DIAGRAM REWRITING AND OPERADS

by

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Abstract. — We give a survey of a diagrammatic syntax for *PROs* and *PROPs*, which are related to the theory of operads and bialgebras. Using *diagram rewriting*, we obtain *presentations of PROs by generators and relations*. In some cases, we even get *convergent rewrite systems*.

Résumé **(Réécriture de diagrammes et opérades).** — Nous donnons un aperçu de la syntaxe diagrammatique pour les *PROs* et les *PROPs*, qui sont liés à la théorie des opérades et des bigèbres. En utilisant la *réécriture de diagrammes*, on obtient des *présentations de PROs par générateurs et relations*. Dans certains cas, on obtient même des *systèmes de réécriture convergents*.

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Except for Sections 4 and 7, most of the material presented in this paper comes from [**Laf03**], which was inspired by [**Bur93**].

1. PROs and PROPs

Definition 1. — *A* PRO *(or* product category*) is a* strict monoidal category*, that is a (small) category* $\mathbf{C}$ *equipped with some associative functor* $* : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *and a unit object, such that the set of objects of* $\mathbf{C}$ *is* $\mathbb{N}$*, and* $p * q = p + q$ *for all* $p, q \in \mathbb{N}$*. In particular, the unit object is* 0*.*

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In order to define a PRO, since objects are already known, it suffices to give the set $\mathbf{C}(p,q)$ of *morphisms* $f : p \to q$ for all $p, q \in \mathbb{N}$, together with:

– a *sequential composition* $g \circ f : p \to r$ for any $f : p \to q$ and $g : q \to r$;
– a *parallel composition* $f * f' : p + p' \to q + q'$ for any $f : p \to q$ and $f' : p' \to q'$;
– an *identity* $\mathrm{id}_p : p \to p$ for all $p \in \mathbb{N}$.

This terminology will be clear in the next section. Of course, those two compositions must be associative, with units:

– $(h \circ g) \circ f = h \circ (g \circ f)$ for any $f : p \to q$, $g : q \to r$, and $h : r \to s$;
– $(f * f') * f'' = f * (f' * f'')$ for any $f : p \to q$, $f' : p' \to q'$, and $f'' : p'' \to q''$;
– $f \circ \mathrm{id}_p = f = \mathrm{id}_q \circ f$ and $f * \mathrm{id}_0 = f = \mathrm{id}_0 * f$ for any $f : p \to q$.

But they must also be compatible (law of *interchange*):

– $(g \circ f) * (g' \circ f') = (g * g') \circ (f * f')$ for any $f : p \to q$, $g : q \to r$, $f' : p' \to q'$, and $g' : q' \to r'$;
– $\mathrm{id}_p * \mathrm{id}_q = \mathrm{id}_{p+q}$ for all $p, q \in \mathbb{N}$.

Here are typical examples:

– the PRO $\mathfrak{F}$, where a morphism $f : p \to q$ is a map from $\{1, \ldots, p\}$ to $\{1, \ldots, q\}$;
– the PRO $\mathfrak{M} \subset \mathfrak{F}$, where a morphism $f : p \to q$ is a monotone map from $\{1, \ldots, p\}$ to $\{1, \ldots, q\}$;
– the PRO $\mathbf{L}(\mathbb{K})$, where a morphism $f : p \to q$ is a $\mathbb{K}$-linear map from $\mathbb{K}^p$ to $\mathbb{K}^q$ (or a $q \times p$ matrix) for any commutative field $\mathbb{K}$.

Compositions are obvious:

– $\circ$ is composition of maps (or product of matrices);
– $*$ is disjoint union (for $\mathfrak{F}$), ordered sum (for $\mathfrak{M}$), or direct sum (for $\mathbf{L}(\mathbb{K})$).

If we remove the object 0 from the PRO $\mathfrak{M}$, then we get the *simplicial category* Δ.

Definition 2. — *A PRO* $\mathbf{C}$ *is* reversible *if all* $\mathbf{C}(p,p)$ *are groups and* $\mathbf{C}(p,q) = \emptyset$ *whenever* $p \neq q$.

In order to define such a PRO, it suffices to give a group $\mathbf{C}_p = \mathbf{C}(p,p)$ for all p, together with a parallel composition $f * g \in \mathbf{C}_{p+q}$ defined for any $f \in \mathbf{C}_p$ and $g \in \mathbf{C}_q$. Note that a reversible PRO is a groupoid, but the condition $\mathbf{C}(p,q) = \emptyset$ for $p \neq q$ is not necessary to get a groupoid. Here are typical examples:

– the PRO $\mathfrak{S} \subset \mathfrak{F}$, where $\mathfrak{S}_p$ is the p-th *symmetric group*;
– the PRO $\mathfrak{B}$, where $\mathfrak{B}_p$ is the p-th *braid group*;
– the PRO $\mathbf{GL}(\mathbb{K}) \subset \mathbf{L}(\mathbb{K})$, where $\mathbf{GL}_p(\mathbb{K})$ is the p-th *linear group over* $\mathbb{K}$;
– the PRO $\mathbf{O} \subset \mathbf{GL}(\mathbb{R})$, where $\mathbf{O}_p \subset \mathbf{GL}_p(\mathbb{R})$ is the p-th *orthogonal group*.

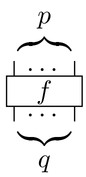
Definition 3. — *A* PROP *(or* product and permutation category*) is a PRO* $\mathbf{C} \supset \mathfrak{S}$.

For instance, both $\mathfrak{F}$ and $\mathbf{L}(\mathbb{K})$ are PROPs, but not $\mathfrak{M}$. PROPs are introduced in [**Mac65**], with a slightly different definition, but of course, our notion is equivalent.

