

TOWARDS A HOMOTOPY THEORY OF HIGHER-DIMENSIONAL AUTOMATA

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ABSTRACT. We introduce a new cofibrantly generated model structure on cubical sets. In this model structure, trivial fibrations are precisely the open maps. In particular, this model structure carries over to the category **HDA** of higher dimensional automata. Bisimilar hda's are thus identified in the homotopy category $\text{Ho}(\mathbf{HDA})$.

1. CUBICAL SETS

Cubical sets are presheaves over a category of elementary cubical shapes.

Definition 1. Let $\square_0 \stackrel{\text{def.}}{=} \{0\}$ and $\square_n \stackrel{\text{def.}}{=} \{0, 1\}^n$ for $n \geq 1$. Let

$$\delta_n^{i,\varepsilon}: \square_{n-1} \longrightarrow \square_n \\ (x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, x_{i-1}, \varepsilon, x_i, \dots, x_{n-1})$$

and

$$\sigma_n^i: \square_n \longrightarrow \square_{n-1} \\ (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{i-1}, x_{i+1})$$

for $n \geq 1$, $1 \leq i \leq n$ and $\varepsilon \in \{0, 1\}$. The category \square of elementary cubes has the \square_n 's as objects. It is generated by the cofaces $\delta_n^{i,\varepsilon}$ and the codegeneracies σ_n^i subject to the cocubical relations

$$\delta_n^{j,\eta} \circ \delta_{n-1}^{i,\varepsilon} = \delta_n^{i,\varepsilon} \circ \delta_{n-1}^{j-1,\eta} \quad i < j \quad (1)$$

$$\sigma_n^j \circ \sigma_{n+1}^i = \sigma_n^i \circ \sigma_{n+1}^{j+1} \quad i \leq j \quad (2)$$

$$\sigma_n^j \circ \delta_{n+1}^{i,\varepsilon} = \begin{cases} \delta_n^{i,\varepsilon} \circ \sigma_{n-1}^{j-1} & i < j \\ \text{id}_{\square_n} & i = j \\ \delta_n^{i-1,\varepsilon} \circ \sigma_{n-1}^j & i > j \end{cases} \quad (3)$$

Definition 2. The category **cSet** of cubical sets is the category of presheaves $\mathbf{Set}^{\square^{\text{op}}}$. The standard n -cube is the representable presheaf

$$\square[n] \stackrel{\text{def.}}{=} \square(-, \square_n)$$

Remark 3. As any category of presheaves, **cSet** is a (Grothendieck) topos.

Remark 4. A cubical set is a family $(K_n)_{n \geq 0}$ of sets equipped with face maps

$$d_n^{i,\varepsilon}: K_n \longrightarrow K_{n-1}$$

and degeneracy maps

$$s_n^i: K_{n-1} \longrightarrow K_n$$

subject to the cubical relations

$$d_{n-1}^{i,\varepsilon} \circ d_n^{j,\eta} = d_{n-1}^{j-1,\eta} \circ d_n^{i,\varepsilon} \quad (4)$$

$$s_{n+1}^i \circ s_n^j = s_{n+1}^{j+1} \circ s_n^i \quad (5)$$

$$d_{n+1}^{i,\varepsilon} \circ s_n^j = \begin{cases} s_{n-1}^{j-1} \circ d_n^{i,\varepsilon} & i < j \\ \text{id}_{K_n} & i = j \\ s_{n-1}^j \circ d_n^{i-1,\varepsilon} & i > j \end{cases} \quad (6)$$

2. HIGHER DIMENSIONAL AUTOMATA

Definition 5. Given a set Σ and $\omega = (\omega_1, \dots, \omega_n) \in \Sigma^*$, let $|\omega| \stackrel{\text{def.}}{=} n$. Suppose from now on Σ totally ordered and let

$$\omega \vDash (\Sigma, \leq) \stackrel{\text{def.}}{\iff} \omega_1 \leq \dots \leq \omega_n$$

Let further $\star \notin \Sigma$, $\omega = (\omega_1, \dots, \omega_n) \in (\Sigma \cup \{\star\})^*$, $\underline{\omega} \subseteq \omega$ be the word obtained from ω by removing all the occurrences of \star and

$$!\Sigma_n \stackrel{\text{def.}}{=} \{\omega \in (\Sigma \cup \{\star\})^* \mid |\omega| = n \wedge \underline{\omega} \vDash (\Sigma, \leq)\}$$

Finally, let

$$\delta_i(\omega_1, \dots, \omega_n) \stackrel{\text{def.}}{=} (\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_n)$$

and

$$\varepsilon_i(\omega_1, \dots, \omega_n) \stackrel{\text{def.}}{=} (\omega_1, \dots, \omega_{i-1}, \star, \omega_i, \dots, \omega_n)$$

Proposition 6. (Goubault) The data of definition 5 assemble to a cubical set.

Definition 7. The category **HDA** of higher dimensional automata (abbreviated hda's) is the comma-category

$$1/\mathbf{cSet}/!\Sigma$$

under 1 and over $!\Sigma$.

3. BISIMULATION OF HIGHER-DIMENSIONAL AUTOMATA

Definition 8. Let K be a cubical set and $A \in K_n$ be an n -cube in K .

