Lattices of paths in higher dimension (i.e. multinomial lattices)

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Keywords, motivations

- concurrency theory, trace theory,
- directed homotopy,
- rewriting on Coxeter systems,
  higher dimensional rewriting,
- algebraic notions of dimension . . .

Outline

In dimension 2

- The binomial lattice

Dimension ≥ 3

- $\mathcal{P}aths(v)$ and its rewrite system
- The lattice $\mathcal{P}erm(v)$
- The congruence lattice of $\mathcal{P}aths(v)$ (and of $\mathcal{P}erm(v)$)

Dimension equations in lattices of paths

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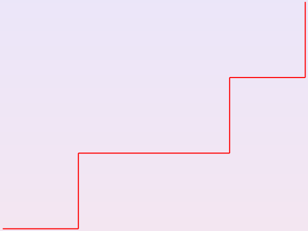
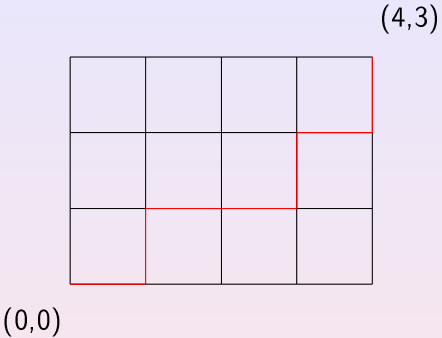
The space $\mathcal{P}aths(4,3)$



A path $\pi \in \mathcal{P}aths(4,3)$

The space $\mathcal{P}aths(4,3)$



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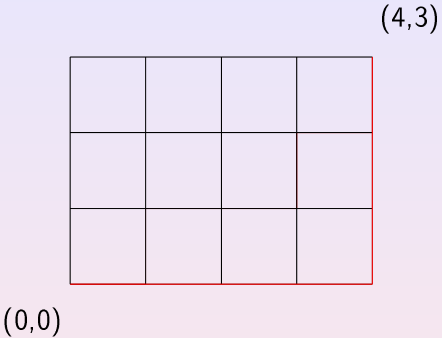

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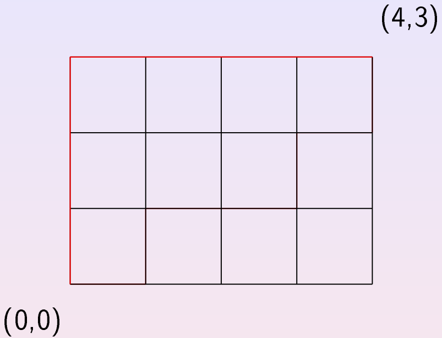

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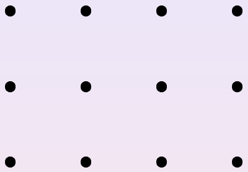

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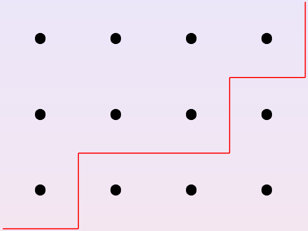

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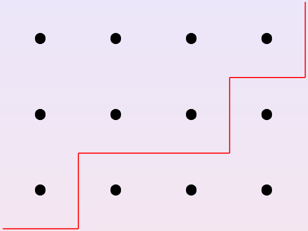

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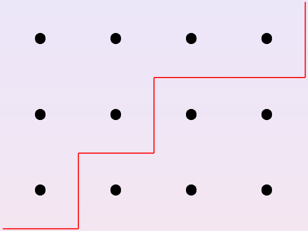

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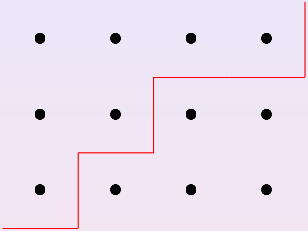

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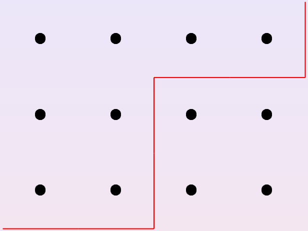

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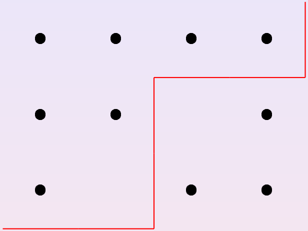

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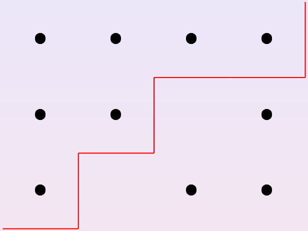

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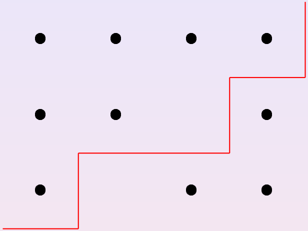

A grid with holes allows the desired commutations

From the geometric representation to lattices

- $\mathcal{P}aths(x, y)$ is a distributive lattice,
- A bullet • is a join irreducible element,
- Directed homotopies in "bijection" with
  lattice theoretic congruences.