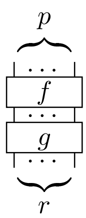
2. Diagrams

We recall the *diagrammatic syntax* of [**Laf03**]:

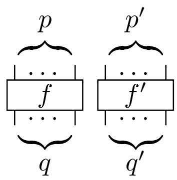
– a morphism $f : p \to q$ is pictured as a box with p inputs and q outputs:



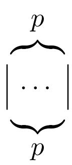
– for $f : p \to q$ and $g : q \to r$, the sequential composition $g \circ f : p \to r$ is pictured as follows:



– for $f : p \to q$ and $f' : p' \to q'$, the parallel composition $f * f' : p + p' \to q + q'$ is pictured as follows:



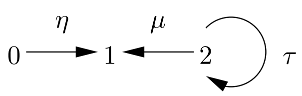
– the identity $\mathrm{id}_p : p \to p$ is pictured as follows:



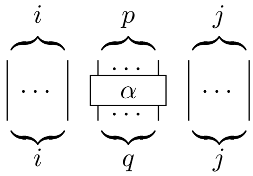
– in particular, $\mathrm{id}_0 : 0 \to 0$ is pictured as an empty diagram.

Definition 4. — *A* signature *is a graph $\mathcal{S}$ with vertices in $\mathbb{N}$. An edge $\alpha : p \to q$ in $\mathcal{S}$ is called a* symbol with p inputs and q outputs.

For instance, the following signature will be introduced in the next section:



Definition 5. — *An* elementary diagram *built over signature $\mathcal{S}$ is a formal parallel composition* $\mathrm{id}_i * \alpha * \mathrm{id}_j : i+p+j \to i+q+j$*, where* $\alpha : p \to q$ *is a symbol of* $\mathcal{S}$*, and* $i, j \in \mathbb{N}$*. It is pictured as follows:*

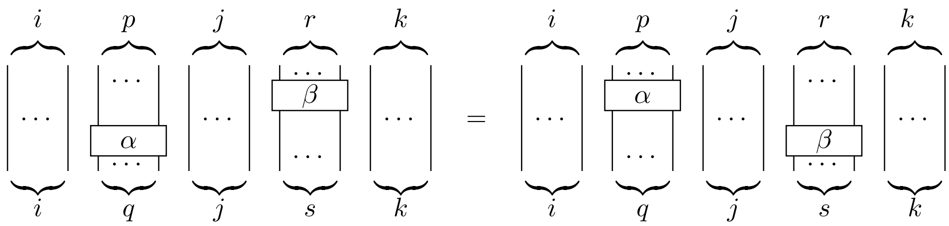


Definition 6. — *A* diagram *built over signature $\mathcal{S}$ is a formal sequential composition* $\phi_n \circ \cdots \circ \phi_1 : p_0 \to p_n$*, where* $\phi_1 : p_0 \to p_1, \phi_2 : p_1 \to p_2, \ldots, \phi_n : p_{n-1} \to p_n$ *are elementary diagrams. In particular, we get* $\mathrm{id}_{p_0} : p_0 \to p_0$ *when* $n = 0$*.*

Definition 7. — *The* free PRO $\mathcal{S}^*$ *consists of all diagrams built over a signature* $\mathcal{S}$*, modulo the* commutation laws*:*

$$(\mathrm{id}_i * \alpha * \mathrm{id}_{j+s+k}) \circ (\mathrm{id}_{i+p+j} * \beta * \mathrm{id}_k) = (\mathrm{id}_{i+q+j} * \beta * \mathrm{id}_k) \circ (\mathrm{id}_i * \alpha * \mathrm{id}_{j+r+k})$$

for any symbols $\alpha : p \to q$ *and* $\beta : r \to s$*, and for all* $i, j, k \in \mathbb{N}$*.*



The commutation laws are necessary to get a PRO, which must satisfy interchange. In fact, a morphism of $\mathcal{S}^*$ can also be considered as a formal (sequential and parallel) composition of symbols modulo associativity, units, and interchange.

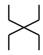
3. Presentations by generators and relations

Definition 8. — *A* relation $\rho = \sigma$ *(over a signature $\mathcal{S}$) is given by two diagrams* $\rho, \sigma : p \to q$ *built over* $\mathcal{S}$*.*

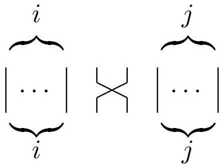
Definition 9. — *A* presentation *of a PRO* $\mathbf{C}$ *consists of a signature $\mathcal{S}$ together with a set $\mathcal{R}$ of relations over $\mathcal{S}$, such that* $\mathbf{C} \simeq \mathcal{S}^* / \leftrightarrow^*_{\mathcal{R}}$*, where* $\leftrightarrow^*_{\mathcal{R}}$ *is the* congruence generated by $\mathcal{R}$*.*

See Section 6 for a rigorous definition of $\leftrightarrow^*_{\mathcal{R}}$. Note that the commutation laws will never appear in presentations, since they are implicit. In fact, generators and relations for PROs have been introduced in the framework of *computads*, also called *polygraphs*: See [**Str87, Pow91, Bur93, Str95**].

For instance, we have a morphism $\tau : 2 \to 2$ in $\mathfrak{S}$, defined by $\tau(1) = 2$ and $\tau(2) = 1$. It is pictured as follows:



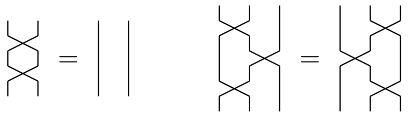
The transposition $\mathrm{id}_i * \tau * \mathrm{id}_j : i + 2 + j \to i + 2 + j$ corresponds to the following elementary diagram:



Since any permutation is a product of such transpositions, τ generates $\mathfrak{S}$, and the following two relations hold:

$$\tau \circ \tau = \mathrm{id}_2, \quad (\tau * \mathrm{id}_1) \circ (\mathrm{id}_1 * \tau) \circ (\tau * \mathrm{id}_1) = (\mathrm{id}_1 * \tau) \circ (\tau * \mathrm{id}_1) \circ (\mathrm{id}_1 * \tau).$$

Those relations are pictured as follows:



Theorem 1. — *The symbol $\tau : 2 \to 2$ together with the above two relations form a presentation of the PRO $\mathfrak{S}$.*

In other words, the following two statements hold:

- any permutation $f : p \to p$ is defined by some diagram $\phi : p \to p$ built over the generator $\tau : 2 \to 2$;
- two diagrams $\phi, \psi : p \to q$ define the same permutation if and only if $\phi \leftrightarrow^*_{\mathcal{R}} \psi$, where $\mathcal{R}$ consists of the above two relations.

