

Lattices of paths in higher dimension (i.e. multinomial lattices)

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Keywords, motivations

- ▶ concurrency theory, trace theory,
- ▶ directed homotopy,
- ▶ rewriting on Coxeter systems,
higher dimensional rewriting,
- ▶ algebraic notions of dimension ...

Outline

In dimension 2

The binomial lattice

Dimension ≥ 3

$\mathcal{Paths}(v)$ and its rewrite system

The lattice $\mathcal{Perm}(v)$

The congruence lattice of $\mathcal{Paths}(v)$ (and of $\mathcal{Perm}(v)$)

Dimension equations in lattices of paths

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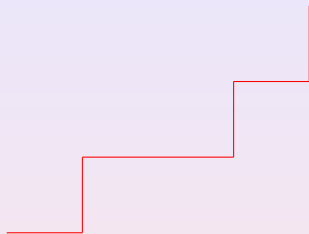
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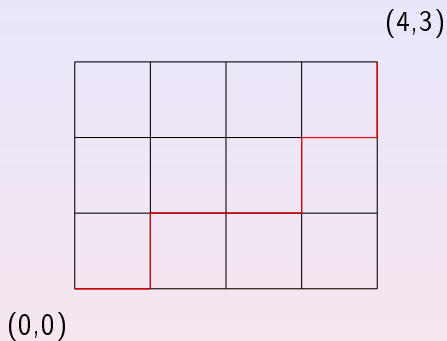
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The space $\mathcal{Paths}(4, 3)$



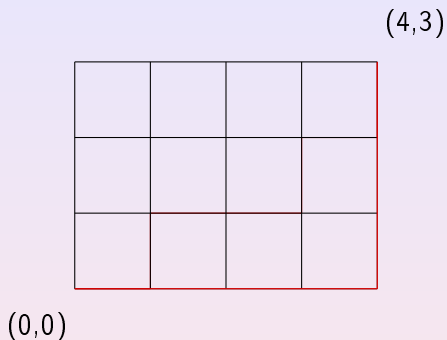
A path $\pi \in \mathcal{Paths}(4, 3)$

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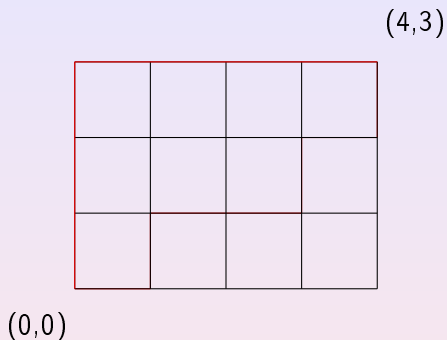
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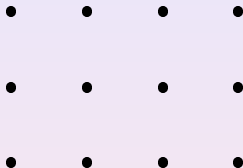
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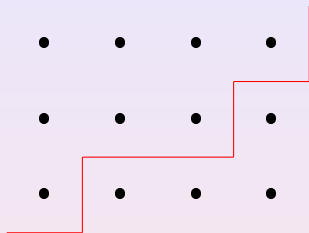


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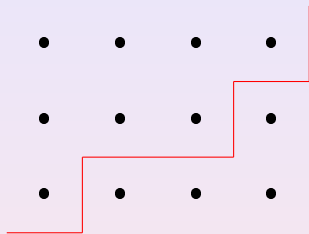
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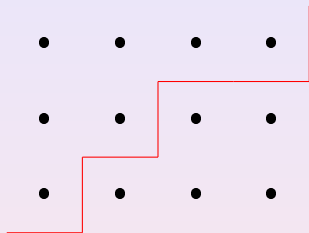


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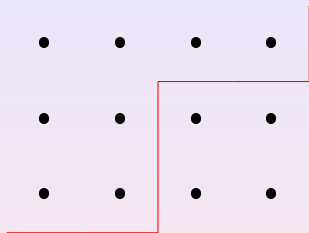
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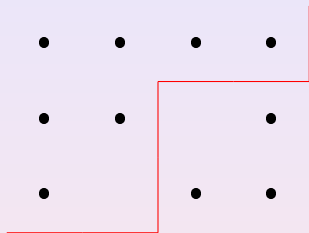
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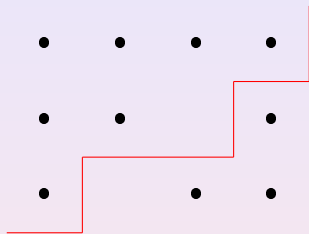
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A grid with holes allows the desired commutations

From the geometric representation to lattices

- ▶ $\mathcal{Paths}(x, y)$ is a distributive lattice,
- ▶ A bullet \bullet is a join irreducible element,
- ▶ Directed homotopies in “bijection” with lattice theoretic congruences.

