Towards a homotopy theory of higher-dimensional automata

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Abstract. We introduce a new cofibrantly generated model structure on cubical sets. In this model structure, trivial fibrations are precisely the open maps. In particular, this model structure carries over to the category **HDA** of higher dimensional automata. Bisimilar hda's are thus identified in the homotopy category Ho(**HDA**).

1. Cubical sets

Cubical sets are presheaves over a category of elementary cubical shapes.

Definition 1. *Let* $\square_0 \overset{\text{def.}}{=\!=} \{0\}$ *and* $\square_n \overset{\text{def.}}{=\!=} \{0,1\}^n$ *for* $n \geqslant 1$. *Let*

$$\begin{array}{rlcl} \delta_n^{i,\varepsilon}\colon & \square_{n-1} & \longrightarrow & \square_n \\ & (x_1,\dots,x_{n-1}) & \longmapsto & (x_1,\dots,x_{i-1},\varepsilon,x_i,\dots,x_{n-1}) \end{array}$$

and

$$\begin{array}{rlcl} \sigma_n^{i}\colon & \square_{n} & \longrightarrow & \square_{n-1} \\ & (x_1,\dots,x_{n}) & \longmapsto & (x_1,\dots,x_{i-1},x_{i+1}) \end{array}$$

for $n \geqslant 1, 1 \leqslant i \leqslant n$ and $\varepsilon \in \{0,1\}$. *The category* $\square$ *of elementary cubes has the* $\square_n$*'s as objects. It is generated by the cofaces* $\delta_n^{i,\varepsilon}$ *and the codegeneracies* σ_n^i *subject to the cocubical relations*

$$\delta_n^{j,\eta} \circ \delta_{n-1}^{i,\varepsilon} = \delta_n^{i,\varepsilon} \circ \delta_{n-1}^{j-1,\eta} \qquad i<j \tag{1}$$

$$\sigma_n^{j} \circ \sigma_{n+1}^{i} = \sigma_n^{i} \circ \sigma_{n+1}^{j+1} \qquad i\leqslant j \tag{2}$$

$$\sigma_n^{j} \circ \delta_{n+1}^{i,\varepsilon} = \begin{cases} \delta_n^{i,\varepsilon} \circ \sigma_{n-1}^{j-1} & i<j \\ \mathrm{id}_{\square_n} & i=j \\ \delta_n^{i-1,\varepsilon} \circ \sigma_{n-1}^{j} & i>j \end{cases} \tag{3}$$

Definition 2. *The category* **cSet** *of cubical sets is the category of presheaves* $\mathbf{Set}^{\square^{\mathrm{op}}}$. *The standard n-cube is the representable preasheaf*

$$\square[n] \overset{\text{def.}}{=\!=} \square(-,\square_n)$$

Remark 3. As any category of presheaves, **cSet** is a (Grothendieck) topos.

Remark 4. A cubical set is a family $(K_n)_{n\geqslant 0}$ of sets equipped with face maps

$$d_n^{i,\varepsilon}\colon K_n \longrightarrow K_{n-1}$$

and degeneracy maps

$$s_n^{i}\colon K_{n-1} \longrightarrow K_n$$

subject to the cubical relations

$$d_{n-1}^{i,\varepsilon} \circ d_n^{j,\eta} = d_{n-1}^{j-1,\eta} \circ d_n^{i,\varepsilon} \tag{4}$$

$$s_{n+1}^{i} \circ s_n^{j} = s_{n+1}^{j+1} \circ s_n^{i} \tag{5}$$

$$d_{n+1}^{i,\varepsilon} \circ s_n^{j} = \begin{cases} s_{n-1}^{j-1} \circ d_n^{i,\varepsilon} & i<j \\ \mathrm{id}_{K_n} & i=j \\ s_{n-1}^{j} \circ d_n^{i-1,\varepsilon} & i>j \end{cases} \tag{6}$$