This folklore result can be proved by means of *diagram rewriting*: See Section 5.

Moreover, the usual presentation of $\mathfrak{S}_p$ as a group can be deduced from this simple presentation of all $\mathfrak{S}_p$, collectively seen as a (reversible) PRO:

Corollary 1. — *The group $\mathfrak{S}_p$ is presented by $p-1$ generators $\tau_1, \ldots, \tau_{p-1}$ together with the following $\frac{p(p-1)}{2}$ relations:*

$$\tau_i^2 = 1 \textit{ (for } i = 1, \ldots, p-1\textit{)}, \quad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \textit{ (for } i = 1, \ldots, p-2\textit{)},$$

$$\tau_i \tau_j = \tau_j \tau_i \textit{ (for } i = 1, \ldots, p-3 \textit{ and } j = i+2, \ldots, p-1\textit{)}.$$

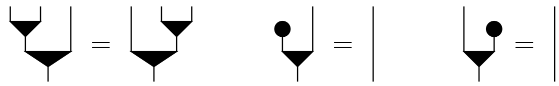
Note that the third family of relations does not come from the relations for the PRO $\mathfrak{S}$, but from the commutation laws.

In the PRO $\mathfrak{M}$, we have two morphisms $\mu : 2 \to 1$ and $\eta : 0 \to 1$, which are pictured as follows:

It is easy to see that μ and η generate $\mathfrak{M}$, and they satisfy the following three relations:

$$\mu \circ (\mu * \mathrm{id}_1) = \mu \circ (\mathrm{id}_1 * \mu), \quad \mu \circ (\eta * \mathrm{id}_1) = \mathrm{id}_1, \quad \mu \circ (\mathrm{id}_1 * \eta) = \mathrm{id}_1.$$

Those relations are pictured as follows:



Theorem 2. — *The symbols $\mu : 2 \to 1$ and $\eta : 0 \to 1$ together with the above three relations form a presentation of the PRO $\mathfrak{M}$.*

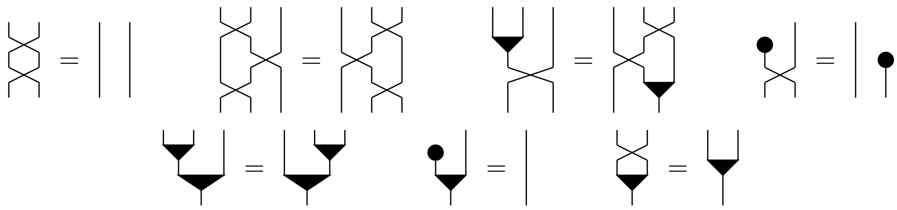
It is possible to extract an infinite presentation of the simplicial category Δ from this finite presentation of the PRO $\mathfrak{M}$.

Since $\mathfrak{F}$ contains $\mathfrak{S}$ and $\mathfrak{M}$, we get morphisms $\tau : 2 \to 2$, $\mu : 2 \to 1$, $\eta : 0 \to 1$ in $\mathfrak{F}$, which are pictured as follows:

Again, it is easy to see that τ, μ and η generate $\mathfrak{F}$, and they satisfy the following extra relations:

$$\tau \circ (\mu * \mathrm{id}_1) = (\mathrm{id}_1 * \mu) \circ (\tau * \mathrm{id}_1) \circ (\mathrm{id}_1 * \tau), \quad \tau \circ (\eta * \mathrm{id}_1) = \mathrm{id}_1 * \eta, \quad \mu \circ \tau = \mu.$$

The relation $\mu \circ (\mathrm{id}_1 * \eta) = \mathrm{id}_1$ becomes superfluous. Hence, we get seven relations, which are pictured as follows:



Theorem 3. — *The symbols $\tau : 2 \to 2$, $\mu : 2 \to 1$, and $\eta : 0 \to 1$ together with the above seven relations form a presentation of the PRO $\mathfrak{F}$.*

This crucial result was first stated and proved in [**Bur93**], where a presentation of a PRO is seen as a 3-*polygraph*. This presentation is minimal: See [**Mas97**].

4. Schur functors and operads

Definition 10. — *If* $\mathbb{K}$ *is a commutative field and* $\mathbf{C}$ *is a PRO, the* Schur functor $\mathbf{C}^{\mathbb{K}}$ *is given by*

$$\mathbf{C}^{\mathbb{K}}(V) = \bigoplus_{p\in\mathbb{N}} \mathbf{C}_p^{\mathbb{K}} \otimes V^{\otimes p} \text{ for any } \mathbb{K}\text{-vector space } V,$$

where $\mathbf{C}_p^{\mathbb{K}}$ *is the free* $\mathbb{K}$*-vector space generated by the set* $\mathbf{C}(p,1)$*.*

So we get:

$$\mathbf{C}^{\mathbb{K}}(\mathbf{C}^{\mathbb{K}}(V)) = \bigoplus_{p,q\in\mathbb{N}} \mathbf{C}_p^{\mathbb{K}} \otimes (\mathbf{C}_q^{\mathbb{K}} \otimes V^{\otimes q})^{\otimes p} \text{ for any } V.$$

We define $\gamma : \mathbf{C}^{\mathbb{K}}(\mathbf{C}^{\mathbb{K}}(V)) \to \mathbf{C}^{\mathbb{K}}(V)$ and $\iota : V \to \mathbf{C}^{\mathbb{K}}(V)$ by the following formulas:

- $\gamma(g \otimes (f_1 \otimes v_1 \otimes \cdots \otimes v_q) \otimes \cdots \otimes (f_p \otimes v_{pq-q+1} \otimes \cdots \otimes v_{pq})) =$ $(g \circ (f_1 * \cdots * f_p)) \otimes v_1 \otimes \cdots \otimes v_{pq}$;
- $\iota(v) = \mathrm{id}_1 \otimes v$.

