Homotopy of computation: a deconstruction of equality

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Equality versus homotopy

- Equality is a trivial notion: a = a

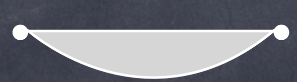
  a•a

- Homotopy is a rich notion: a ~ a′

  a •———• a′

  - because you can compose homotopies:

  - and there are homotopies between homotopies:



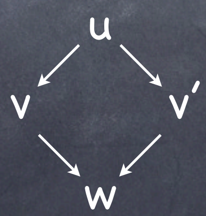
Convergent rewriting

Motivation: symbolic computation in some algebraic structure (group, monoid, …)

- **orient** equality ($u \to v$ instead of $u = v$)
- **termination**: no infinite computation

$$u_0 \to u_1 \to \cdots \to u_n \to \cdots$$

- **confluence**: resolution of « conflicts »



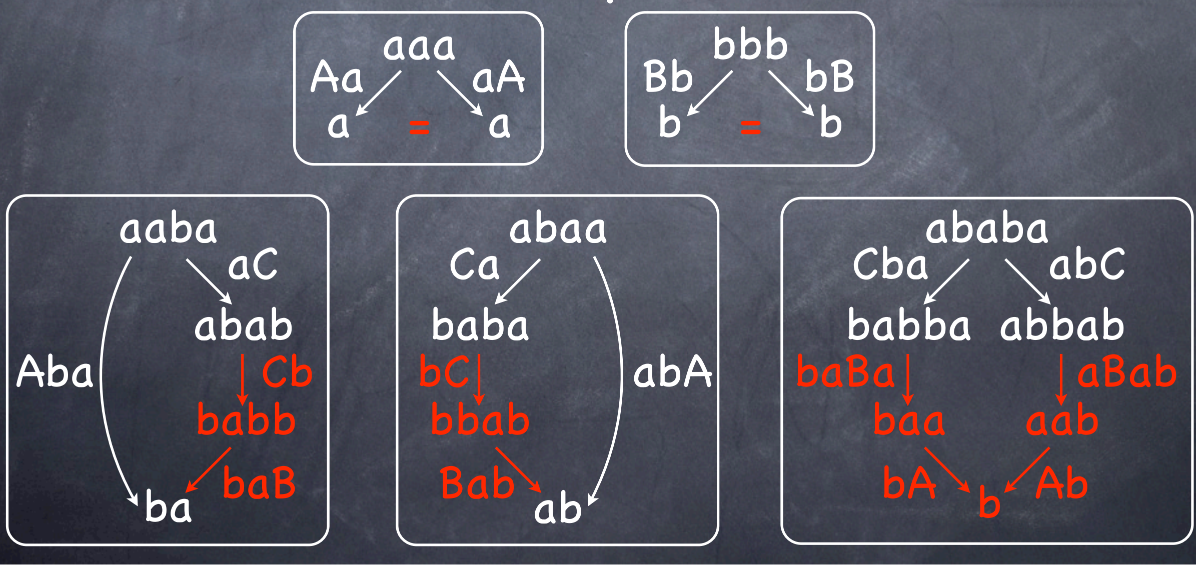
➨ existence and uniqueness of the **normal form** $\hat{u}$

Example of convergent rewriting

Symmetric group $S_3 = \langle a,b \mid a^2, b^2, ababab \rangle$

Rewrite rules : $aa \xrightarrow{A} 1$, $bb \xrightarrow{B} 1$, $aba \xrightarrow{C} bab$

Critical peaks:



Squier theory

Theorem (Squier 1987)
Any finite convergent presentation of a monoid M yields a partial resolution of **Z** by free **Z**M-modules:

$$0 \leftarrow \mathbf{Z} \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow F_3$$

where the F_i (including F_3) have finite dimension.

Idea: 

(modulo critical diagrams)

Corollary
If a monoid M has a finite convergent presentation, then its homology group $H_3(M)$ has finite type.

Squier theory

Theorem (Kobayashi 1990)
Any finite convergent presentation of a monoid M yields a full resolution of **Z** by free **Z**M-modules:

$$0 \leftarrow \mathbf{Z} \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow \cdots$$

where all the F_i have finite dimension.

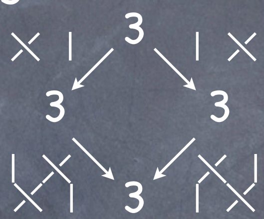
Idea: consider higher dimensional critical peaks

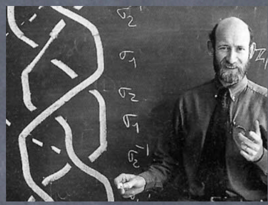
Corollary
If a monoid M has a finite convergent presentation, then all its homology groups $H_n(M)$ have finite type.

