta, A, \Theta' \;;\; \Gamma} \mathrm{a} \qquad \frac{}{\vdash \Theta \;;\; A, A^{\perp}} \mathrm{id} \qquad \frac{\vdash \Theta \;;\; A, \Gamma \qquad \vdash \Theta \;;\; A^{\perp}, \Delta}{\vdash \Theta \;;\; \Gamma, \Delta} \mathrm{cut}$$

$$\frac{\vdash \Theta \;;\; A, \Gamma \qquad \vdash \Theta \;;\; B, \Delta}{\vdash \Theta \;;\; A \otimes B, \Gamma, \Delta} \otimes \qquad \frac{\vdash \Theta \;;\; A, B, \Gamma}{\vdash \Theta \;;\; A \parr B, \Gamma} \parr \qquad \frac{}{\vdash \Theta \;;\; \mathbf{1}} \mathbf{1} \qquad \frac{\vdash \Theta \;;\; \Gamma}{\vdash \Theta \;;\; \perp, \Gamma} \perp$$

$$\frac{\vdash \Theta \;;\; A, \Gamma \qquad \vdash \Theta \;;\; B, \Gamma}{\vdash \Theta \;;\; A \,\&\, B, \Gamma} \& \qquad \frac{\vdash \Theta \;;\; A, \Gamma}{\vdash \Theta \;;\; A \oplus B, \Gamma} \oplus_1 \qquad \frac{\vdash \Theta \;;\; B, \Gamma}{\vdash \Theta \;;\; A \oplus B, \Gamma} \oplus_2 \qquad \frac{}{\vdash \Theta \;;\; \top, \Gamma} \top$$

$$\frac{\vdash \Theta \;;\; A}{\vdash \Theta \;;\; !A} ! \qquad \frac{\vdash \Theta, A \;;\; \Gamma}{\vdash \Theta \;;\; ?A, \Gamma} ? \qquad \frac{\vdash \Theta \;;\; A, \Gamma}{\vdash \Theta \;;\; \forall\xi.A, \Gamma} \forall \qquad \frac{\vdash \Theta \;;\; A[\tau/\xi], \Gamma}{\vdash \Theta \;;\; \exists\xi.A, \Gamma} \exists$$

The second structural rule is called *absorption*. In the $\forall$-rule, ξ must have no free occurrence in Θ, Γ.

A sequent $\vdash \Theta \;;\; \Gamma$ is provable in the hybrid calculus if and only if $\vdash ?\Theta, \Gamma$ is provable in the one-sided calculus. In particular, $\vdash \Gamma$ is provable in the one-sided calculus if and only if it is provable in the hybrid calculus. Note that, in practise, Θ can be considered as a finite set of formulas.

Intuitionistic Linear Logic

Formulas are built as in (Classical) Linear Logic, except that there is no *negation*, no *why not*, and *par* is replaced by *linear implication*. Sequents are of the form $\Gamma \vdash A$ where Γ is a finite sequence of formulas and A is a formula. The rules are the following:

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\,\text{x} \qquad \frac{}{A \vdash A}\,\text{id} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,\text{cut}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\vdash\multimap \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,\multimap\vdash$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\vdash\otimes \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\vdash \qquad \frac{}{\vdash \mathbf{1}}\,\vdash\mathbf{1} \qquad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}\,\mathbf{1}\vdash$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\vdash\& \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C}\,\&_1\vdash \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C}\,\&_2\vdash \qquad \frac{}{\Gamma \vdash \top}\,\vdash\top$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,\oplus\vdash \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\,\vdash\oplus_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,\vdash\oplus_2 \qquad \frac{}{\Gamma, \mathbf{0} \vdash C}\,\mathbf{0}\vdash$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\vdash! \qquad \frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\,!\text{d}\vdash \qquad \frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C}\,!\text{c}\vdash \qquad \frac{\Gamma \vdash C}{\Gamma, !A \vdash C}\,!\text{w}\vdash$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall\xi.A}\,\vdash\forall \qquad \frac{\Gamma, A[\tau/\xi] \vdash C}{\Gamma, \forall\xi.A \vdash C}\,\forall\vdash \qquad \frac{\Gamma, A \vdash C}{\Gamma, \exists\xi.A \vdash C}\,\exists\vdash \qquad \frac{\Gamma \vdash A[\tau/\xi]}{\Gamma \vdash \exists\xi.A}\,\vdash\exists$$

In the $\vdash\forall$-rule and in the $\exists\vdash$-rule, ξ must have no free occurrence in Γ (respectively in Γ and C).

If A is provable in Intuitionistic Linear Logic, then it is obviously provable in Linear Logic. The converse holds if A contains no additive unit ($\top$, $\mathbf{0}$) and no second order quantifier, but not in general: Linear Logic is *not* a conservative extension of Intuitionistic Linear Logic (H. Schellinx). For instance, the formula $((\alpha \multimap \beta) \multimap \mathbf{0}) \multimap \alpha \otimes \top$ is provable in Linear Logic, but not in Intuitionistic Linear Logic.

More structural rules

Linear Logic does not allow (unrestricted) contraction and weakening:

$$\frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}\,\text{c} \qquad \frac{\vdash \Gamma}{\vdash A, \Gamma}\,\text{w}$$

If we add them, we essentially get *Classical Logic* with modalities. In that case, we have $A \otimes B \equiv A \& B$, $A ⅋ B \equiv A \oplus B$, $\mathbf{1} \equiv \top$, and $\bot \equiv \mathbf{0}$. Furthermore, the $\bot$-rule becomes redundant, as well as the ?c-rule and the ?w-rule.