2. Higher dimensional automata

Definition 5. *Given a set* Σ *and* $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma^*$, *let* $|\omega| \overset{\text{def.}}{=} n$. *Suppose from now on* Σ *totally ordered and let*

$$\omega \vDash (\Sigma, \leq) \overset{\text{def}}{\Longleftrightarrow} \omega_1 \leq \cdots \leq \omega_n$$

Let further $\star \notin \Sigma$, $\omega = (\omega_1, \ldots, \omega_n) \in (\Sigma \cup \{\star\})^*$, $\underline{\omega} \subseteq \omega$ *be the word obtained from* ω *by removing all the occurences of* $\star$ *and*

$$!\Sigma_n \overset{\text{def.}}{=} \{\omega \in (\Sigma \cup \{\star\})^* \mid |\omega| = n \wedge \underline{\omega} \vDash (\Sigma, \leq)\}$$

Finally, let

$$\delta_i(\omega_1, \ldots, \omega_n) \overset{\text{def.}}{=} (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n)$$

and

$$\varepsilon_i(\omega_1, \ldots, \omega_n) \overset{\text{def.}}{=} (\omega_1, \ldots, \omega_{i-1}, \star, \omega_i, \ldots, \omega_n)$$

Proposition 6. (Goubault) *The data of definition 5 assemble to a cubical set.*

Definition 7. *The category* ***HDA*** *of higher dimensional automata (abbreviated hda's) is the comma-category*

$$1/\mathbf{cSet}/!\Sigma$$

under 1 *and over* $!\Sigma$.

3. Bisimulation of higher-dimensional automata

Definition 8. *Let* K *be a cubical set and* $A \in K_n$ *be an* n*-cube in* K.

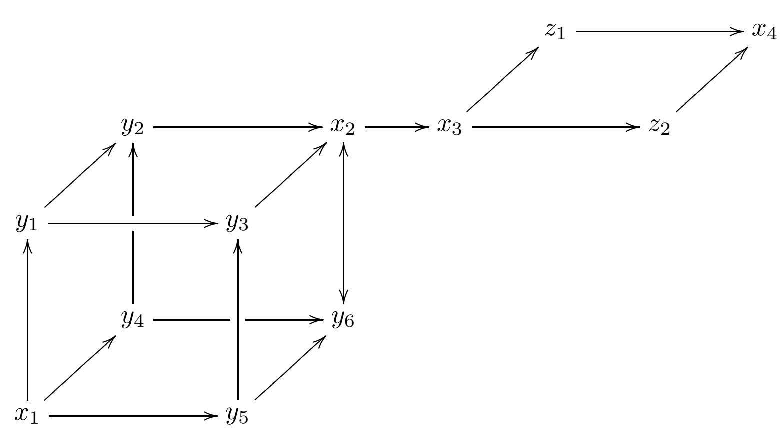
i. $\operatorname{dom}(A) \overset{\text{def.}}{=} \underbrace{\partial_0^- \ldots \partial_0^-}_{n\times}$;

ii. $\operatorname{cod}(A) \overset{\text{def.}}{=} \underbrace{\partial_0^+ \ldots \partial_0^+}_{n\times}$;

Remark 9. Given a family of $(n_i)_{1 \leqslant i \leqslant r}$ of positive integers, the corresponding family $(\square[n_i])_{1 \leqslant i \leqslant r}$ of standard n_i-cubes can be glued to the cubical set

$$\oint_{i=1}^{r} n_i \overset{\text{def.}}{=} \operatorname{coequ}_{1 \leqslant i < r} \left\{ \operatorname{cod}(\square[n_i]), \operatorname{dom}(\square[n_{i+1}]) \colon 1 \rightrightarrows \sum_{i=1}^{r} \square[n_i] \right\}$$

Example 10. $\oint (3, 1, 2)$ can be depicted as



Definition 11. *A cubical set as in remark 9 is called a path from*

$$\mathrm{dom}\left(\oint_{i=1}^{r} n_i\right) \overset{\text{def.}}{=} \mathrm{dom}(\square[n_1])$$

to

$$\mathrm{cod}\left(\oint_{i=1}^{r} n_i\right) \overset{\text{def.}}{=} \mathrm{cod}(\square[n_r])$$

A path extension is a mono

$$m\colon \oint_{j=1}^{p} n_j \rightarrowtail \oint_{i=1}^{r} n_i$$

in ***cSet*** *commuting with the operation* dom.