Gaussian groups

Example: **braid group B_3** = $\langle a,b \mid abab^{-1}a^{-1}b^{-1} \rangle$

Analogue of confluence:





Theorem (Dehornoy & Lafont 2003)

Any finite gaussian presentation of a group G yields a full resolution of **Z** by free **Z**G-modules:

$$0 \leftarrow \mathbf{Z} \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow \cdots$$

where all the F_i have finite dimension.

Idea: follow Kobayashi

Omega-categories

0-cells 1-cells 2-cells 3-cells …

0-composition

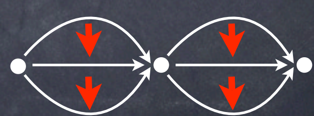
…

1-composition 2-composition …

…

Associativity and units

Interchange



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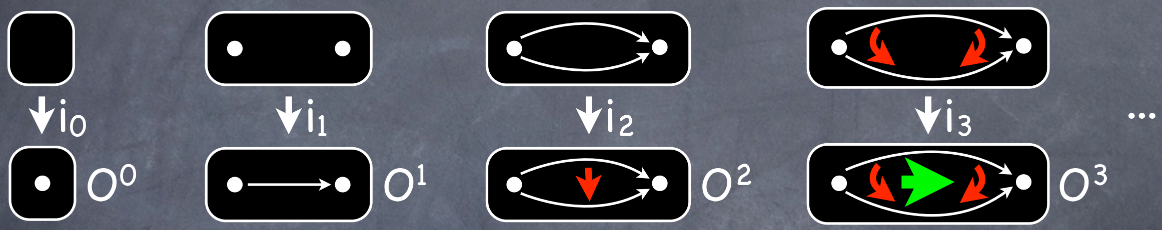
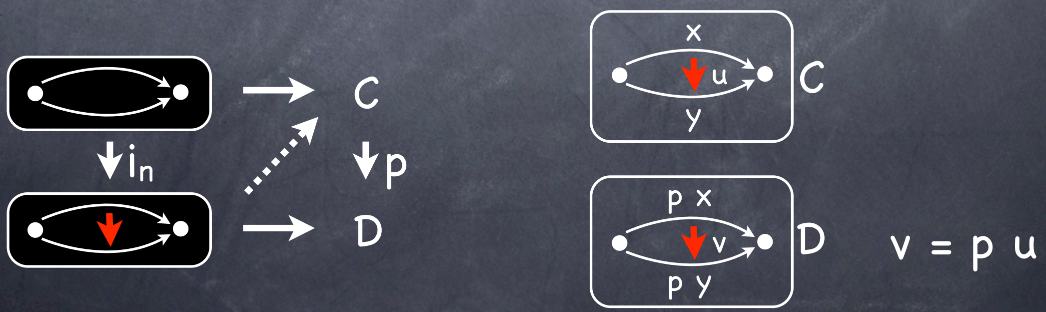
Omega-categories

Canonical inclusions:

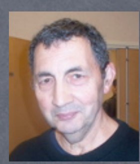


Acyclic fibrations:



Polygraphs (or computads)

Definition (Burroni 1993):



A polygraph is an ω-category S^* of the form

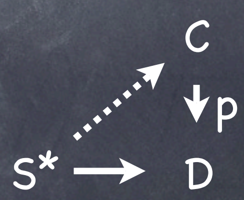
$$S_0^* \leftleftarrows S_1^* \leftleftarrows S_2^* \leftleftarrows S_3^* \leftleftarrows \cdots$$

where each S_n^* is freely generated by a set S_n.

Polygraphs can be seen as:

- higher dimensional rewrite systems;
- directed cellular complexes.

Lemma: Polygraphs are **cofibrant**.



The converse holds (Métayer 2008).

Polygraphic resolutions

Theorem (Métayer 2003)

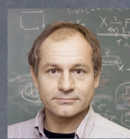
Any ω-category C has a **polygraphic resolution**:

p : S*→C (where p is an acyclic fibration).

Theorem (Lafont & Métayer 2009)

The homology of a monoid M is obtained by abelianization of a polygraphic resolution of M.

A model structure



Theorem (Lafont, Métayer & Worytkiewicz 2010)

There is a (Quillen) model structure on the category ω-**Cat** of ω-categories such that:

- generating cofibrations are canonical inclusions;
- the class **W** of weak equivalences is minimal.

The folk model structures on **Cat** (Joyal & Tierney 1990) and 2-**Cat** (Lack 2002) are derivable from this structure.

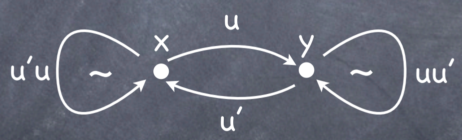
Reversibility

This notion is crucial for the definition of the class W of **weak equivalences**.

Coinductive definition:

$x \sim y$ iff there are $u : x \to y$ and $u' : y \to x$ such that

$u'u \sim id_x$ and $uu' \sim id_y$.



In that case, we say that u is **reversible**.

This is a deconstruction of the notion of isomorphism!

References

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