By the axioms of PRO, we get a monad structure on $\mathbf{C}^{\mathbb{K}}$, which is the *(nonsymmetric) operad* associated with $\mathbf{C}$.

This definition forgets most of the structure of $\mathbf{C}$, since it only uses the sets $\mathbf{C}(p,1)$. For instance, $\mathfrak{S}^{\mathbb{K}}$ is trivial. However, if $\mathbf{C}$ is a PROP, we define a right action of $\mathfrak{S}_p$ on $\mathbf{C}_p^{\mathbb{K}}$ by $f \cdot g = f \circ g$ for any $f : p \to 1$ and $g \in \mathfrak{S}_p$, which by hypothesis, can be considered as a morphism $g : p \to p$ in $\mathbf{C}$. In that case, $\mathbf{C}^{\mathbb{K}}$ is a *symmetric operad*.

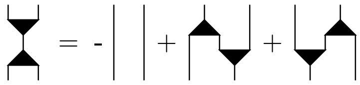
Here are typical examples:

- $\mathfrak{M}^{\mathbb{K}}$ is the (nonsymmetric) operad $\mathbf{Ass}^{\mathbb{K}}$ of *(unital) associative* $\mathbb{K}$*-algebras*;
- $\mathfrak{F}^{\mathbb{K}}$ is the (symmetric) operad $\mathbf{Com}^{\mathbb{K}}$ of *(unital) commutative* $\mathbb{K}$*-algebras*.

See [**Lod08**] for more details.

In fact, $\mathfrak{M}^{\mathbb{K}}$ is the same functor as $\mathfrak{F}^{\mathbb{K}}$, since all $\mathfrak{M}(p,1)$ are singletons, and similarly for the $\mathfrak{F}(p,1)$. But the PRO $\mathfrak{M}$ also extends to a PROP $\mathfrak{M} \otimes \mathfrak{S}$, which defines a (symmetric) operad $(\mathfrak{M} \otimes \mathfrak{S})^{\mathbb{K}}$, and this operad is not $\mathfrak{F}^{\mathbb{K}}$. The PRO $\mathfrak{M} \otimes \mathfrak{S}$ is presented by the (disjoint) union of a presentation of $\mathfrak{M}$ with a presentation of $\mathfrak{S}$, together with some *commutation relations*. Essentially, we get the presentation of $\mathfrak{F}$, but without the *commutativity* $\mu \circ \tau = \mu$.

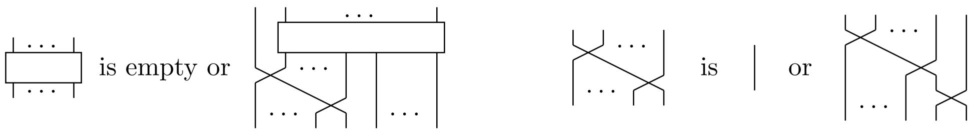
Note that [**Lod08**] uses Σ-*diagrams*, which are linear combinations of diagrams, rather than diagrams. Hence, they are *enriched* in the category of Abelian groups. For instance, here is the *compatibility relation in the infinitesimal unital case*:



Therefore, we should introduce Σ-diagram rewriting, as in [**Ran09**].

5. Canonical forms

To prove Theorem 1, we introduce *canonical forms*, which are recursively defined by the following *grammar*:



In other words, a canonical form is a diagram $\phi : p \to p$ of the following form:

$$\begin{cases} \mathrm{id}_0 & \text{if } p = 0, \\ (\tau_q * \mathrm{id}_{p-q}) \circ (\mathrm{id}_1 * \psi) & \text{if } p > 0, \text{ where} \end{cases}$$

$$1 \le q \le p \text{ and } \tau_q : q \to q \text{ is } \begin{cases} \mathrm{id}_1 & \text{if } q = 1, \\ (\mathrm{id}_{q-2} * \tau) \circ \tau_{q-1} & \text{if } q > 1, \end{cases}$$

$$\psi : p-1 \to p-1 \text{ is a canonical form.}$$

Lemma 1. — id_p *is a canonical form for all* $p \in \mathbb{N}$.

This is proved by induction on p:

- id_0 is a canonical form by definition;
- if $p > 0$, then $\mathrm{id}_p = \mathrm{id}_p \circ \mathrm{id}_p = (\mathrm{id}_1 * \mathrm{id}_{p-1}) \circ (\mathrm{id}_1 * \mathrm{id}_{p-1})$ is a canonical form by induction hypothesis.

Lemma 2. — *Any permutation* $f : p \to p$ *corresponds to a unique canonical form.*

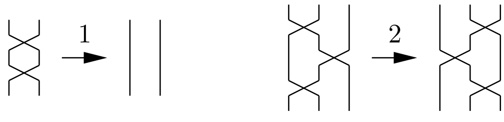
This is proved by induction on p:

- if $p = 0$, then $f = \mathrm{id}_0$;
- if $p > 0$, then $f = (\tau_q * \mathrm{id}_{p-q}) \circ (\mathrm{id}_1 * g)$ where $q = f(1)$ and $g : p-1 \to p-1$ is defined by

$$g(i) = \begin{cases} f(i+1) & \text{if } f(i+1) < q, \\ f(i+1) - 1 & \text{if } f(i+1) > q. \end{cases}$$

By induction hypothesis, g corresponds to a unique canonical form. So we get a unique canonical form for f. In particular, τ generates $\mathfrak{S}$.