If we only add (unrestricted) weakening, we get *Affine Linear Logic*. In that case, we have $\mathbf{1} \equiv \top$, $\bot \equiv \mathbf{0}$, and the following sequents become provable:

$$A \otimes B \vdash A \,\&\, B, \qquad A \oplus B \vdash A ⅋ B.$$

Furthermore, the $\bot$-rule becomes redundant, as well as the ?w-rule.

If we only add (unrestricted) contraction, we get *Contractive Linear Logic*. In that case, the following sequents become provable:

$$A \,\&\, B \vdash A \otimes B, \qquad A ⅋ B \vdash A \oplus B.$$

Furthermore, the ?c-rule becomes redundant.

Contractive Linear Logic must not be confused with *Relevant Logic*, which has an artificial distributivity law, and therefore, no good property of cut elimination.

Cut elimination (key cases)

A cut with an identity is eliminated as follows:

$$\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,\text{id} \qquad \vdash A, \Gamma}{\vdash A, \Gamma}\,\text{cut} \quad\rightarrow\quad \vdash A, \Gamma$$

A cut between two matching logical rules is eliminated as follows:

$$\dfrac{\dfrac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,\otimes \qquad \dfrac{\vdash A^{\perp}, B^{\perp}, \Theta}{\vdash A^{\perp} \parr B^{\perp}, \Theta}\,\parr}{\vdash \Gamma, \Delta, \Theta}\,\text{cut} \quad\rightarrow\quad \dfrac{\vdash A, \Gamma \qquad \dfrac{\vdash B, \Delta \qquad \vdash A^{\perp}, B^{\perp}, \Theta}{\vdash A^{\perp}, \Delta, \Theta}\,\text{cut}}{\vdash \Gamma, \Delta, \Theta}\,\text{cut}$$

$$\dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1} \qquad \dfrac{\vdash \Gamma}{\vdash \bot, \Gamma}\,\bot}{\vdash \Gamma}\,\text{cut} \quad\rightarrow\quad \vdash \Gamma$$

$$\dfrac{\dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\,\& \qquad \dfrac{\vdash A^{\perp}, \Delta}{\vdash A^{\perp} \oplus B^{\perp}, \Delta}\,\oplus_1}{\vdash \Gamma, \Delta}\,\text{cut} \quad\rightarrow\quad \dfrac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,\text{cut}$$

$$\dfrac{\dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\,\& \qquad \dfrac{\vdash B^{\perp}, \Delta}{\vdash A^{\perp} \oplus B^{\perp}, \Delta}\,\oplus_2}{\vdash \Gamma, \Delta}\,\text{cut} \quad\rightarrow\quad \dfrac{\vdash B, \Gamma \qquad \vdash B^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,\text{cut}$$

$$\dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \qquad \dfrac{\vdash A^{\perp}, \Delta}{\vdash ?A^{\perp}, \Delta}\,?\text{d}}{\vdash ?\Gamma, \Delta}\,\text{cut} \quad\rightarrow\quad \dfrac{\vdash A, ?\Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash ?\Gamma, \Delta}\,\text{cut}$$

$$\dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \qquad \dfrac{\vdash ?A^{\perp}, ?A^{\perp}, \Delta}{\vdash ?A^{\perp}, \Delta}\,?\text{c}}{\vdash ?\Gamma, \Delta}\,\text{cut} \quad\rightarrow\quad \dfrac{\dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \qquad \dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \qquad \vdash ?A^{\perp}, ?A^{\perp}, \Delta}{\vdash ?A^{\perp}, ?\Gamma, \Delta}\,\text{cut}}{\vdash ?\Gamma, ?\Gamma, \Delta}\,\text{cut}}{\vdash ?\Gamma, \Delta}\,?\text{C}$$

$$\dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,! \qquad \dfrac{\vdash \Delta}{\vdash ?A^{\perp}, \Delta}\,?\text{w}}{\vdash ?\Gamma, \Delta}\,\text{cut} \quad\rightarrow\quad \dfrac{\vdash \Delta}{\vdash ?\Gamma, \Delta}\,?\text{W}$$

$$\dfrac{\dfrac{\vdash A, \Gamma}{\vdash \forall\xi.A, \Gamma}\,\forall \qquad \dfrac{\vdash A^{\perp}[\tau/\xi], \Delta}{\vdash \exists\xi.A^{\perp}, \Delta}\,\exists}{\vdash \Gamma, \Delta}\,\text{cut} \quad\rightarrow\quad \dfrac{\vdash A[\tau/\xi], \Gamma \qquad \vdash A^{\perp}[\tau/\xi], \Delta}{\vdash \Gamma, \Delta}\,\text{cut}$$

Cut elimination (commutative cases)

