

① (ABELIAN) CHAIN COMPLEXES

$$C: C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2 \xleftarrow{\partial_2} C_3 \dots$$

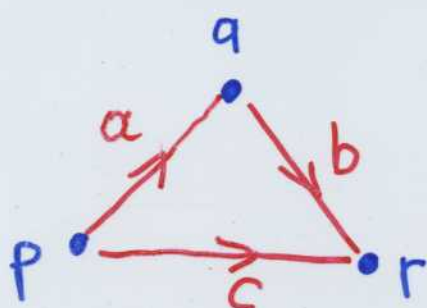
(abelian groups)

$$\partial_0 \circ \partial_1 = 0$$

$$\partial_1 \circ \partial_2 = 0$$

$$\boxed{\partial_{n-1} \circ \partial_n = 0}$$

Example 1 (unfilled triangle)



$$C_0 = \mathbb{Z}^3 \simeq \mathbb{Z}_p + \mathbb{Z}_q + \mathbb{Z}_r$$

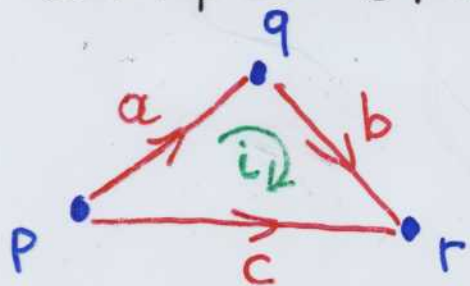
$$C_1 = \mathbb{Z}^3 \simeq \mathbb{Z}_a + \mathbb{Z}_b + \mathbb{Z}_c$$

$$C_2 = 0$$

$$\partial_0 a = q - p \quad \partial_0 b = r - q \quad \partial_0 c = r - p$$

$$\partial_0 (a + b - c) = q - p + r - q + p - r = 0 \quad a + b - c \text{ is a cycle}$$

Example 2 (filled triangle)



$$C_2 = \mathbb{Z} \simeq \mathbb{Z}_i$$

$$C_3 = 0$$



$$\partial_1 i = a + b - c$$

$a + b - c$ is a boundary



Remark: we use free \mathbb{Z} -modules.


EXACT COMPLEXES & HOMOLOGY

$$\partial_{n-1} \circ \partial_n = 0 \iff \text{im } \partial_n \subset \text{ker } \partial_{n-1}$$

Every boundary is a cycle:  \Rightarrow 

• Exactness: $\text{im } \partial_n = \text{ker } \partial_{n-1}$

Every cycle is a boundary:  \Rightarrow 
(no holes)

• Homology: $H_n(C) = \text{ker } \partial_{n-1} / \text{im } \partial_n$ 
(counting holes)