Lemma 3. — *Any diagram* $\phi : p \to p$ *reduces to a canonical form by the following two rewrite rules:*



Rewrite rules are applied from left to right. See Section 6 for a precise definition of diagram rewriting.

This lemma is proved by double induction on the *width* p and on the *size* n of ϕ, that is the length of the sequence $\phi_1, \dots, \phi_n$ of elementary diagrams defining ϕ:

- if $n = 0$, then $\phi = \mathrm{id}_p$ is a canonical form by Lemma 1.
- if $n > 0$, then $\phi = (\mathrm{id}_i * \tau * \mathrm{id}_j) \circ \phi'$ where ϕ' has width $p = i + 2 + j$ and size $n - 1$.

By induction hypothesis for size $n - 1$, we may assume that ϕ' is a canonical form $(\tau_q * \mathrm{id}_{p-q}) \circ (\mathrm{id}_1 * \psi)$ where ψ has width $p - 1$. There are four cases (see Figure 1):

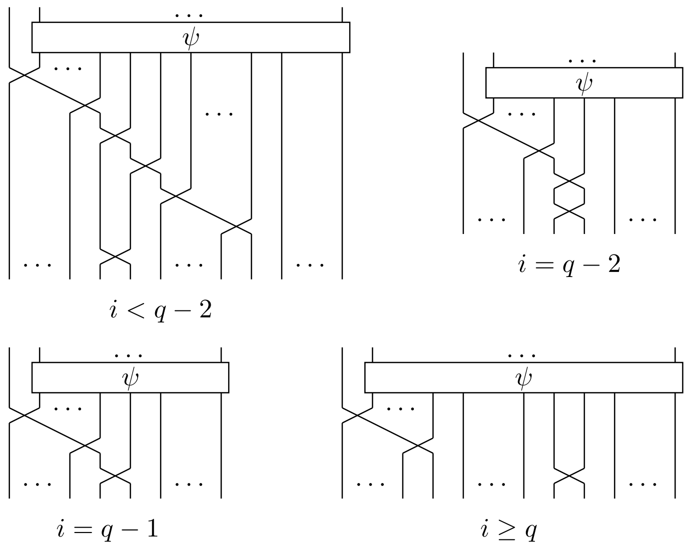
- if $i < q - 2$, apply commutation, rule 2, commutation again, and the induction hypothesis for width $p - 1$;
- if $i = q - 2$, apply rule 1;
- if $i = q - 1$, then ϕ is a canonical form;
- if $i \geq q$, apply commutation and the induction hypothesis for width $p - 1$.

Hence, we have proved Lemma 3.

In particular, if two diagrams $\phi, \psi : p \to p$ define the same permutation, then ϕ reduces to a canonical form $\hat{\phi}$ and ψ reduces to a canonical form $\hat{\psi}$. Since ϕ, $\hat{\phi}$, ψ, and $\hat{\psi}$ represent the same permutation, then $\hat{\phi} = \hat{\psi}$ by Lemma 2. Hence, ϕ is equivalent to ψ modulo the above relations, and we have proved Theorem 1.

Theorems 2 and 3 are proved by the same method, using some suitable notions of canonical forms: See [**Laf03**].

FIGURE 1. the four cases of Lemma 3



6. Convergent rewriting

The notions introduced in this section (*rewrite rules*, *termination*, *confluence*, and *convergence*) are usually given in the case of *strings* (or *words*), or of *terms*: See [**KN85b, BN98**]. Here, we consider a generalization to diagrams.

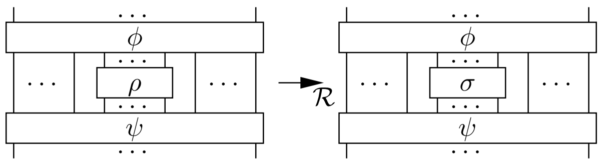
Definition 11. — *A* rewrite rule *over signature $\mathcal{S}$ is a directed relation $\rho \to \sigma$, where $\rho, \sigma : p \to q$ are diagrams built over $\mathcal{S}$.*

Definition 12. — *A* rewrite system *for* $\mathbf{C}$ *is given by a signature $\mathcal{S}$ together with a set $\mathcal{R}$ of rewrite rules over $\mathcal{S}$ such that $\mathcal{S}, \mathcal{R}$ form a presentation of* $\mathbf{C}$.

For instance, the two rules of Lemma 3 form a rewrite system for $\mathfrak{S}$. Again, such a rewrite system is a 3-*polygraph*.

If $\mathcal{R}$ is a rewrite system, we introduce *one-step* and *many-step reduction*:

– $\to_{\mathcal{R}}$ is defined by $\psi \circ (\mathrm{id}_i * \rho * \mathrm{id}_j) \circ \phi \to_{\mathcal{R}} \psi \circ (\mathrm{id}_i * \sigma * \mathrm{id}_j) \circ \phi$ for any rule $\rho \to \sigma$ in $\mathcal{R}$ with $\rho, \sigma : p \to q$, for all $i, j \in \mathbb{N}$, and for any diagrams $\phi : r \to i + p + j$ and $\psi : i + q + j \to s$;



– $\to^*_{\mathcal{R}}$ is defined by $\phi \to^*_{\mathcal{R}} \psi$ if $\phi = \phi_0 \to_{\mathcal{R}} \phi_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} \phi_{n-1} \to_{\mathcal{R}} \phi_n = \psi$. In particular, we get $\phi \to^*_{\mathcal{R}} \phi$.

$\to^*_{\mathcal{R}}$ is the *transitive closure* of $\to_{\mathcal{R}}$, that is the smallest ordering containing $\mathcal{R}$ which is compatible with $\circ$ and $*$. Note that if $\phi \to^*_{\mathcal{R}} \psi$ with $\phi : p \to q$, we get $\psi : p \to q$.