In a distributive lattice a lattice theoretic congruence is a subset of missing join irreducible elements.

- What happens with more than 2 dimensions ?

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$\mathcal{P}aths(v)$, its rewrite system

Let $v \in \mathbb{N}^n$, define :

$$\begin{aligned}\mathcal{P}aths(v) &= \{\,\text{stepwise 1-increasing paths from 0 to } v \text{ in } \mathbb{N}^n\,\}\\ &= \{\, w \in \{\,\sigma_1, \ldots, \sigma_n\,\} \mid |w|_{\sigma_i} = v_i \text{ for } i = 1, \ldots, n\,\},\end{aligned}$$

and the rewrite system $\rightarrow$:

$$w\sigma_i\sigma_j u \rightarrow w\sigma_j\sigma_i u \qquad i < j$$

$\leq$:
the order generated by this confluent-terminating rewrite system.

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The lattice $\mathcal{P}erm(v)$

If

$$v = \underbrace{(1, 1, \ldots, 1)}_{n-\text{times}}$$

then

$$\begin{aligned}(\mathcal{P}aths(v), \leq) &= \mathcal{P}erm(n) \\ &= \text{set of permutations with the weak Bruhat order}\end{aligned}$$

$\mathcal{P}erm(n)$ is a (generally non distributive) lattice.
For $\sigma, \sigma' \in \mathcal{P}erm(\sigma)$

$$D(\sigma) = \{\, \{\, x, y \,\} \mid 1 \leq x < y \leq n,\ \sigma^{-1}(x) > \sigma^{-1}(y) \,\}$$
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$$\sigma \leq \sigma' \ \text{ iff } \ D(\sigma) \subseteq D(\sigma')\,.$$

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$\mathcal{P}aths(v)$ is a lattice

Bennet-Birkhoff-McNamara coding :

$$\chi : \mathcal{P}aths(v) \longrightarrow \mathcal{P}erm(\sum_{i=1\ldots n} v_i)\,.$$

Example with $\chi : \mathcal{P}aths(3,2,1) \longrightarrow \mathcal{P}erm(6)$:

$$\chi(aaabbc) = 123456$$
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$\mathcal{P}aths(v)$ is order isomorphic to a principal ideal of $\mathcal{P}erm(\sum_{i=1\ldots n} v_i)$

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Projections generalize inversions/disagreements.

Examples:

$$\mathtt{pr}_{ab}(acbbaa) = abbaa,$$

Proposition

Let $w, u \in \mathcal{P}aths(v)$, then $\mathtt{pr}_{\sigma_i \sigma_j}(w) \in \mathcal{P}aths(v_i, v_j)$, and $w \leq u$ if and only if

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The join dependency relation

Let j, k be join irreducible elements, and define

$$jDk \text{ iff } j \neq k, \text{ and}$$
$$\text{there exists a meet irreducible element } m \text{ s.t.}$$
$$j \nearrow m \quad \text{and} \quad m \searrow k\,.$$

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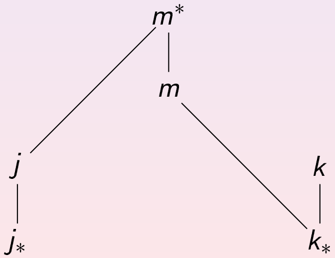

The congruence latticeof a lattice

A finite lattice is bounded if and only if
it is semidistributive and the relation D is acyclic.

For a finite bounded lattice L

$$Con(L) \sim \mathcal{I}(J(L))$$

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The congruence lattice of $\mathcal{P}erm(v)$

Proposition

Explicit characterisation of $\nearrow$, $\searrow$,
and of the join dependency relation D in $\mathcal{P}aths(v)$.

Example : the join indipendence relation on $\mathcal{P}erm(n)$.
A join irreducible: (i, j, P) with

- $0 \leq i < j \leq n$, and
- P is a binary partition of the open interval (i, j).