Examples: • unfilled triangle $H_1(C) \cong \mathbb{Z}$

• filled triangle $H_1(C) \cong 0$

The complex for the filled triangle is exact

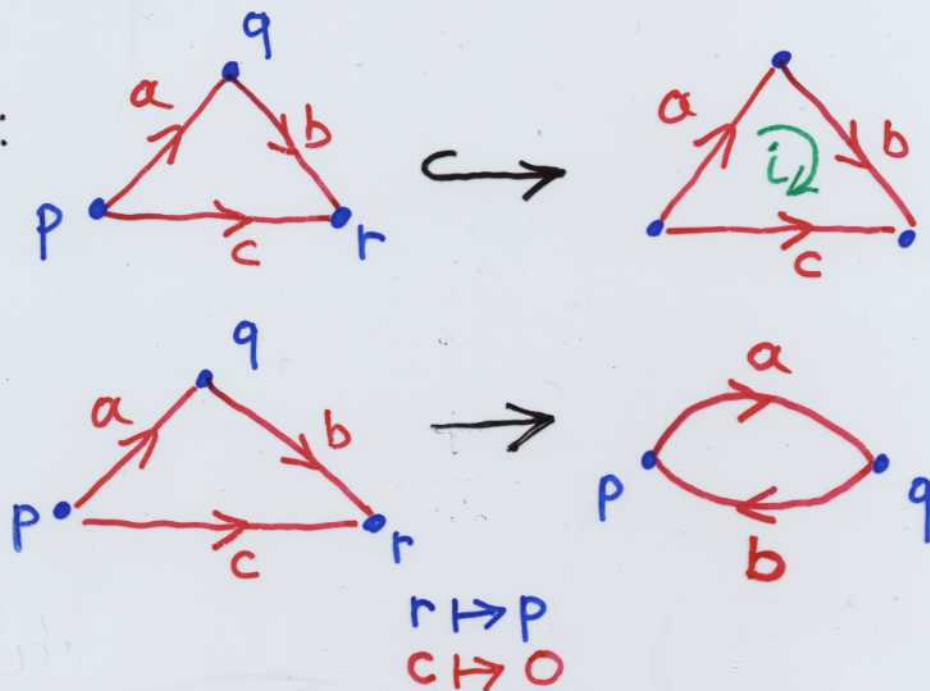
MORPHISMS & HOMOLOGISMS

• Morphism:

$$\begin{array}{c}
 C : C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2 \xleftarrow{\partial_2} C_3 \dots \\
 f \downarrow \quad \downarrow f_0 = \downarrow f_1 = \downarrow f_2 = \downarrow f_3 \dots \\
 D : D_0 \xleftarrow{\partial_0} D_1 \xleftarrow{\partial_1} D_2 \xleftarrow{\partial_2} D_3 \dots
 \end{array}$$

$$f_n \circ \partial_n = \partial_n \circ f_{n+1}$$

Examples:

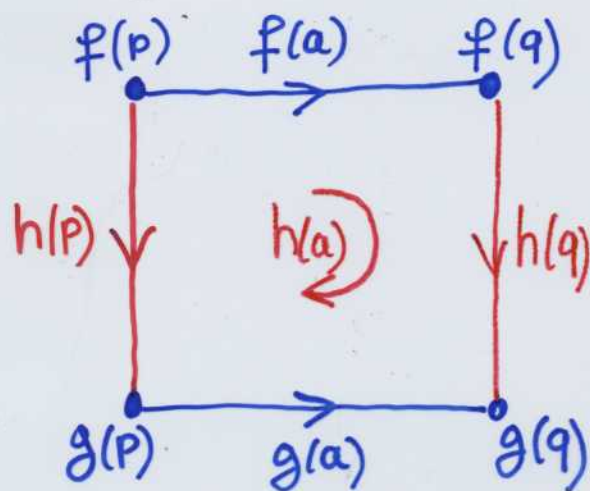


• Homologism: $\begin{cases} \partial_n x = 0 \\ f_n x = \partial_n y \end{cases} \Rightarrow \exists z \begin{cases} x = \partial_n z \\ f_{n+1} z = y \end{cases} (+ f_0 \text{ onto})$



- Exercise:
1. Prove that each f_n is onto.
 2. Prove that each cycle is the image of a cycle.
 3. Prove that $H_n(C) \cong H_n(D)$.

④ HOMOTOPY & HOMOLOGY

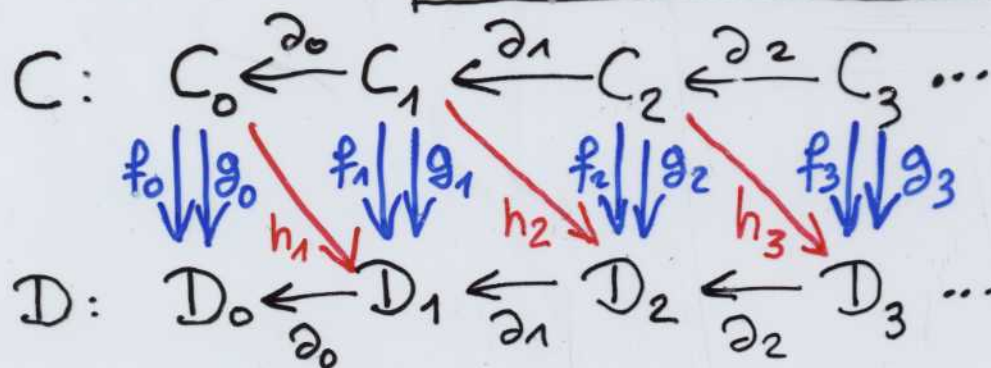


$$f \stackrel{h}{\sim} g$$

$$\begin{aligned} \partial_1 h_2(a) &= f_1(a) + h_1(q) - g_1(a) - h_1(p) \\ &= (f_1 - g_1)(a) + h_1(q - p) \\ &= (f_1 - g_1)(a) + h_1 \partial_0(a) \end{aligned}$$

$$\partial_1 h_2 - h_1 \partial_0 = f_1 - g_1$$

More generally: $\partial_n h_{n+1} - h_n \partial_{n-1} = f_n - g_n$ ($+ \partial_0 h_1 = f_0 - g_0$)



If $f \sim g$, then $H_n(f) = H_n(g)$ for all n .