Commutation with most logical rules is straigthforward. Here are some typical examples:

$$\dfrac{\dfrac{\vdash A, B, C, \Gamma}{\vdash A, B \parr C, \Gamma}\,\parr \qquad \vdash A^\perp, \Delta}{\vdash B \parr C, \Gamma, \Delta}\,\text{cut} \quad \rightarrow \quad \dfrac{\dfrac{\vdash A, B, C, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash B, C, \Gamma, \Delta}\,\text{cut}}{\vdash B \parr C, \Gamma, \Delta}\,\parr$$

$$\dfrac{\dfrac{\vdash A, B, \Gamma \qquad \vdash C, \Theta}{\vdash A, B \otimes C, \Gamma, \Theta}\,\otimes \qquad \vdash A^\perp, \Delta}{\vdash B \otimes C, \Gamma, \Delta, \Theta}\,\text{cut} \quad \rightarrow \quad \dfrac{\dfrac{\vdash A, B, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash B, \Gamma, \Delta}\,\text{cut} \qquad \vdash C, \Theta}{\vdash B \otimes C, \Gamma, \Delta, \Theta}\,\otimes$$

$$\dfrac{\dfrac{\vdash A, B, \Gamma \qquad \vdash A, C, \Gamma}{\vdash A, B \,\&\, C, \Gamma}\,\& \qquad \vdash A^\perp, \Delta}{\vdash B \,\&\, C, \Gamma, \Delta}\,\text{cut} \quad \rightarrow \quad \dfrac{\dfrac{\vdash A, B, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash B, \Gamma, \Delta}\,\text{cut} \qquad \dfrac{\vdash A, C, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash C, \Gamma, \Delta}\,\text{cut}}{\vdash B \,\&\, C, \Gamma, \Delta}\,\&$$

$$\dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,\top \qquad \vdash A^\perp, \Delta}{\vdash \top, \Gamma, \Delta}\,\text{cut} \quad \rightarrow \quad \dfrac{}{\vdash \top, \Gamma, \Delta}\,\top$$

In the case of the *promotion* rule, the commutation is the following:

$$\dfrac{\dfrac{\vdash ?A, B, ?\Gamma}{\vdash ?A, !B, ?\Gamma}\,! \qquad \dfrac{\vdash A^\perp, ?\Delta}{\vdash !A^\perp, ?\Delta}\,!}{\vdash !B, ?\Gamma, ?\Delta}\,\text{cut} \quad \rightarrow \quad \dfrac{\dfrac{\vdash ?A, B, ?\Gamma \qquad \dfrac{\vdash A^\perp, ?\Delta}{\vdash !A^\perp, ?\Delta}\,!}{\vdash B, ?\Gamma, ?\Delta}\,\text{cut}}{\vdash !B, ?\Gamma, ?\Delta}\,!$$

Expansion of identities

Identities are expanded into atomic ones as follows:

$$\dfrac{}{\vdash A \otimes B, A^\perp \parr B^\perp}\,\text{id} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id} \qquad \dfrac{}{\vdash B, B^\perp}\,\text{id}}{\vdash A \otimes B, A^\perp, B^\perp}\,\otimes}{\vdash A \otimes B, A^\perp \parr B^\perp}\,\parr$$

$$\dfrac{}{\vdash \mathbf{1}, \perp}\,\text{id} \quad \rightarrow \quad \dfrac{\dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}}{\vdash \mathbf{1}, \perp}\,\perp$$

$$\dfrac{}{\vdash A \,\&\, B, A^\perp \oplus B^\perp}\,\text{id} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id}}{\vdash A, A^\perp \oplus B^\perp}\,\oplus_1 \qquad \dfrac{\dfrac{}{\vdash B, B^\perp}\,\text{id}}{\vdash B, A^\perp \oplus B^\perp}\,\oplus_2}{\vdash A \,\&\, B, A^\perp \oplus B^\perp}\,\&$$

$$\dfrac{}{\vdash \top, \mathbf{0}}\,\text{id} \quad \rightarrow \quad \dfrac{}{\vdash \top, \mathbf{0}}\,\top$$

$$\dfrac{}{\vdash !A, ?A^\perp}\,\text{id} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id}}{\vdash A, ?A^\perp}\,?\text{d}}{\vdash !A, ?A^\perp}\,!$$

$$\dfrac{}{\vdash \forall\xi.A, \exists\xi.A^\perp}\,\text{id} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,\text{id}}{\vdash A, \exists\xi.A^\perp}\,\exists}{\vdash \forall\xi.A, \exists\xi.A^\perp}\,\forall$$

About provable formulas

In a cut-free proof of a formula A, the last rule must be a logical rule. Therefore:

- $\bot$ is not provable (*strong consistency*);
- $\mathbf{0}$ is not provable (*weak consistency*);
- if $A \otimes B$ is provable, then A and B are provable (*splitting property*);
- if $A \oplus B$ is provable, then A or B is provable (*disjunction property*);
- if $\exists\xi.A$ is provable, then $A[\tau/\xi]$ is provable for some τ (*existence property*).

To sum up:

- $\mathbf{1}$, $\top$ are provable, but not $\bot$, $\mathbf{0}$, α, $\alpha^\perp$;
- $A \otimes B$, as well as $A \& B$, is provable if and only if A and B are provable;
- $A ⅋ B$ is provable if and only if the sequent $\vdash A, B$ is provable;
- $A \oplus B$ is provable if and only if A or B is provable;
- $!A$, as well as $\forall\xi.A$, is provable if and only if A is provable;
- $?A$ is provable whenever A is provable;
- $\exists\xi.A$ is provable if and only if $A[\tau/\xi]$ is provable for some τ.

It may happen that:

- A and B are provable, but not $A ⅋ B$ (take $A = B = \mathbf{1}$);
- $A ⅋ B$ is provable but neither A nor B is provable (take $A = \alpha$ and $B = \alpha^\perp$);
- $?A$ is provable but not A (take $A = \alpha \oplus \alpha^\perp$).