Lemma

$(i, j, P) D (k, l, Q)$ iff

- *$[k, l] \subset [i, j]$,*
- *P agrees with Q on (k, l).*

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The join indipendence relation on $\mathcal{P}aths(v)$

If $v = (v_1, \ldots, v_n)$, call n the dimension of $\mathcal{P}aths(v)$.

Proposition

The lattice $\mathcal{P}aths(v)$ is bounded, and the length k of a sequence

$$j_0 D j_1 D j_2 \ldots j_{k-1} D j_k$$

has length at most $n - 2$ (it has Krull dimension at most $n - 1$).

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Dimension equations in lattices of paths

Meet semidistributivity on finite lattices

$$x \wedge y = x \wedge z \quad \text{implies } x \wedge (y \vee z) = x \wedge y \qquad \mathrm{SD}(\wedge)$$

$$y_0 = y \qquad z_0 = z$$

$$y_{n+1} = y \vee (x \wedge z_n) \qquad z_{n+1} = z \vee (x \wedge y_n) .$$

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D-sequences and n-meet-distributivity

Proposition

Let L be a finite bounded lattice.
Let ℓ be the length of the longest sequence

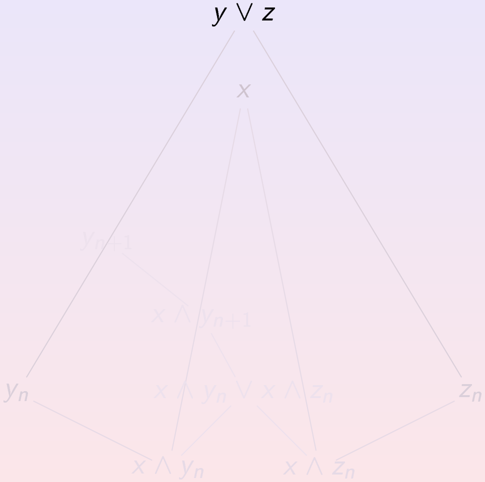
$$j_0 \, D \, j_1 \, D \, j_2 \ldots j_{k-1} \, D \, j_k \, .$$

of join irreducible elements in L.
Then L satifies $SD_n(\wedge)$, for $n \geq \ell + 1$.

Corollary

$SD_{n-1}(\wedge)$ holds in $\mathcal{P}aths(v_1, \ldots, v_n)$.

Proof of the Proposition (sketch)



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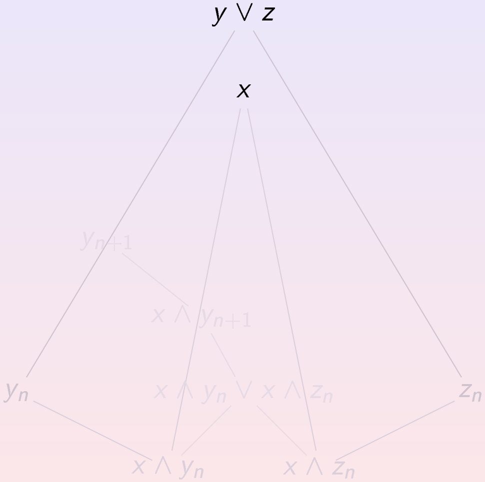

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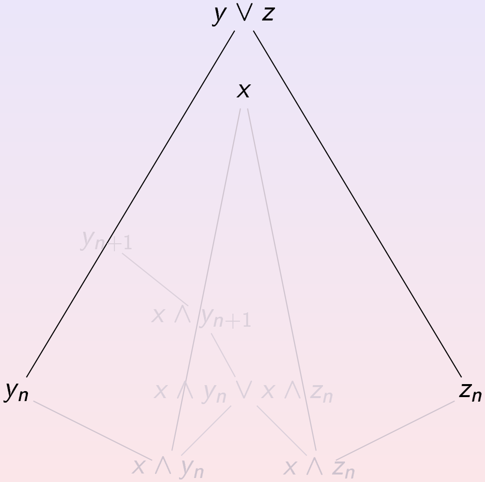

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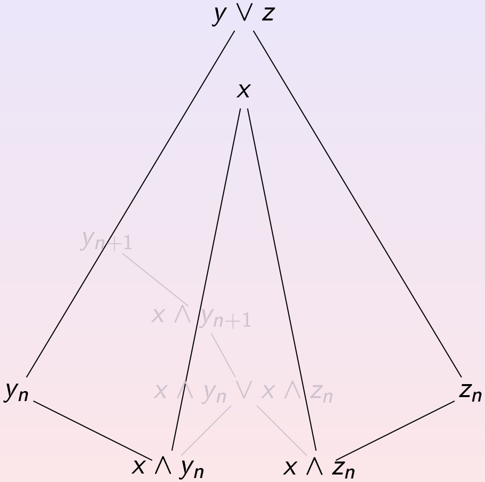