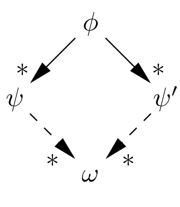
Similarly, we define $\leftrightarrow_{\mathcal{R}}$ by $\phi \leftrightarrow_{\mathcal{R}} \psi$ whenever $\phi \to_{\mathcal{R}} \psi$ or $\psi \to_{\mathcal{R}} \phi$, and $\leftrightarrow^*_{\mathcal{R}}$ is the transitive closure of $\leftrightarrow_{\mathcal{R}}$, that is the smallest congruence containing $\mathcal{R}$.

Definition 13. — *A rewrite system $\mathcal{R}$ is* convergent *if:*

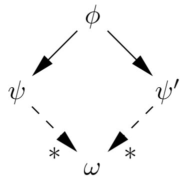
– *$\mathcal{R}$ is* Noetherian *(or* terminating*), which means there is no infinite reduction:*

$$\phi_0 \to_{\mathcal{R}} \phi_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} \phi_{n-1} \to_{\mathcal{R}} \phi_n \to_{\mathcal{R}} \phi_{n+1} \cdots$$

– *$\mathcal{R}$ is* confluent*, which means that for any diagrams ϕ, ψ, ψ' such that $\phi \to^*_{\mathcal{R}} \psi$ and $\phi \to^*_{\mathcal{R}} \psi'$, there is some diagram ω such that $\psi \to^*_{\mathcal{R}} \omega$ and $\psi' \to^*_{\mathcal{R}} \omega$:*



In the Noetherian case, confluence is equivalent to *local confluence*:



Therefore, we shall consider all possible *conflicts* between rules, which are also called *critical pairs*, or *branchings*.

Definition 14. — *A diagram ϕ is* normal *if there is no reduction $\phi \to_{\mathcal{R}} \psi$.*

In the Noetherian case, any diagram reduces to a normal one, which is unique in the confluent case:

Definition 15. — *In the convergent case, the* normal form *of ϕ is the unique normal diagram $\hat{\phi}$ such that $\phi \to_{\mathcal{R}}^{*} \hat{\phi}$.*

Proposition 1. — *In the convergent case, we have $\phi \leftrightarrow_{\mathcal{R}}^{*} \psi$ if and only if $\hat{\phi} = \hat{\psi}$.*

Hence, if $\mathcal{R}$ is a finite convergent rewrite system for $\mathbf{C}$, we get an algorithm solving the *word problem* for $\mathbf{C}$. Moreover, the *Knuth-Bendix completion* transforms a rewrite system into a convergent one: See [**KN85b**]. But this completion may loop.

Lemma 4. — *The rewrite system for $\mathfrak{S}$ is Noetherian.*

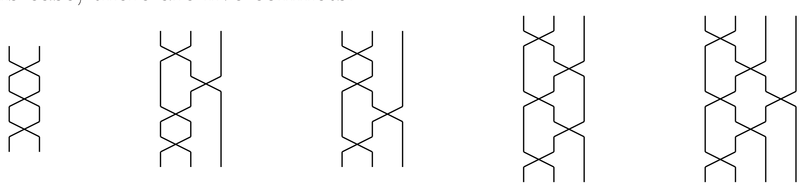
To prove this, it suffices to define $|\phi| \in \mathbb{N}$ for any diagram ϕ, in such a way that $|\phi| > |\psi|$ whenever $\phi \to_{\mathcal{R}} \psi$:

$$|\mathrm{id}_p| = 0, \quad |(\mathrm{id}_i * \tau * \mathrm{id}_j) \circ \phi| = j + 1 + |\phi|.$$

In fact, this method works because any diagram with p inputs also has p outputs. A general method for proving termination of diagram rewriting is given in [**Gui06**].

Lemma 5. — *The rewrite system for $\mathfrak{S}$ is confluent.*

In this case, there are five conflicts:

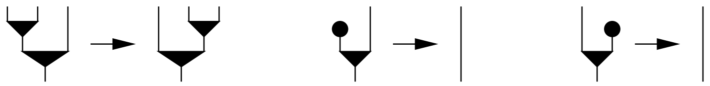


The first four ones are obtained by superposing left members of rules, just as in *string rewriting*: See [**KN85b**]. But the last one is a *global conflict*, because the rightmost occurrence of τ is not involved in any rewriting process. This complication comes from the commutation laws: See appendix A of [**Laf03**]. In fact, it may happen that a diagram rewrite system produces infinitely many (global) conflicts: See [**GM09**].

Fortunately, we only get one global conflict in that case, and the five conflicts are confluent (Figure 2). To sum up:

– the above rewrite system is convergent;
– the canonical forms of Section 5 are the normal diagrams for this rewrite system.

Similarly, we have a rewrite system with three rules for $\mathfrak{M}$:



Termination comes from the fact that some occurrence of μ moves to the right (in the first case) or disappears (in the two other cases). Furthermore, there are five conflicts, which are confluent (Figure 3). Hence, this system is convergent. Note that the first two diagrams of Figure 3 are *Mac Lane's coherence conditions*: See [**Mac71**].

Finally, we have a rewrite system with 12 rules for $\mathfrak{F}$: See Figure 4. In this case, there are 68 conflicts: See [**Laf03**]. Again, this rewrite system is convergent.