In a distributive lattice a lattice theoretic congruence is a subset of missing join irreducible elements.

- ▶ What happens with more than 2 dimensions ?

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$\mathcal{P}aths(v)$, its rewrite system

Let $v \in \mathbb{N}^n$, define :

$$\begin{aligned}\mathcal{P}aths(v) &= \{ \text{stepwise 1-increasing paths from } 0 \text{ to } v \text{ in } \mathbb{N}^n \} \\ &= \{ w \in \{ \sigma_1, \dots, \sigma_n \} \mid |w|_{\sigma_i} = v_i \text{ for } i = 1, \dots, n \},\end{aligned}$$

and the rewrite system \rightarrow :

$$w\sigma_i\sigma_ju \rightarrow w\sigma_j\sigma_iu \qquad i < j$$

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the order generated by this confluent-terminating rewrite system.

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The lattice $\mathcal{P}erm(v)$

If

$$v = \underbrace{(1, 1, \dots, 1)}_{n\text{-times}}$$

then

$$\begin{aligned}(\mathcal{P}aths(v), \leq) &= \mathcal{P}erm(n) \\ &= \text{set of permutations with the weak Bruhat order}\end{aligned}$$

$\mathcal{P}erm(n)$ is a (generally non distributive) lattice.

For $\sigma, \sigma' \in \mathcal{P}erm(n)$

$$D(\sigma) = \{ \{x, y\} \mid 1 \leq x < y \leq n, \sigma^{-1}(x) > \sigma^{-1}(y) \}$$

$$A(\sigma) = \{ \{x, y\} \mid 1 \leq x < y \leq n, \sigma^{-1}(x) < \sigma^{-1}(y) \}$$

$$\sigma \leq \sigma' \text{ iff } D(\sigma) \subseteq D(\sigma').$$

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Bennet-Birkhoff-McNamara coding :

$$\chi : \mathcal{Paths}(v) \longrightarrow \mathcal{Perm}\left(\sum_{i=1\dots n} v_i\right).$$

Example with $\chi : \mathcal{Paths}(3, 2, 1) \longrightarrow \mathcal{Perm}(6)$:

$$\chi(aaabbc) = 123456$$

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$\mathcal{Paths}(v)$ is order isomorphic to a principal ideal of $\mathcal{Perm}\left(\sum_{i=1\dots n} v_i\right)$

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Projections

Projections generalize inversions/disagreements.

Examples:

$$\text{pr}_{ab}(acbbaa) = abbaa,$$

Proposition

Let $w, u \in \text{Paths}(v)$, then $\text{pr}_{\sigma_i \sigma_j}(w) \in \text{Paths}(v_i, v_j)$, and
 $w \leq u$ if and only if

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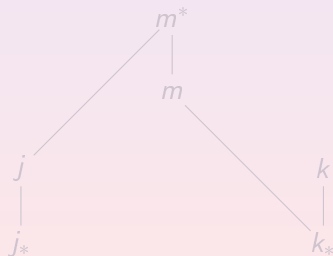
The join dependency relation

Let j, k be join irreducible elements, and define

jDk iff $j \neq k$, and

there exists a meet irreducible element m s.t.

$j \nearrow m$ and $m \searrow k$.



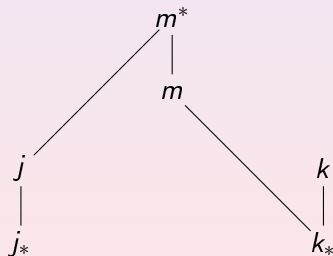
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The congruence lattice of a lattice

A finite lattice is bounded if and only if it is semidistributive and the relation D is acyclic.

For a finite bounded lattice L

$$\text{Con}(L) \sim \mathcal{I}(J(L))$$

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The congruence lattice of $\mathcal{P}erm(v)$

Proposition

*Explicit characterisation of \nearrow , \searrow ,
and of the join dependency relation D in $\mathcal{P}aths(v)$.*

Example : the join independence relation on $\mathcal{P}erm(n)$.

A join irreducible: (i, j, P) with

- ▶ $0 \leq i < j \leq n$, and
- ▶ P is a binary partition of the open interval (i, j) .

Lemma

$(i, j, P)D(k, l, Q)$ iff

- ▶ $[k, l] \subset [i, j]$,
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The join independence relation on $\mathcal{P}aths(v)$

If $v = (v_1, \dots, v_n)$, call n the dimension of $\mathcal{P}aths(v)$.