⑤ RESOLUTIONS & HOMOLOGY OF MONOIDS

- R ring \rightarrow complexes of R -modules
- V R -module \rightarrow resolution of V by free R -modules

exact complex: $0 \leftarrow V \xleftarrow{\varepsilon} RS_0 \xleftarrow{\partial_0} RS_1 \xleftarrow{\partial_1} RS_2 \dots$

\rightarrow homologism:

$$\begin{array}{ccccccc} RS_0 & \xleftarrow{\partial_0} & RS_1 & \xleftarrow{\partial_1} & RS_2 & \xleftarrow{\partial_2} & RS_3 \dots \\ \downarrow \varepsilon & & \downarrow & & \downarrow & & \downarrow \\ V & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 \dots \end{array}$$

- existence of resolutions
- uniqueness up to homotopical equivalence.

Crucial property of free R -modules
(projectivity)

$$\begin{array}{c} \nearrow U \\ \downarrow \\ RS \rightarrow V \end{array}$$

- M monoid \rightarrow ring of M : $\mathbb{Z}M$
 $\rightarrow \mathbb{Z}$ seen as a $\mathbb{Z}M$ -module (trivial action)

\rightarrow resolution of \mathbb{Z} by free $\mathbb{Z}M$ -modules

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}MS_0 \xleftarrow{\partial_0} \mathbb{Z}MS_1 \xleftarrow{\partial_1} \mathbb{Z}MS_2 \dots$$

\rightarrow complex $\mathbb{Z}S$: $\mathbb{Z}S_0 \xleftarrow{\tilde{\partial}_0} \mathbb{Z}S_1 \xleftarrow{\tilde{\partial}_1} \mathbb{Z}S_2 \dots$

Definition: $H_n(M) = H_n(\mathbb{Z}S)$.

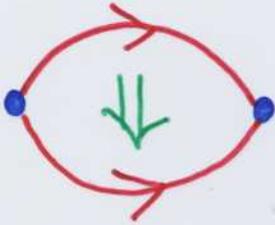
⑥ NON ABELIAN COMPLEXES

$$C: C_0 \begin{array}{c} \xleftarrow{\sigma_0} \\ \xleftarrow{\tau_0} \end{array} C_1 \begin{array}{c} \xleftarrow{\sigma_1} \\ \xleftarrow{\tau_1} \end{array} C_2 \begin{array}{c} \xleftarrow{\sigma_2} \\ \xleftarrow{\tau_2} \end{array} C_3 \dots$$

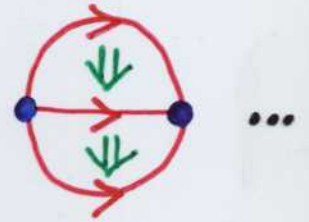
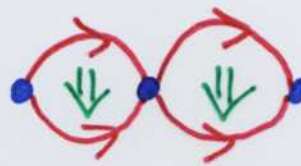
(sets)

$$\begin{cases} \sigma_0 \circ \sigma_1 = \sigma_0 \circ \tau_1 \\ \tau_0 \circ \sigma_1 = \tau_0 \circ \tau_1 \\ \vdots \end{cases}$$

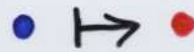
$$\boxed{\begin{cases} \sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n \\ \tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n \end{cases}}$$



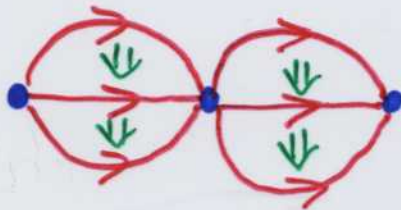
+ compositions



+ identities



+ properties of associativity, left & right unit, exchange



C_0 is a set

$C_0 \rightleftarrows C_1$ is a category

$C_0 \rightleftarrows C_1 \rightleftarrows C_2$ is a 2-category

⋮

C is a (strict) ∞ -category.