Notation 12. *$\mathcal{P}$ stands for the collection of all path extensions.*

Definition 13. *A cubical map $h\colon M \to K$ is open provided $h \in \mathrm{rlp}(\mathcal{P})$.*

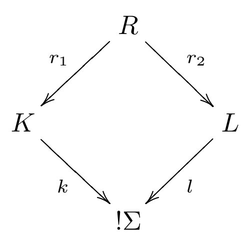
Remark 14. An open map is surjective in all degrees. The notion of open map has it's obvious counterparts in comma categories on **cSet**.

Definition 15. *Let $K, L \in \mathbf{HDA}$. A bisimulation $R\colon K \leftrightsquigarrow L$ is a span*

$$K \longleftarrow R \longrightarrow L$$

of open maps. Hda's K and L are bismilar if there is a witnessing bisimulation $K \leftrightsquigarrow L$.

Remark 16. A bisimulation is a commuting diagram



in $1/\mathbf{cSet}$ with r_1 and r_2 open.

4. The model structure

Definition 17. *Let $\mathbb{C}$ be a category. A collection of morphisms $S \subseteq \mathbb{C}_1$ is saturated if*

i. all isos belong to S;

ii. S is stable by

- *retracts;*
- *coproducts;*
- *pushouts;*
- *countable compositions.*

Let $A \subseteq \mathbb{C}_1$ be a collection of morphisms. Its saturation $\bar{A}$ is the smallest saturated collection of morphisms containing A.

Definition 18. *Cubical maps $f, g\colon X \longrightarrow Y$ are naively homotopic if there is a cubical map*

$$h\colon X \times \square[1] \longrightarrow Y$$

such that the diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle i_0}\downarrow & \searrow^{f} & \\ X\times\square[1] & \xrightarrow{\;h\;} & Y \\ {\scriptstyle i_1}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

commutes.

Notation 19. $\mathbf{cSet}_I$ *stands for* $\mathbf{cSet}$ *quotiented by the congruence relation induced by naive homotopy.*

Definition 20. *Let*

$$J \stackrel{\text{def.}}{=\!=} \{P \hookrightarrow \square[n] \mid n\in\mathbb{N}, P\in\mathcal{P}, \mathrm{dom}(P)=\mathrm{dom}(\square[n], \mathrm{cod}(P)=\mathrm{cod}(\square[n]\}$$

and

$$I \stackrel{\text{def.}}{=\!=} \{\mathrm{dom}\colon 1 \hookrightarrow \square[n] \mid n\in\mathbb{N}\}$$

be sets of cubical maps.

1. *A cubical set* X *is formally fibrant if* $X \stackrel{!}{\longrightarrow} 1 \in \mathrm{rlp}\big(\overline{I\cup J}\big)$*;*
2. *A formal weak equivalence is a cubical map* $f\colon K\longrightarrow L$ *inducing a bijection*

$$f^*\colon \mathbf{cSet}_I(L,X) \stackrel{\approx}{\longrightarrow} \mathbf{cSet}_I(K,X)$$

for all formally fibrant cubical sets X*.*

Theorem 21. I *and* J *are the generating cofibrations respectively the generating trivial cofibrations of a cofibrantly generated model structure on* $\mathbf{cSet}$*. We have*

- $\mathcal{C} = \{\text{monos}\}$*;*
- $\mathcal{W} = \{\text{formal weak equivalences}\}$*.*

Remark 22. It is easy to see that $\mathrm{rlp}(\mathcal{P}) = \mathrm{rlp}(I)$, hence the trivial fibrations in this model structure are precisely the open maps.

Remark 23. This model structure carries over to **HDA**. In particular, bisimilar hda's are identified in the homotopy category. **HDA** being pointed (i.e. the initial and the terminal objects are isomorphic), the homotopy category is (pre-)triangulated and so a good platform for homology theories.