Note also that:

- the empty sequent is not provable, so that A and $A^\perp$ cannot be both provable;
- the sequent $\vdash A_1, \ldots, A_n$ is provable if and only if the formula $A_1 ⅋ \cdots ⅋ A_n$ is provable.

Polarities

We say that a formula A is *positive* (respectively *negative*) if $A \equiv\ !A$ (respectively $A \equiv\ ?A$). Obviously, A is positive if and only if $A^\perp$ is negative. Furthermore:

- $\mathbf{1}$ and $\mathbf{0}$ are positive, as well as $!A$ for any A;
- if A and B are positive, so are $A \otimes B$ and $A \oplus B$;
- if A is positive, so is $\exists\xi.A$.

By duality, we get:

- $\bot$ and $\top$ are negative, as well as $?A$ for any A;
- if A and B are negative, so are $A ⅋ B$ and $A \& B$;
- if A is negative, so is $\forall\xi.A$.

We say that A is *regular* if $A \equiv\ ?!A$. We have:

- if A is positive, then $?A$ is regular;
- $\bot$ is regular, as well as $?!A$ for any A;
- if A and B are regular, so is $A ⅋ B$.

Intuitionistic Logic

Intuitionistic formulas are built from atoms (α, or more generally, $\alpha(t_1, \ldots, t_n)$) and units ($\top$, $\bot$) using binary connectives ($\Rightarrow$, $\wedge$, $\vee$) and quantifiers ($\forall$, $\exists$). Sequents are of the form $\Gamma \vdash C$ where Γ is a sequence of formulas and C is a formula. The rules are the following:

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\,\mathrm{x} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\,\mathrm{c} \qquad \frac{\Gamma \vdash C}{\Gamma, A \vdash C}\,\mathrm{w} \qquad \frac{}{A \vdash A}\,\mathrm{id} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,\mathrm{cut}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\vdash\Rightarrow \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \Rightarrow B \vdash C}\Rightarrow\vdash$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\vdash\wedge \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\wedge_1\vdash \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge_2\vdash \qquad \frac{}{\Gamma \vdash \top}\vdash\top$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\vee\vdash \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vdash\vee_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vdash\vee_2 \qquad \frac{}{\Gamma, \bot \vdash C}\bot\vdash$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall\xi.A}\vdash\forall \qquad \frac{\Gamma, A[\tau/\xi] \vdash C}{\Gamma, \forall\xi.A \vdash C}\forall\vdash \qquad \frac{\Gamma, A \vdash C}{\Gamma, \exists\xi.A \vdash C}\exists\vdash \qquad \frac{\Gamma \vdash A[\tau/\xi]}{\Gamma \vdash \exists\xi.A}\vdash\exists$$

In the $\vdash\forall$-rule (respectively in the $\exists\vdash$-rule), ξ must have no free occurrence in Γ (respectively in Γ, C). Alternatively, the rules for $\wedge$ and $\top$ can be formulated as follows (*multiplicative* version):

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\vdash\wedge \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge\vdash \qquad \frac{}{\vdash \top}\vdash\top$$

The translation $A \mapsto A^*$ from Intuitionistic Logic into Linear Logic is defined by

$$\alpha^* = \alpha, \qquad (A \Rightarrow B)^* = {!A^*} \multimap B^*, \qquad (A \wedge B)^* = A^* \mathbin{\&} B^*, \qquad (A \vee B)^* = {!A^*} \oplus {!B^*},$$

$$\top^* = \top, \qquad \bot^* = \mathbf{0}, \qquad (\forall\xi.A)^* = \forall\xi.A^*, \qquad (\exists\xi.A)^* = \exists\xi.!A^*.$$

A sequent $\Gamma \vdash C$ is provable in Intuitionistic Logic if and only if its translation $!\Gamma^* \vdash C^*$ is provable in Linear Logic. The *only if* direction is easy. Conversely, it is clear that $\Gamma \vdash C$ is provable in Intuitionistic Logic whenever its translation $!\Gamma^* \vdash C^*$ is provable in *Intuitionistic* Linear Logic: It suffices to consider the obvious translation $A \mapsto A_*$ from Intuitionistic Linear Logic into Intuitionistic Logic, defined by

$$\alpha_* = \alpha, \qquad (A \multimap B)_* = A_* \Rightarrow B_*, \qquad (A \otimes B)_* = (A \mathbin{\&} B)_* = A_* \wedge B_*, \qquad (A \oplus B)_* = A_* \vee B_*,$$

$$\mathbf{1}_* = \top_* = \top, \qquad \mathbf{0}_* = \bot, \qquad (!A)_* = A_*, \qquad (\forall\xi.A)_* = \forall\xi.A_*, \qquad (\exists\xi.A)_* = \exists\xi.A_*.$$

But since Linear Logic is not a conservative extension of Intuitionistic Linear Logic, it is more difficult to show that $\Gamma \vdash C$ is provable in Intuitionistic Logic whenever its translation is provable in Linear Logic (H. Shellinx).