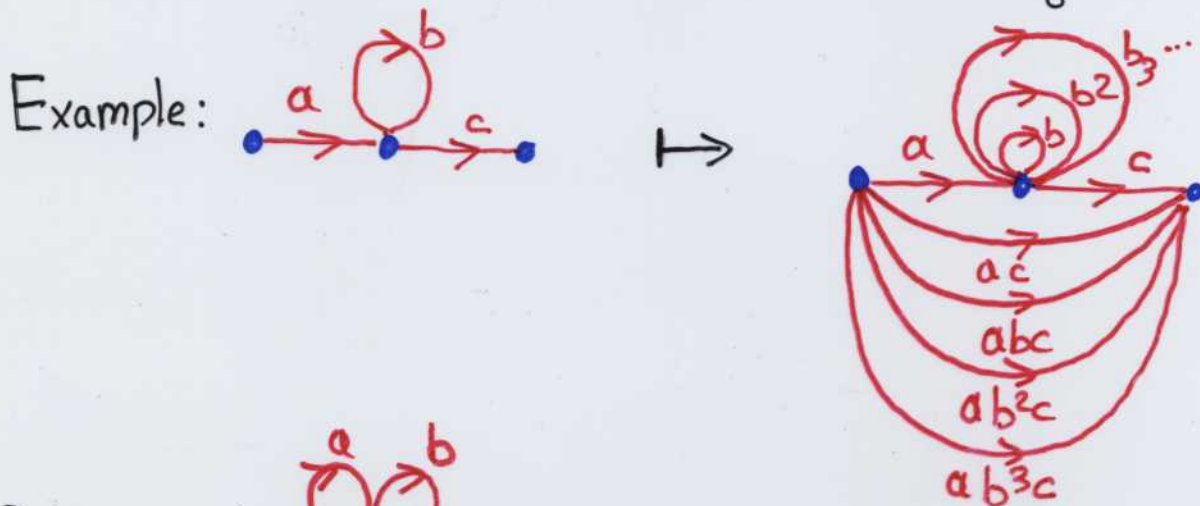
Special case: M is a monoid (or a group)

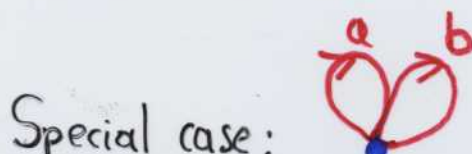
$$M: 1 \begin{array}{c} \xleftarrow{\sigma_0} \\ \xleftarrow{\tau_0} \end{array} M \begin{array}{c} \xleftarrow{\sigma_1} \\ \xleftarrow{\tau_1} \end{array} M \begin{array}{c} \xleftarrow{\sigma_2} \\ \xleftarrow{\tau_2} \end{array} M \dots$$

⑦ FREE COMPLEXES OR POLYGRAPHS

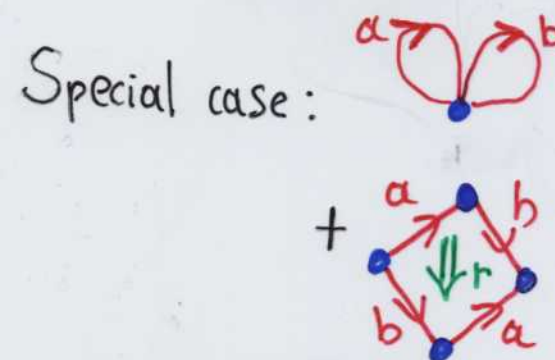
• Set $S_0 \mapsto$ set $S_0^* = S_0$

• Graph $S_0 \xrightleftharpoons[\tau_0]{\sigma_0} S_1 \mapsto$ Category $S_0^* \xrightleftharpoons[\tau_0]{\sigma_0} S_1^*$



Special case:  \mapsto free monoid $\{a, b\}^*$

2-Graph $S_0^* \xrightleftharpoons[\tau_0]{\sigma_0} S_1^* \xrightleftharpoons[\tau_1]{\sigma_1} S_2 \mapsto$ 2-Category $S_0^* \xrightleftharpoons[\tau_0]{\sigma_0} S_1^* \xrightleftharpoons[\tau_1]{\sigma_1} S_2^*$

