A folk model structure on omega-cat

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joint work with
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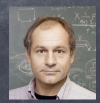


HOCAT 2008
CRM, Barcelona

History

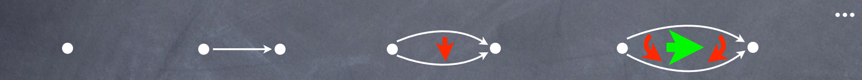
- Homology of rewriting (Anick/Squier 1987)
- Finite derivation type (Squier 1994)

- Computads/polygraphs (Street & Power/Burroni 1991)
- Polygraphic resolutions (Métayer 2003)

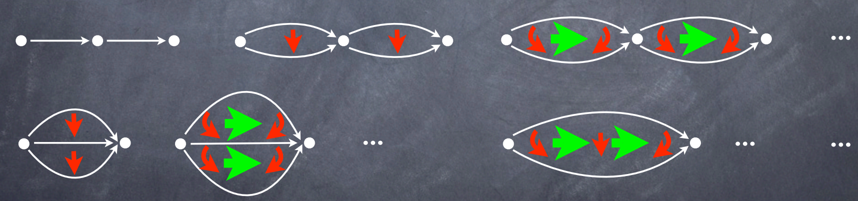
- Model structure on **Cat** (Joyal & Tierney 1990)
- Model structure on 2-**Cat** (Lack 2002)

Strict omega-categories

0-cells 1-cells 2-cells 3-cells ...

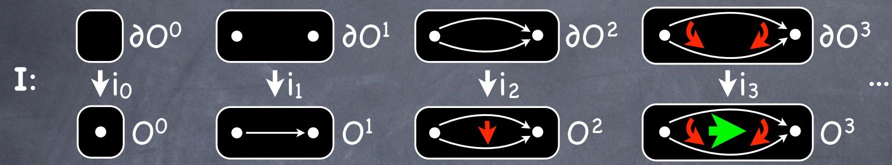


Compositions

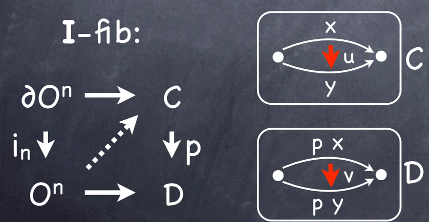


Laws of associativity, units, and interchange

Generating cofibrations



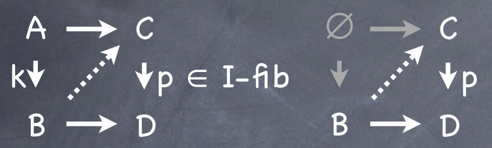
Acyclic fibrations



For any v : p x → p y in D,
there is u : x → y in C
such that p u = v.

Cofibrations

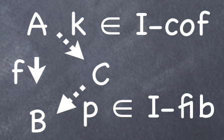
I-cof:



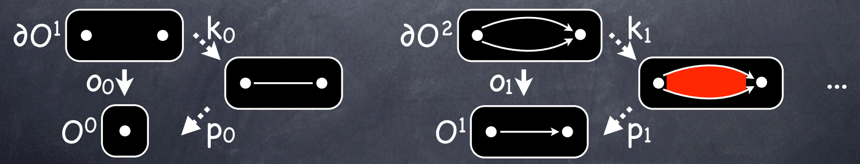
Theorem (Métayer)
Cofibrant objects are polygraphs.

Factorization

Any ω-functor factorizes as follows:



Crucial example:



Our main result

Theorem (Lafont, Métayer & Worytkiewicz):

There is a model structure on ω-**Cat** such that:

- **I** is a set of generating cofibrations for this structure;
- the class **W** of weak equivalences is minimal.

The following model structures are derivable from this:

- on **Cat** (Joyal & Tierney 1990);
- on 2-**Cat** (Lack 2002);
- on n-**Cat**.

They are obtained by Quillen adjunctions (Beke 2001)

Our strategy

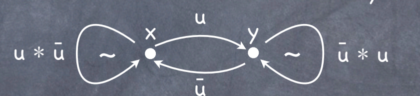
- Define the class **W** of weak equivalences.
- Prove the following properties (Smith):
  - **I**-fib ⊂ **W**;
  - **W** has the 3 for 2 property;
  - **I**-cof ∩ **W** is closed under pushout, retract, and tranfinite composition;
  - **I** and **W** satisfy the solution set condition.

Tools: ω-equivalence & reversibility, connections, gluing factorization, immersions, generic squares.

ω-equivalence & reversibility

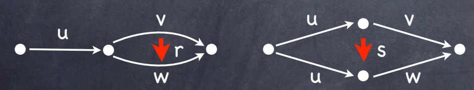
Definition (by coinduction)

- $x \sim y$ iff there exists a reversible cell $u : x \to y$;
- $u : x \to y$ is reversible iff there exists $\bar{u} : y \to x$ such that $u * \bar{u} \sim id_x$ and $\bar{u} * u \sim id_y$.



Lemma

If u is reversible, it satisfies a (left) division property:



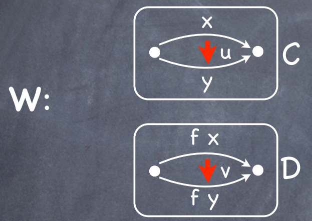
For all $s : u * v \to u * w$, there is $r : v \to w$ such that $u * r \sim s$.