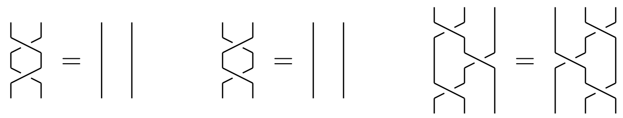
7. The braided case

In the PRO $\mathfrak{B}$, we have a generator $\tau : 2 \to 2$ and its inverse $\overline{\tau} : 2 \to 2$:

This generator satisfies the *Yang-Baxter equation*:

$$(\tau * \mathrm{id}_1) \circ (\mathrm{id}_1 * \tau) \circ (\tau * \mathrm{id}_1) = (\mathrm{id}_1 * \tau) \circ (\tau * \mathrm{id}_1) \circ (\mathrm{id}_1 * \tau).$$

To sum up, we get the following relations:



Theorem 4. — *The symbols $\tau : 2 \to 2$ and $\overline{\tau} : 2 \to 2$ together with the above three relations form a presentation of the PRO $\mathfrak{B}$.*

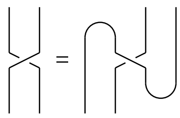
Corollary 2. — *The group $\mathfrak{B}_p$ is presented by $p-1$ generators $\tau_1, \ldots, \tau_{p-1}$ together with the following $\frac{(p-1)(p-2)}{2}$ relations:*

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \text{ (for } i = 1, \ldots, p-2\text{)},$$

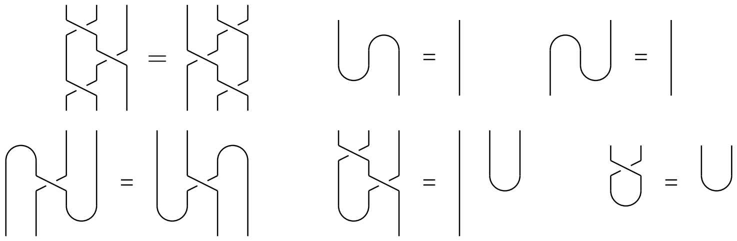
$$\tau_i \tau_j = \tau_j \tau_i \text{ (for } i = 1, \ldots, p-3 \text{ and } j = i+2, \ldots, p-1\text{)}.$$

The PRO $\mathfrak{B}$ extends to the (non reversible) PRO $\mathfrak{T}$ of *tangles*. In $\mathfrak{T}$, we have three morphisms $\tau : 2 \to 2$, $\varepsilon : 2 \to 0$, and $\eta : 0 \to 2$, which are pictured as follows:

In particular, a *knot* can be seen as a tangle $\phi : 0 \to 0$: See [**JS93, KRT97**]. Note also that the inverse $\overline{\tau} : 2 \to 2$ becomes definable:



Those generators satisfy the following relations, corresponding to *Reidemeister moves*:



Theorem 5. — *The symbols $\tau : 2 \to 2$, $\varepsilon : 2 \to 0$, and $\eta : 0 \to 2$ together with the above six relations form a presentation of the PRO $\mathfrak{T}$.*

But no convergent rewrite system is known for $\mathfrak{B}$ or for $\mathfrak{T}$. In fact, there is already a problem with the monoid $\mathfrak{B}_3^+$ of *positive braids*: See [**KN85a**].

8. Going further

More examples of presentations of PROs are given in [**Laf03**], for instance:

– a finite convergent rewrite system for the PRO $\mathbf{L}(\mathbb{Z}_2)$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$;
– a finite presentation for the reversible PRO $\mathbf{GL}(\mathbb{Z}_2)$ of $\mathbb{Z}_2$-linear permutations;
– a convergent rewrite system for the reversible PRO $\mathbf{O}$ of orthogonal transforms.

The third one was motivated by the theory of *quantum boolean circuits*: See [**LR08**], where the connections with *Euler angles* and *Zamolodchikov equation* are explained.

This diagrammatic syntax is not only useful for practical computations, but also for theoretical results. In fact, rewriting is strongly related to homotopical algebra: It can be used to compute homological invariants of algebraic structures, as in [**Squ87**], or to prove coherence results, as in [**Mac71**]. Connections with homology are also explained in [**Laf07**].

Note also that diagrams are informally related to *proof-nets* and *interaction nets*, which were introduced in the context of proof theory: See [**Gir87, Laf97, ER06**].