The above translation is sometimes called the *call-by-name translation.* There is also a *call-by-value translation*, defined by

$$\alpha^* = !\alpha, \qquad (A \Rightarrow B)^* = !(A^* \multimap B^*), \qquad (A \wedge B)^* = A^* \otimes B^*, \qquad (A \vee B)^* = A^* \oplus B^*,$$

$$\top^* = \mathbf{1}, \qquad \bot^* = \mathbf{0}, \qquad (\forall\xi.A)^* = !\forall\xi.A^*, \qquad (\exists\xi.A)^* = \exists\xi.A^*.$$

In that case, a proof of $\Gamma \vdash C$ is translated into a proof of $\Gamma^* \vdash C^*$, using the fact that A^* is positive for any A.

Classical Logic

Formulas are built from atoms (α, $\neg\alpha$, or more generally, $\alpha(t_1,\ldots,t_n)$, $\neg\alpha(t_1,\ldots,t_n)$) and units ($\top$, $\bot$) using binary connectives ($\wedge$, $\vee$) and quantifiers ($\forall$, $\exists$). Negation is defined on atoms and extended to all formulas by De Morgan equations. Implication is defined by $A \Rightarrow B = \neg A \vee B$. Sequents are of the form $\vdash \Gamma$ where Γ is a sequence of formulas. The rules are the following:

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,\mathrm{x} \qquad \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}\,\mathrm{c} \qquad \frac{\vdash \Gamma}{\vdash A, \Gamma}\,\mathrm{w} \qquad \frac{}{\vdash A, \neg A}\,\mathrm{id} \qquad \frac{\vdash A, \Gamma \qquad \vdash \neg A, \Delta}{\Gamma, \Delta}\,\mathrm{cut}$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \wedge B, \Gamma}\,\wedge \qquad \frac{\vdash A, \Gamma}{\vdash A \vee B, \Gamma}\,\vee_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \vee B, \Gamma}\,\vee_2 \qquad \frac{}{\vdash \top, \Gamma}\,\top$$

$$\frac{\vdash A, \Gamma}{\vdash \forall\xi.A, \Gamma}\,\forall \qquad \frac{\vdash A[\tau/\xi], \Gamma}{\vdash \exists\xi.A, \Gamma}\,\exists$$

In the $\forall$-rule, ξ must have no free occurrence in Γ. Alternatively, the rules for $\wedge$, $\vee$, and $\top$ can be formulated as follows (*multiplicative* version):

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \wedge B, \Gamma, \Delta}\,\wedge \qquad \frac{\vdash A, B, \Gamma}{\vdash A \vee B, \Gamma}\,\vee \qquad \frac{}{\vdash \top}\,\top$$

A translation $A \mapsto A^*$ of (cut-free) Classical Logic into Linear Logic is given by

$$\alpha^* = \alpha, \qquad (\neg\alpha)^* = \alpha^\perp \qquad (A \wedge B)^* = ?A^* \otimes ?B^*, \qquad (A \vee B)^* = A^* \oplus B^*,$$

$$\top^* = \mathbf{1}, \qquad \bot^* = \mathbf{0}, \qquad (\forall\xi.A)^* = \forall\xi.?A^*, \qquad (\exists\xi.A)^* = \exists\xi.A^*.$$

A variant is given by $(A \wedge B)^* = ?A^* \mathbin{\&} ?B^*$ and $\top^* = \top$. In both cases, a sequent $\vdash \Gamma$ is provable in Classical Logic if and only if its translation $\vdash ?\Gamma^*$ is provable in Linear Logic. The *only if* direction is easy: It suffices to consider *cut-free* proofs. Conversely, it is clear that $\vdash \Gamma$ is provable in Classical Logic whenever its translation $\vdash ?\Gamma^*$ is provable in Linear Logic: It suffices to consider the obvious translation $A \mapsto A_*$ from Linear Logic into Classical Logic, defined by

$$\alpha_* = \alpha, \qquad (\alpha^\perp)_* = \neg\alpha, \qquad (A \otimes B)_* = (A \mathbin{\&} B)_* = A_* \wedge B_*, \qquad (A \parr B)_* = (A \oplus B)_* = A_* \vee B_*,$$

$$\mathbf{1}_* = \top_* = \top, \qquad \bot_* = \mathbf{0}_* = \bot, \qquad (!A)_* = (?A)_* = A_*, \qquad (\forall\xi.A)_* = \forall\xi.A_*, \qquad (\exists\xi.A)_* = \exists\xi.A_*.$$

An alternative translation of (cut-free) Classical Logic into Linear Logic is given by

$$\alpha^* = ?\alpha, \qquad (\neg\alpha)^* = ?\alpha^\perp \qquad (A \wedge B)^* = A^* \mathbin{\&} B^*, \qquad (A \vee B)^* = A^* \parr B^*,$$

$$\top^* = \top, \qquad \bot^* = \bot, \qquad (\forall\xi.A)^* = \forall\xi.A^*, \qquad (\exists\xi.A)^* = ?\exists\xi.A^*.$$

In that case, a cut-free proof of $\vdash \Gamma$ is translated into a proof of $\vdash \Gamma^*$, using the fact that A^* is negative for any A.

A translation of (full) Classical Logic into Linear Logic is given by

$$\alpha^* = ?!\alpha, \qquad (\neg\alpha)^* = ?!?\alpha^\perp, \qquad (A \wedge B)^* = ?(!A^* \otimes !B^*), \qquad (A \vee B)^* = A^* \parr B^*,$$

$$\top^* = ?\mathbf{1}, \qquad \bot^* = \bot, \qquad (\forall\xi.A)^* = ?!\forall\xi.A^*, \qquad (\exists\xi.A)^* = ?\exists\xi.!A^*.$$

In that case, a proof of $\vdash \Gamma$ is translated into a proof of $\vdash \Gamma^*$, using the fact that A^* is regular and that $(\neg A)^* \equiv ?A^{*\perp}$ for any A.