(General case: u is a 1-cell and s is a n+1-cell)

Weak equivalences

Definition

By a relaxed version of the right lifting property:



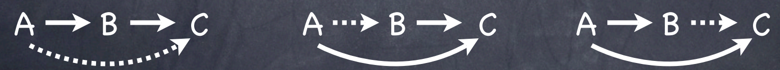
For any v : f x → f y in D

there is u : x → y in C

such that f u ~ v (not =)

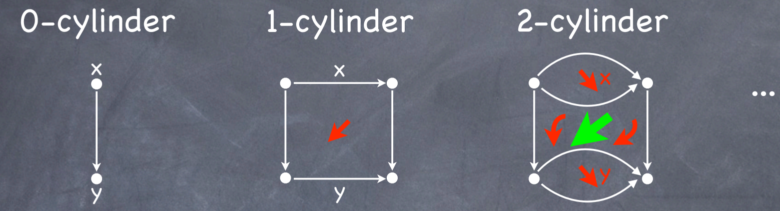
Proposition

The class **W** of weak equivalences satisfies 3 for 2.



(Some machinery is needed for the third case)

Cylinders & connections

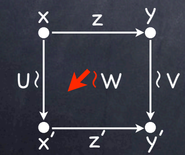


Definition

A connection U : x ☎ y is a reversible cylinder.

Lemma

There is a topdown transport for parallel connections:



For all z : x → y, z′ is weakly unique.

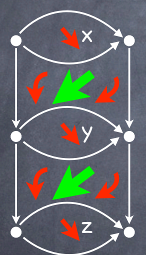
there is z′ : x′ → y′

and a connection W : U → V | z ☎ z′.

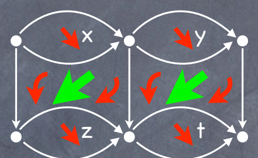
Operations on connections

Theorem: Connections in C form an ω-category Γ(C).

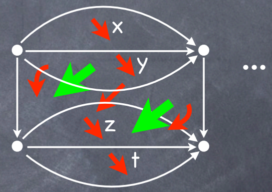
concatenation



0-composition



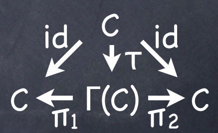
1-composition



Concatenation is needed to define n-compositions.

trivial cylinder

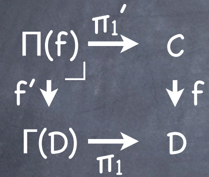
We get the following commutative diagram:

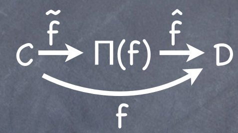


Gluing factorization

Definition

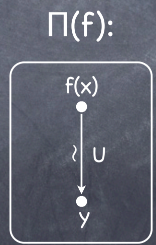
The gluing factorization $f = \hat{f} \circ \tilde{f}$ is defined by a pullback:

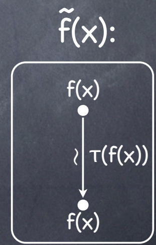




where $\hat{f} = \pi_2 \circ f'$

Explicit descriptions:





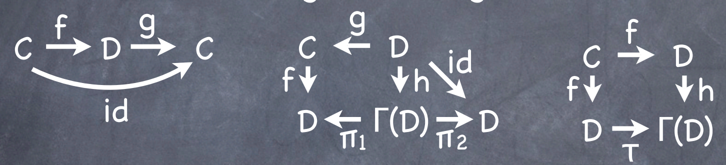
Lemma

$f \in$ **W** if and only if $\hat{f} \in$ **I**-fib.

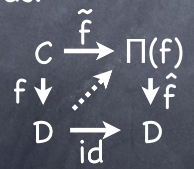
Immersions

Definition

$f : C \to D$ is in **Z** if there are $g : D \to C$ and $h : D \to \Gamma(D)$ such that the following three diagrams commute:



In other words:



Corollary

I-cof ∩ **W** is closed under pushout.

Lemma

I-cof ∩ **W** ⊂ **Z** ⊂ **W**.

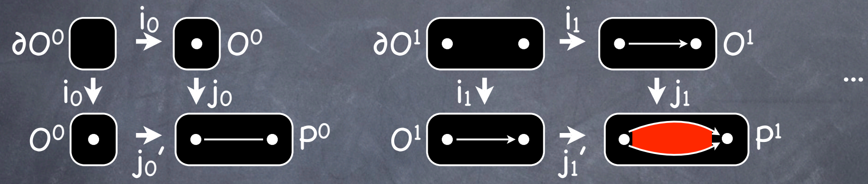
Lemma

Z is closed under pushout.

Generic squares

Definition

The generic squares are the following diagrams:



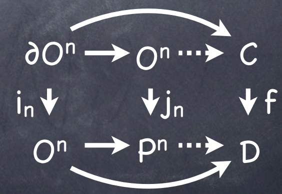
Lemma

Those squares are universal for ω-equivalence.

Corollary

$f \in$ **W** iff we have factorizations:

(solution set condition)



Conclusion

We have proved the following properties:

- **I**-fib ⊂ **W**;
- **W** has the 3 for 2 property;
- **I**-cof ∩ **W** is closed under pushout, retract, and tranfinite composition;
- **I** and **W** satisfy the solution set condition.

References

- Y. Lafont, François Métayer & Krzysytof Worytkiewicz, A folk model structure on omega-cat (ArXiv, 2007)
- Y. Lafont & François Métayer, Polygraphic resolutions and homology of monoids (submitted)
- François Métayer, Cofibrant objects among higher dimensional categories (HHA, 2008)