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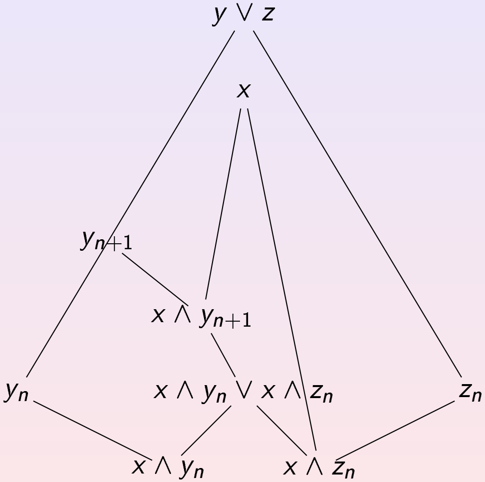

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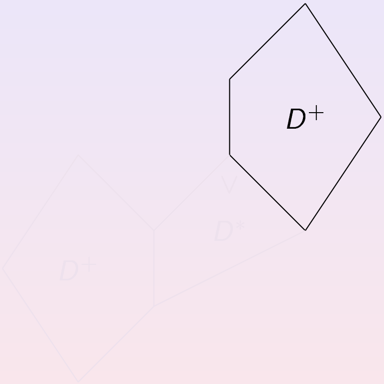

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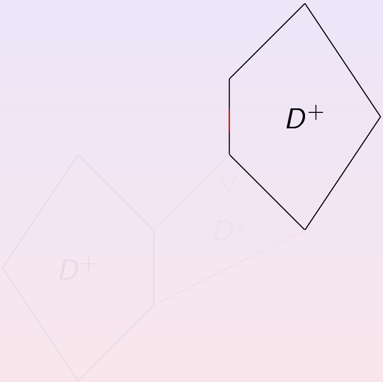

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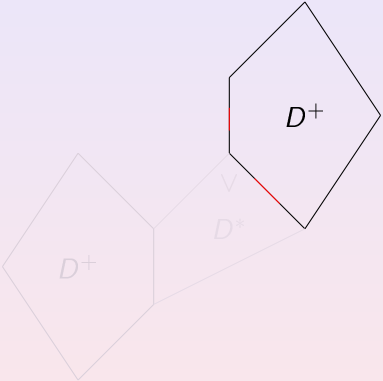

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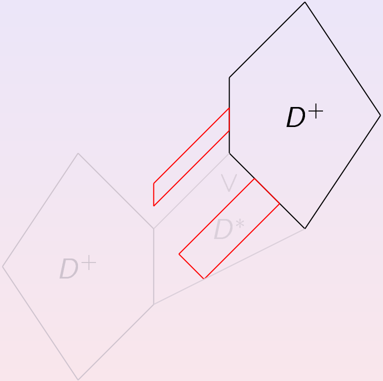
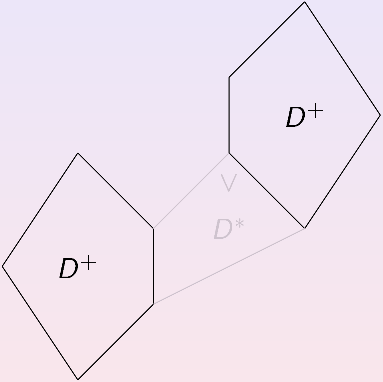
Proof of the Proposition (sketch)



$\square$

Proof of the Proposition (sketch)



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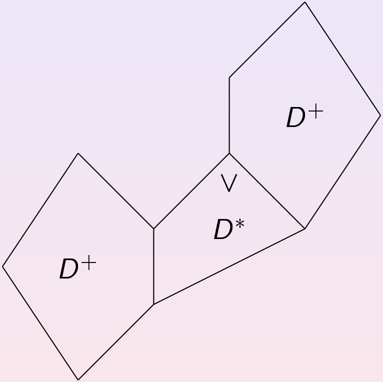
Proof of the Proposition (sketch)



□

Proposition

$SD_{n-2}(\wedge)$ fails in $\mathcal{P}aths(v)$ if $v = (v_1, \ldots, v_n)$ and $v_i > 0$ for $i = 1, \ldots, n$.

Proof.

It suffice to prove it for $\mathcal{P}erm(n)$.
Choose

$$x = 23 \ldots n1$$
$$y = 214365 \ldots$$
$$z = 1325476 \ldots$$

□