Invariants

Here, we consider propositional formulas in the *multiplicative* fragment of Linear Logic, built from a set $\mathcal{P}$ of propositional variables. If Γ is a sequent, $[\Gamma]_\otimes$ stands for the number of occurrences of $\otimes$ in Γ. Similarly, we define $[\Gamma]_{⅋}$, $[\Gamma]_{\mathbf{1}}$, $[\Gamma]_\perp$, $[\Gamma]_\alpha$, and $[\Gamma]_{\alpha^\perp}$. Note that the following equation holds for any sequent of length n:

$$[\Gamma]_\otimes + [\Gamma]_{⅋} + n = [\Gamma]_{\mathbf{1}} + [\Gamma]_\perp + \sum_{\alpha \in \mathcal{P}} ([\Gamma]_\alpha + [\Gamma]_{\alpha^\perp}).$$

Furthermore, a provable sequent satisfies the following equations:

$$[\Gamma]_\alpha = [\Gamma]_{\alpha^\perp}, \qquad [\Gamma]_{⅋} + n = [\Gamma]_\perp + \sum_{\alpha \in \mathcal{P}} [\Gamma]_\alpha + 1, \qquad [\Gamma]_{⅋} + [\Gamma]_{\mathbf{1}} + n = [\Gamma]_\otimes + [\Gamma]_\perp + 2.$$

This is easily checked by induction on proofs. Note that the last equation is a consequence of the previous ones. In the case of a provable formula, we get:

$$[A]_\alpha = [A]_{\alpha^\perp}, \qquad [A]_{⅋} = [A]_\perp + \sum_{\alpha \in \mathcal{P}} [A]_\alpha, \qquad [A]_{⅋} + [A]_{\mathbf{1}} = [A]_\otimes + [A]_\perp + 1.$$

Those conditions are not sufficient: Take for instance $A = (\mathbf{1} ⅋ \mathbf{1}) \otimes \perp$, or $A = (\alpha ⅋ \alpha) \otimes (\alpha^\perp ⅋ \alpha^\perp)$.

About exponential rules

If A is a formula, $A^{(n)}$ stands for the sequent $A, \ldots, A$ of length n. The following rules are derivable (*weak promotion*, *digging*, *absorption*, and *multiplexing*):

$$\frac{\vdash A, \Gamma}{\vdash !A, ?\Gamma} \qquad \frac{\vdash ??A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash ?A, A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash A^{(n)}, \Gamma}{\vdash ?A, \Gamma}$$

Furthermore, *promotion* is derivable from *weak promotion* and *digging*.

Multiplexing is invertible in certain circumstances: A sequent $\vdash ?A, \Gamma$ with no occurrence of &, !, or second order $\exists$ is provable if and only if $\vdash A^{(n)}, \Gamma$ is provable for some n. This is easily checked by induction on cut-free proofs. To see that this does not hold in general, take for instance $A = \alpha^\perp$ and $\Gamma = \alpha \,\&\, \mathbf{1}$, or $\Gamma = !\alpha$. Similarly, a sequent $\vdash ?A, \Gamma$ with no occurrence of ! or second order $\exists$ is provable if and only if $\vdash (A \oplus \perp)^{(n)}, \Gamma$ is provable for some n.

If A is a formula, $!_n A$ stands for the formula $(A \& \mathbf{1}) \otimes \cdots \otimes (A \& \mathbf{1})$ (n times) and $?_n A$ for the formula $(A \oplus \perp) ⅋ \cdots ⅋ (A \oplus \perp)$ (n times). The latter result can be generalized a follows: Consider a provable sequent with p occurrences of !, q occurrences of ?, and no second order $\exists$. Then, for all $m_1, \ldots, m_p \in \mathbb{N}$, there are $n_1, \ldots, n_q \in \mathbb{N}$ such that the sequent obtained by replacing the p occurrences of ! by $!_{m_1}, \ldots, !_{m_p}$ and the q occurrences of ? by $?_{n_1}, \ldots, ?_{n_q}$ is provable (*approximation theorem*).

Note that the following rule is *not* admissible:

$$\frac{C \vdash A \qquad C \vdash C \otimes C \qquad C \vdash \mathbf{1}}{C \vdash !A}$$

For instance, if $A = C = \alpha \otimes !(\alpha \multimap \alpha \otimes \alpha) \otimes !(\alpha \multimap \mathbf{1})$, the three premisses are provable but not the conclusion.

Phase spaces

If M is a (multiplicative) monoid and $X, Y \subset M$, we write XY for the set $\{xy \mid x \in X \text{ and } y \in Y\}$ and $X \multimap Y$ for the set $\{z \in M \mid xz \in Y \text{ for all } x \in X\}$.