Proposition

The lattice $\mathcal{P}aths(v)$ is bounded, and the length k of a sequence

$$j_0 Dj_1 Dj_2 \dots j_{k-1} Dj_k$$

has length at most $n - 2$ (it has Krull dimension at most $n - 1$).

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Meet semidistributivity on finite lattices

$$x \wedge y = x \wedge z \quad \text{implies} \quad x \wedge (y \vee z) = x \wedge y \quad \text{SD}(\wedge)$$

$$\begin{array}{ll} y_0 = y & z_0 = z \\ y_{n+1} = y \vee (x \wedge z_n) & z_{n+1} = z \vee (x \wedge y_n). \end{array}$$

$$x \wedge y_n = x \wedge (y \vee z) \quad \text{SD}_n(\wedge)$$

SD(\wedge) holds in a finite lattice L iff SD $_n(\wedge)$ holds in L
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D -sequences and n -meet-distributivity

Proposition

Let L be a finite bounded lattice.

Let ℓ be the length of the longest sequence

$$j_0 D j_1 D j_2 \dots j_{k-1} D j_k.$$

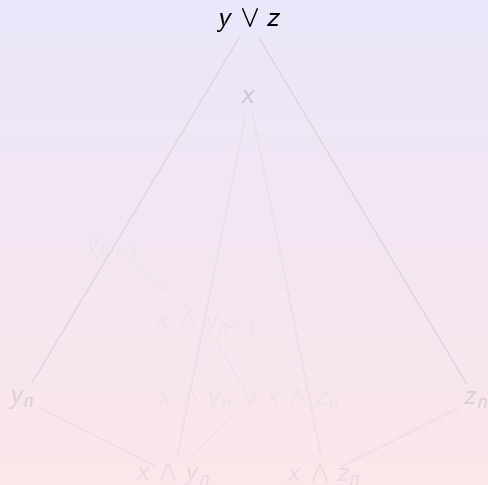
of join irreducible elements in L .

Then L satisfies $SD_n(\wedge)$, for $n \geq \ell + 1$.

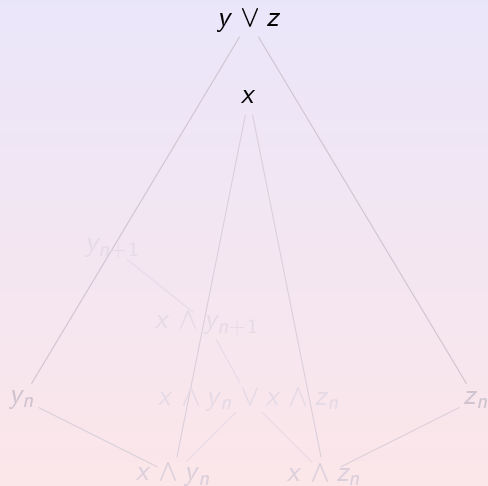
Corollary

$SD_{n-1}(\wedge)$ holds in $\mathcal{Paths}(v_1, \dots, v_n)$.