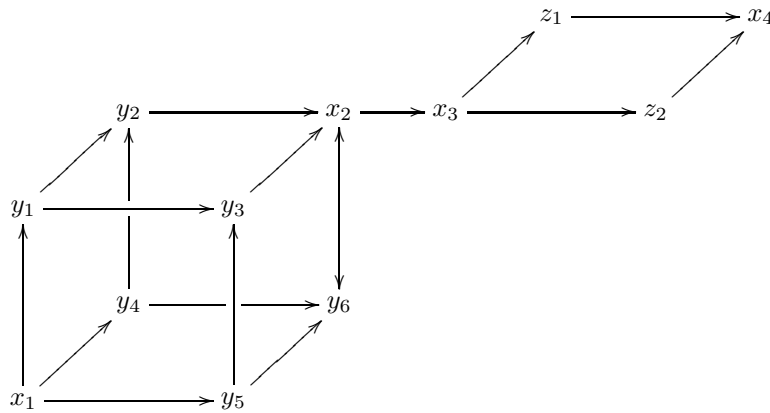
i. $\text{dom}(A) \stackrel{\text{def.}}{=} \underbrace{\partial_0^- \dots \partial_0^-}_{n \times}$;

ii. $\text{cod}(A) \stackrel{\text{def.}}{=} \underbrace{\partial_0^+ \dots \partial_0^+}_{n \times}$;

Remark 9. Given a family of $(n_i)_{1 \leq i \leq r}$ of positive integers, the corresponding family $(\square[n_i])_{1 \leq i \leq r}$ of standard n_i -cubes can be glued to the cubical set

$$\bigoplus_{i=1}^r n_i \stackrel{\text{def.}}{=} \text{coequ}_{1 \leq i < r} \left\{ \text{cod}(\square[n_i]), \text{dom}(\square[n_{i+1}]) : 1 \rightrightarrows \sum_{i=1}^r \square[n_i] \right\}$$

Example 10. $\phi(3, 1, 2)$ can be depicted as



Definition 11. A cubical set as in remark 9 is called a path from

$$\text{dom}\left(\bigoplus_{i=1}^r n_i\right) \stackrel{\text{def.}}{=} \text{dom}(\square[n_1])$$

to

$$\text{cod}\left(\bigoplus_{i=1}^r n_i\right) \stackrel{\text{def.}}{=} \text{cod}(\square[n_r])$$

A path extension is a mono

$$m: \bigoplus_{j=1}^p n_j \rightarrow \bigoplus_{i=1}^r n_i$$

in \mathbf{cSet} commuting with the operation dom .

Notation 12. \mathcal{P} stands for the collection of all path extensions.

Definition 13. A cubical map $h: M \rightarrow K$ is open provided $h \in \text{rlp}(\mathcal{P})$.

Remark 14. An open map is surjective in all degrees. The notion of open map has its obvious counterparts in comma categories on \mathbf{cSet} .

Definition 15. Let $K, L \in \mathbf{HDA}$. A bisimulation $R: K \rightsquigarrow L$ is a span

$$K \longleftarrow R \longrightarrow L$$

of open maps. Hda's K and L are bismilar if there is a witnessing bisimulation $K \rightsquigarrow L$.

Remark 16. A bisimulation is a commuting diagram

$$\begin{array}{ccc} & R & \\ r_1 \swarrow & & \searrow r_2 \\ K & & L \\ k \searrow & & \swarrow l \\ & !\Sigma & \end{array}$$

in $1/\mathbf{cSet}$ with r_1 and r_2 open.

4. THE MODEL STRUCTURE

Definition 17. Let \mathbb{C} be a category. A collection of morphisms $S \subseteq \mathbb{C}_1$ is saturated if

- i. all isos belong to S ;
- ii. S is stable by
 - retracts;
 - coproducts;
 - pushouts;
 - countable compositions.

Let $A \subseteq \mathbb{C}_1$ be a collection of morphisms. Its saturation \bar{A} is the smallest saturated collection of morphisms containing A .

Definition 18. Cubical maps $f, g: X \rightarrow Y$ are naively homotopic if there is a cubical map

$$h: X \times \square[1] \rightarrow Y$$

such that the diagram

$$\begin{array}{ccc}
X & & \\
i_0 \downarrow & \searrow f & \\
X \times \square[1] & \xrightarrow{h} & Y \\
i_1 \uparrow & \nearrow g & \\
X & &
\end{array}$$

commutes.

Notation 19. \mathbf{cSet}_I stands for \mathbf{cSet} quotiented by the congruence relation induced by naive homotopy.

Definition 20. Let

$$J \stackrel{\text{def.}}{=} \{P \hookrightarrow \square[n] \mid n \in \mathbb{N}, P \in \mathcal{P}, \text{dom}(P) = \text{dom}(\square[n]), \text{cod}(P) = \text{cod}(\square[n])\}$$

and

$$I \stackrel{\text{def.}}{=} \{\text{dom}: 1 \hookrightarrow \square[n] \mid n \in \mathbb{N}\}$$

be sets of cubical maps.

1. A cubical set X is formally fibrant if $X \xrightarrow{!} 1 \in \text{rlp}(\overline{I \cup J})$;
2. A formal weak equivalence is a cubical map $f: K \rightarrow L$ inducing a bijection

$$f^*: \mathbf{cSet}_I(L, X) \xrightarrow{\approx} \mathbf{cSet}_I(K, X)$$

for all formally fibrant cubical sets X .

Theorem 21. I and J are the generating cofibrations respectively the generating trivial cofibrations of a cofibrantly generated model structure on \mathbf{cSet} . We have

- $\mathcal{C} = \{\text{monos}\}$;
- $\mathcal{W} = \{\text{formal weak equivalences}\}$.

Remark 22. It is easy to see that $\text{rlp}(\mathcal{P}) = \text{rlp}(I)$, hence the trivial fibrations in this model structure are precisely the open maps.

Remark 23. This model structure carries over to **HDA**. In particular, bisimilar hda's are identified in the homotopy category. **HDA** being pointed (i.e. the initial and the terminal objects are isomorphic), the homotopy category is (pre-)triangulated and so a good platform for homology theories.