A *phase space* is a pair $(M, \bot^M)$ where M is a commutative (multiplicative) monoid and $\bot^M \subset M$. If $X \subset M$, we write $X^\bot$ for $X \multimap \bot^M$. It is easy to prove the following properties:

$$X \subset Y^\bot \text{ if and only if } XY \subset \bot^M, \qquad XX^\bot \subset \bot^M, \qquad \text{if } X \subset Y \text{ then } Y^\bot \subset X^\bot, \qquad X \subset X^{\bot\bot},$$

$$X^{\bot\bot\bot} = X^\bot, \qquad (X^{\bot\bot}Y)^\bot = (XY)^\bot = X \multimap Y^\bot, \qquad (X^{\bot\bot} \cup Y)^\bot = (X \cup Y)^\bot = X^\bot \cap Y^\bot.$$

A *fact* is an $X \subset M$ such that $X = X^{\bot\bot}$, or equivalently, $X = Y^\bot$ for some $Y \subset M$. For instance, $\bot^M = \{1\}^\bot$ is a fact, as well as

$$\mathbf{1}^M = (\bot^M)^\bot = \{1\}^{\bot\bot}, \qquad \top^M = M = \emptyset^\bot, \qquad \mathbf{0}^M = (\top^M)^\bot = M^\bot = \emptyset^{\bot\bot}.$$

Note also that:

- if $X \subset M$ and Y is a fact, then $X \multimap Y = (XY^\bot)^\bot$ is a fact;
- if $(X_i)_{i \in I}$ is a family of facts, then $\bigcap_{i \in I} X_i = (\bigcup_{i \in I} X_i^\bot)^\bot$ is a fact;
- if $X \subset M$, then $X^{\bot\bot}$ is the smallest fact containing X.

We write $x \sqsubseteq y$ if $\{y\}^\bot \subset \{x\}^\bot$, or equivalently, $\{x\}^{\bot\bot} \subset \{y\}^{\bot\bot}$. We write $x \equiv y$ if $x \sqsubseteq y$ and $y \sqsubseteq x$, or equivalently, $\{x\}^\bot = \{y\}^\bot$.

We define the following operations on facts:

$$X \parr Y = X^\bot \multimap Y = (X^\bot Y^\bot)^\bot, \qquad X \otimes Y = (X^\bot \parr Y^\bot)^\bot = (X \multimap Y^\bot)^\bot = (XY)^{\bot\bot},$$

$$X \,\&\, Y = X \cap Y = (X^\bot \cup Y^\bot)^\bot, \qquad X \oplus Y = (X^\bot \,\&\, Y^\bot)^\bot = (X^\bot \cap Y^\bot)^\bot = (X \cup Y)^{\bot\bot},$$

$$?X = (X^\bot \cap \mathbf{I}^M)^\bot, \qquad !X = (?X^\bot)^\bot = (X \cap \mathbf{I}^M)^{\bot\bot},$$

where $\mathbf{I}^M = \{x \in \mathbf{1}^M \mid x = x^2\}$. In the last two definitions, $\mathbf{I}^M$ may be replaced by any submonoid $\mathbf{K}^M$ of $\mathbf{J}^M = \{x \in \mathbf{1}^M \mid x \equiv x^2\} = \{x \in M \mid x \sqsubseteq x^2 \text{ and } x \sqsubseteq 1\}$. Such a $\mathbf{K}^M$ is called an *exponential structure* (Y. Lafont). For instance:

$$\mathbf{K}^M = \{1\} \ (\textit{trivial structure}), \qquad \mathbf{K}^M = \mathbf{I}^M \ (\textit{standard structure}), \qquad \mathbf{K}^M = \mathbf{J}^M \ (\textit{full structure}).$$

Anyway, the notion of *topolinear space* is definitively obsolete.

Phase models

Here we consider propositional formulas built from a set $\mathcal{P}$ of propositional variables. A *phase model* is a phase space (possibly with an exponential structure) together with a fact α^M for each $\alpha \in \mathcal{P}$. The interpretation, which is already defined for units and propositional variables is extended to all formulas in the obvious way:

$$(\alpha^\perp)^M = (\alpha^M)^\perp, \qquad (A \parr B)^M = A^M \parr B^M, \qquad (A \otimes B)^M = A^M \otimes B^M,$$

$$(A \,\&\, B)^M = A^M \,\&\, B^M, \qquad (A \oplus B)^M = A^M \oplus B^M, \qquad (?A)^M = ?A^M, \qquad (!A)^M = !A^M.$$

Since A^M is always a fact, we also get $(A^\perp)^M = (A^M)^\perp$ and $(A \multimap B)^M = (A^M)^\perp \parr B^M = A^M \multimap B^M$.

We say that A *holds* in the model if $1 \in A^M$. Note that the formula $A \multimap B$ holds if and only if $A^M \subset B^M$. More generally, if Γ is a sequence of formulas $A_1, \ldots, A_n$, we write Γ^M for $(A_1 \parr \cdots \parr A_n)^M$, and we say that the sequent $\vdash \Gamma$ *holds* in the model if $1 \in \Gamma^M$, or equivalently, $(A_1^M)^\perp \cdots (A_n^M)^\perp \subset \perp^M$. By induction on proofs, it is easy to see that if a sequent is provable, then it holds in any phase model (*soundness*).