Special case:  \mapsto 2-monoid of reductions $u \rightarrow^* v$

3-Graph $S_0^* \xrightleftharpoons[\tau_0]{\sigma_0} S_1^* \xrightleftharpoons[\tau_1]{\sigma_1} S_2^* \xrightleftharpoons[\tau_2]{\sigma_2} S_3 \mapsto$ 3-Category $S_0^* \xrightleftharpoons[\tau_0]{\sigma_0} S_1^* \xrightleftharpoons[\tau_1]{\sigma_1} S_2^* \xrightleftharpoons[\tau_2]{\sigma_2} S_3^*$

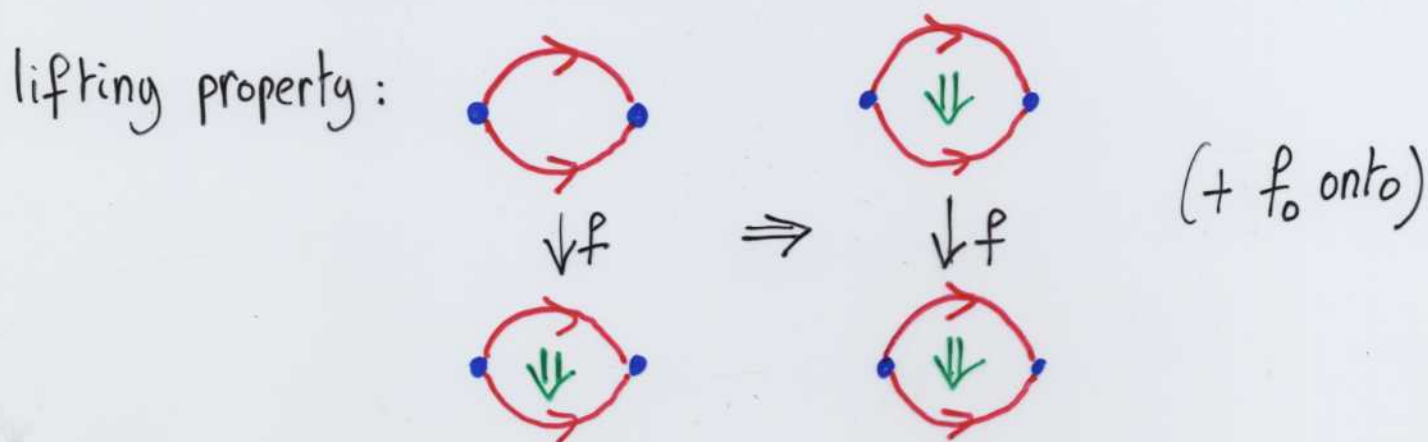
Free complex: $S: S_0^* \xrightleftharpoons[\tau_0]{\sigma_0} S_1^* \xrightleftharpoons[\tau_1]{\sigma_1} S_2^* \xrightleftharpoons[\tau_2]{\sigma_2} S_3^* \dots$

3) TRIVIAL FIBRATIONS & RESOLUTIONS

morphism:

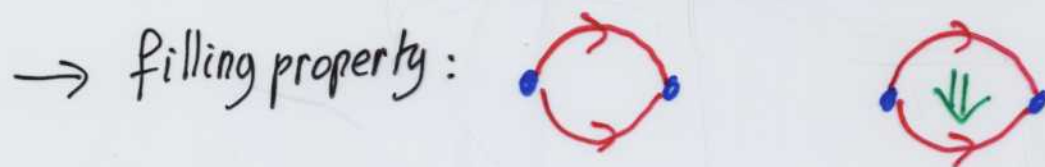
$$\begin{array}{cccc}
 C_0 & \xleftarrow{\sigma_0} & C_1 & \xleftarrow{\sigma_1} & C_2 & \xleftarrow{\sigma_2} & C_3 \\
 \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
 D_0 & \xleftarrow{\tau_0} & D_1 & \xleftarrow{\tau_1} & D_2 & \xleftarrow{\tau_2} & D_3
 \end{array}$$

f preserves source, target, composition, identity.



In that case, f is called a trivial fibration.

Special case: D is the singleton $1 \xleftarrow{\tau} 1 \xleftarrow{\tau} 1 \xleftarrow{\tau} 1 \dots$ (terminal object)



resolution of C : trivial fibration $p: S^* \rightarrow C$

- existence
- uniqueness up to homotopical equivalence

homology of C : $H_n(C) = H_n(\mathbb{Z}S)$ where

$\mathbb{Z}S$ is the abelianization of S^* .