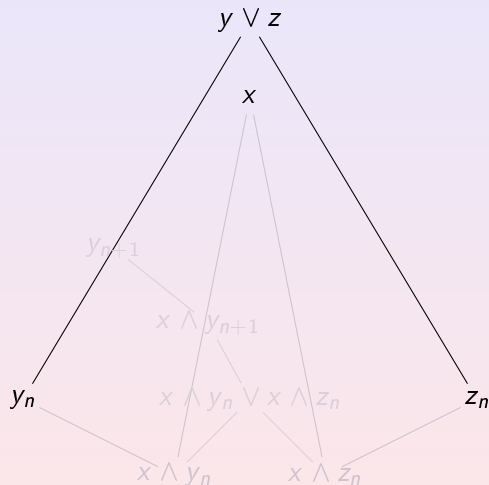
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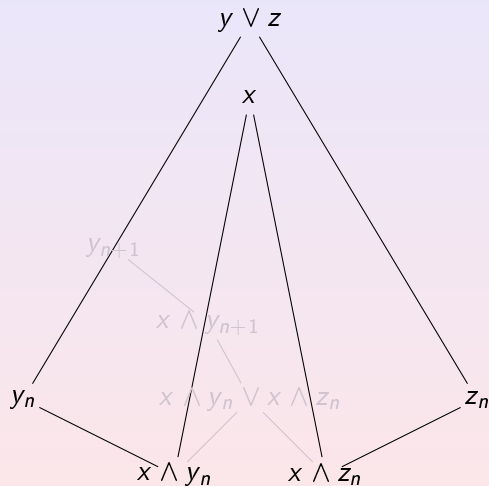
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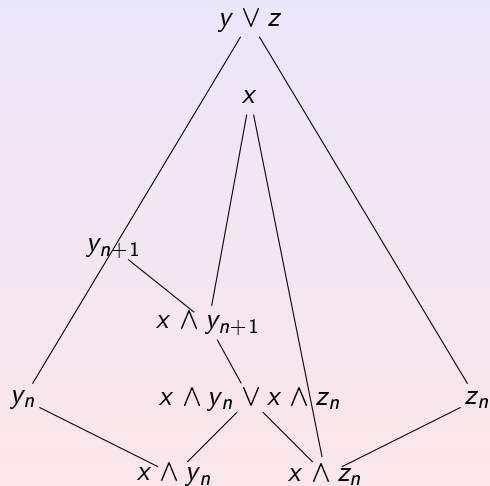
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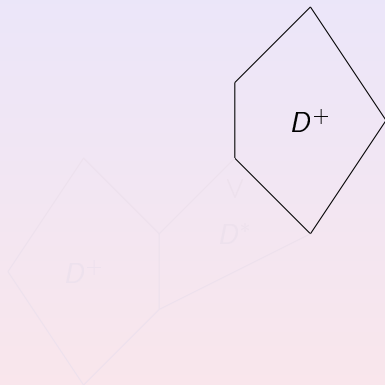
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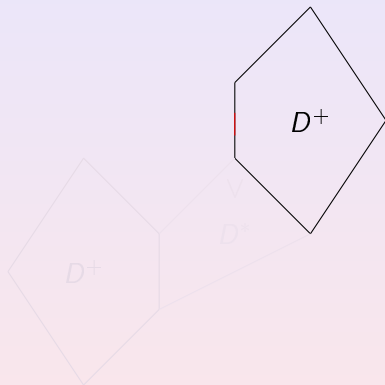
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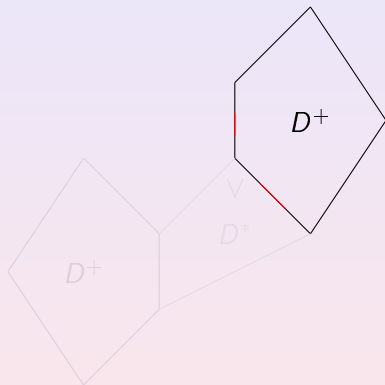
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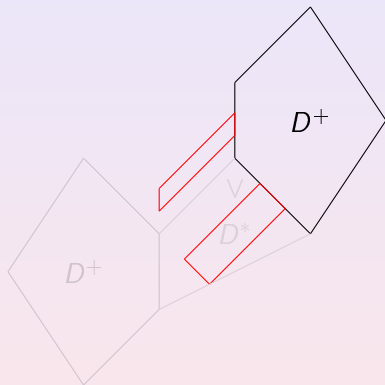
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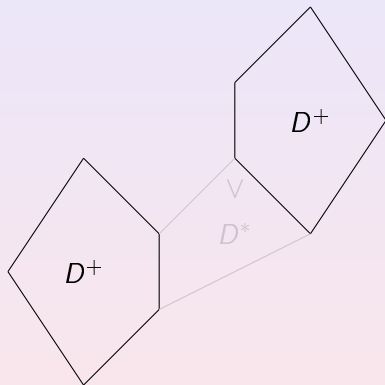
Proof of the Proposition (sketch)



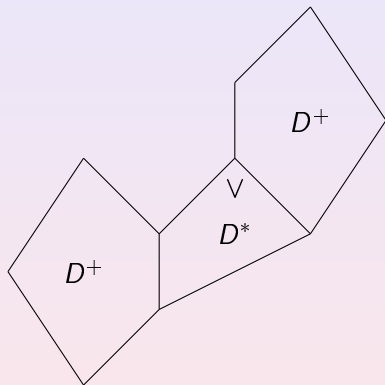
Proof of the Proposition (sketch)



Proof of the Proposition (sketch)



Proof of the Proposition (sketch)



Proposition

$SD_{n-2}(\wedge)$ fails in $\mathcal{Paths}(v)$ if $v = (v_1, \dots, v_n)$ and $v_i > 0$ for $i = 1, \dots, n$.

Proof.

It suffice to prove it for $\mathcal{Perm}(n)$.

Choose

$$x = 23 \dots n1$$

$$y = 214365 \dots$$

$$z = 1325476 \dots$$