Here are some basic examples:

- If $\perp^M = \emptyset$, then $\perp^M$ and $\top^M$ are the only facts, and we get a Boolean model. In that case, soundness means that any provable formula is classically valid.
- Let $\alpha \in \mathcal{P}$. If $M = \mathbb{Z}$ with addition, $\perp^M = \{0\}$, $\alpha^M = \{1\}$, and $\beta^M = \{0\}$ for each $\beta \in \mathcal{P} \setminus \{\alpha\}$, then for any multiplicative formula A, the set A^M consists of the single element $[A]_\alpha - [A]_{\alpha^\perp}$. In that case, soundness means that $[A]_\alpha = [A]_{\alpha^\perp}$ whenever A is provable.
- If $M = \mathbb{Z}$ with addition, $\perp^M = \{1\}$, and $\alpha^M = \{1\}$ for each $\alpha \in \mathcal{P}$, then for any multiplicative formula A, the set A^M consists of the single element $[A]_\perp + \sum_{\alpha \in \mathcal{P}} [A]_\alpha - [A]_\parr$. In that case, soundness means that $[A]_\parr = [A]_\perp + \sum_{\alpha \in \mathcal{P}} [A]_\alpha$ whenever A is provable.

A *logical congruence* $\sim$ on a phase model M is a congruence such that $\perp^M$ is closed for $\sim$: if $x \in \perp^M$ and $x \sim y$, then $y \in \perp^M$. For instance, $=$ is the finest logical congruence and $\equiv$ is the coarsest one. If $\sim$ is a logical congruence, then all facts are closed for $\sim$, and the canonical projection $\pi : M \mapsto M/\sim$ induces a structure of phase model on the quotient monoid $M/\sim$:

$$\perp^{M/\sim} = \pi(\perp^M), \qquad \alpha^{M/\sim} = \pi(\alpha^M) \text{ for each } \alpha \in \mathcal{P}, \qquad \mathbf{K}^{M/\sim} = \pi(\mathbf{K}^M).$$

It is easy to see that $A^{M/\sim} = \pi(A^M)$ for any formula A. In particular, A holds in M if and only if it holds in $M/\sim$. Note that if $\mathbf{K}^M$ is the *standard* exponential structure on M, then $\pi(\mathbf{K}^M)$ is not necessarily the standard exponential structure on $M/\sim$.

Syntactical model

The *syntactical model* is defined as follows:

- M is the free commutative monoid generated by all formulas. In other words, M is the set of all sequents, considered as finite multisets, with multiset union;
- $\perp^M$ is the set of all *cut-free* provable sequents, and $\alpha^M = \{\alpha\}^\perp$ for each $\alpha \in \mathcal{P}$. In other words, $\alpha^M = \{\Gamma \in M \mid \vdash \alpha, \Gamma \text{ is cut-free provable}\}$;
- $\mathbf{K}^M$ is the set of all sequents of the form $?\Gamma$.

By induction on formulas, $A^M \subset \{A\}^\perp$ for any A (M. Okada). In particular, if A holds in the syntactical model, then the empty sequent belongs to $\{A\}^\perp$, which means that A is (cut-free) provable (*completeness*).

Note the following points:

- By cut elimination, $\perp^M$ is the set of all provable sequents. However, it is essential to consider cut-free proofs if one wants to use this model for proving cut elimination.
- In fact, $A^M = \{A\}^\perp$ for any A, but in the cut-free version of the syntactical model, only the inclusion can be proved directly, and this is enough for proving completeness and cut elimination.
- M can be replaced by the free commutative monoid generated by all subformulas of A. Therefore, completeness holds for phase models whose underlying monoid is *finitely generated*.
- M can also be replaced by $M/\sim$, where $\sim$ is the smallest congruence such that $\Gamma, \Gamma \sim \Gamma$ for any $\Gamma \in \mathbf{K}^M$, so that $\pi(\mathbf{K}^M) = \mathbf{I}^{M/\sim}$. Therefore, completeness holds for phase models with the *standard* exponential structure.
- Completeness does not hold for phase models with the *trivial* exponential structure: For instance, $\alpha \oplus (!\alpha \multimap \mathbf{0})$ holds in any such phase model, but it is not provable.

Finite models

We say that that a model M is *finite* if it has finitely many facts. In that case, $M/\equiv$ is a finite monoid and satisfies the same formulas as M. We say that a fragment of Linear Logic satisfies the *finite model property* if any formula of this fragment that hold in all finite models is provable. It is easy to see that such a fragment is decidable.

The *multiplicative additive* fragment satisfies the finite model property (Y. Lafont). To see that, it suffices to consider a finite version of the syntactical model. For instance:

- M is the free commutative monoid generated by all subformulas of A;
- $\perp^M = \{\Gamma \in M \mid \vdash \Gamma \text{ is cut-free provable or } |\Gamma| > |A|\}$, where $|\Gamma| = [\Gamma]_{\parr} + n$ for any sequent Γ of length n, and $\alpha^M = \{\alpha\}^\perp$ for each $\alpha \in \mathcal{P}$.

The *multiplicative exponential* fragment does *not* satisfy the finite model property: For instance, the formula $!\alpha \otimes !(\alpha \otimes \beta) \otimes !(\alpha \otimes \beta \multimap \mathbf{1}) \multimap \beta$ holds in any finite model, but it is not provable.

We say that that a model M is *affine* if the set $\perp^M$ is an ideal of M, that is $\perp^M M \subset \perp^M$. This amonts to say that $\perp^M = \mathbf{0}^M$, or equivalently, $\mathbf{1}^M = \top^M$. In that case, any fact is an ideal. There is a completeness theorem for Affine Linear Logic with respect to affine models, and Affine Linear Logic satisfies a finite model property. This comes from the fact that if M is a finitely generated free commutative monoid, then all its ideals are finitely generated. Consequently, Affine Linear Logic is decidable (A. Kopylov).