Figure 2. confluence of the rewrite system for $\mathfrak{S}$

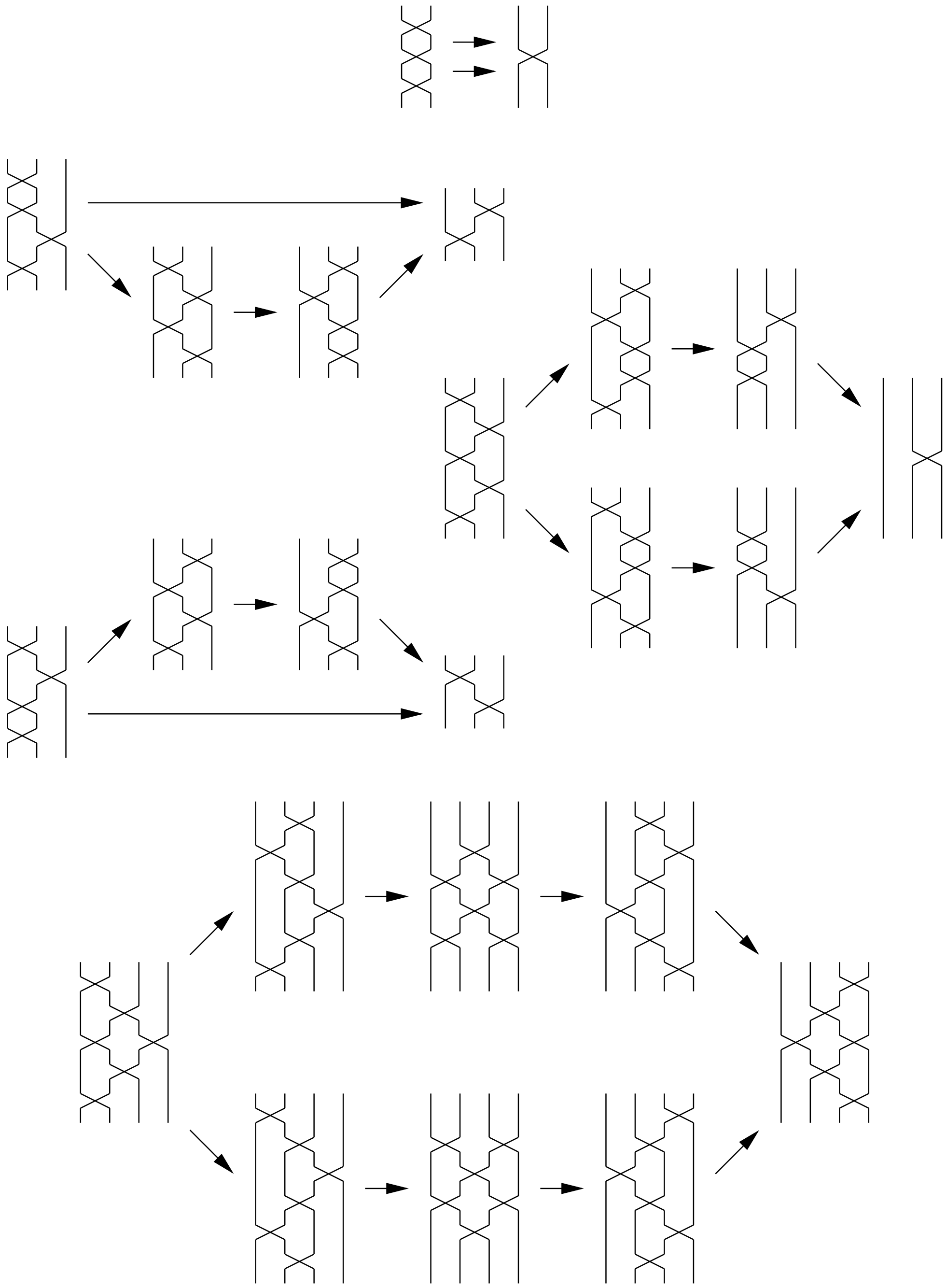


FIGURE 3. confluence of the rewrite system for $\mathfrak{M}$

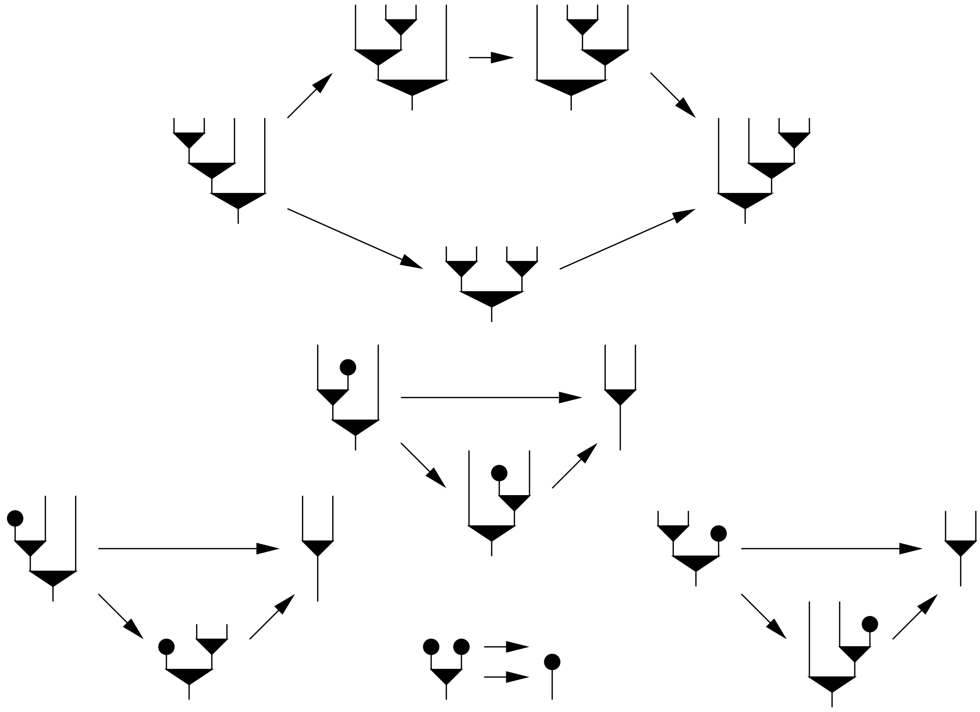
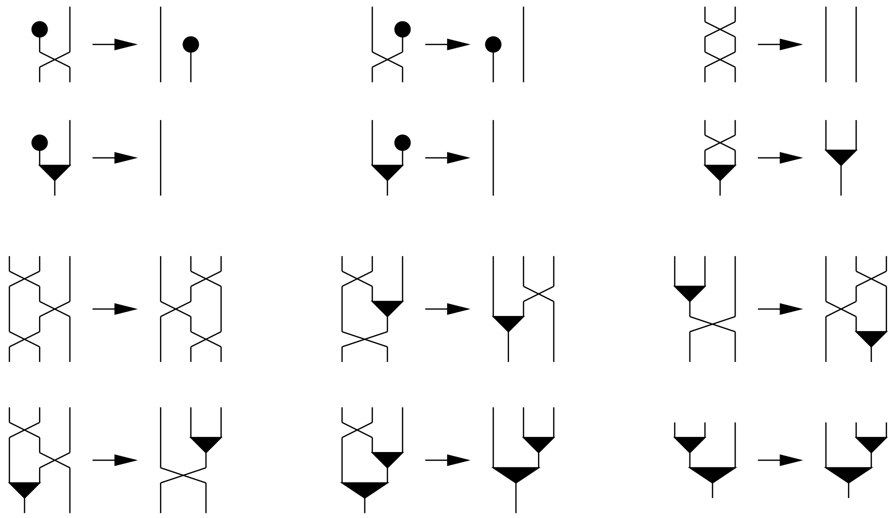


FIGURE 4. rewrite system for $\mathfrak{F}$



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