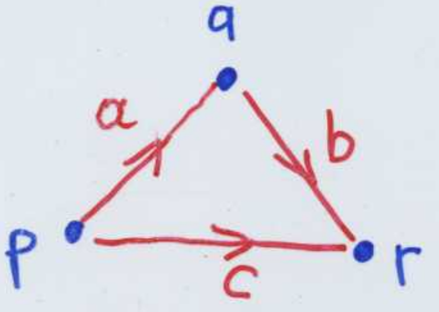
① (ABELIAN) CHAIN COMPLEXES

$$C: C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2 \xleftarrow{\partial_2} C_3 \cdots$$

(abelian groups)

$\partial_0 \circ \partial_1 = 0$
$\partial_1 \circ \partial_2 = 0$
$\vdots$
$\boxed{\partial_{n-1} \circ \partial_n = 0}$
$\vdots$

Example 1 (unfilled triangle)

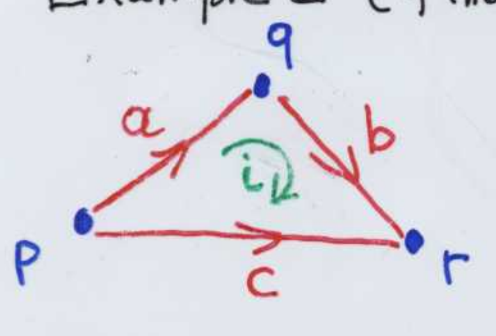


$C_0 = \mathbb{Z}^3 \simeq \mathbb{Z}p + \mathbb{Z}q + \mathbb{Z}r$
$C_1 = \mathbb{Z}^3 \simeq \mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c$
$C_2 = 0$

$\partial_0 a = q - p \quad \partial_0 b = r - q \quad \partial_0 c = r - p$

$\partial_0 (a+b-c) = q-p+r-q+p-r = 0$ $\quad$ $a+b-c$ is a cycle

Example 2 (filled triangle)



$C_2 = \mathbb{Z} \simeq \mathbb{Z}i$
$C_3 = 0$

$\partial_1 i = a + b - c$ $\quad$ $a+b-c$ is a boundary

Remark: we use free $\mathbb{Z}$-modules.

2 EXACT COMPLEXES & HOMOLOGY

$\partial_{n-1} \circ \partial_n = 0 \iff \operatorname{im} \partial_n \subset \ker \partial_{n-1}$

Every boundary is a cycle: 

- Exactness: $\operatorname{im} \partial_n = \ker \partial_{n-1}$

  Every cycle is a boundary: 
  (no holes)

- Homology: $H_n(C) = \ker \partial_{n-1} / \operatorname{im} \partial_n$ 
  (counting holes)

Examples:
- unfilled triangle $H_1(C) \simeq \mathbb{Z}$
- filled triangle $H_1(C) \simeq 0$

The complex for the filled triangle is exact

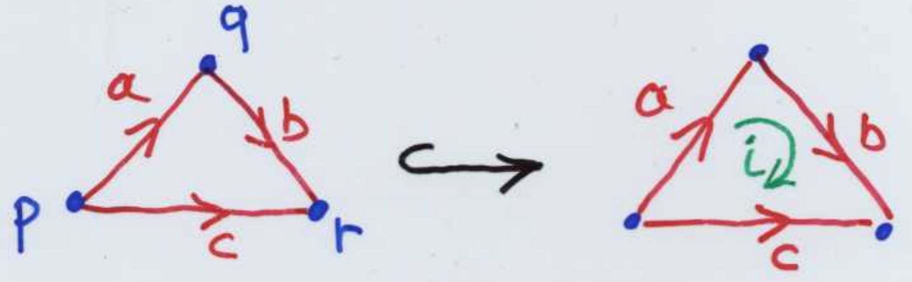
MORPHISMS & HOMOLOGISMS

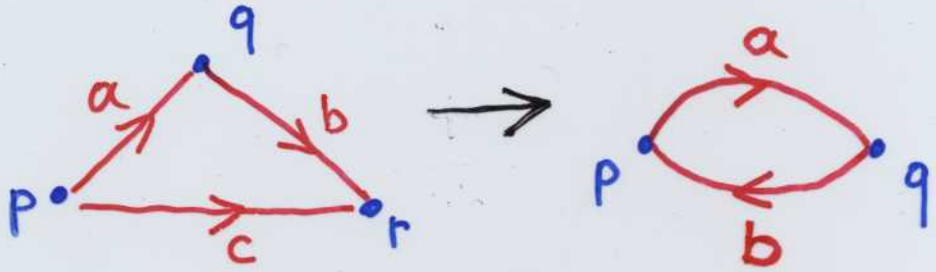
- Morphism:

$$\begin{tikzcd}
C : & C_0 & C_1 \arrow[l, "\partial_0"'] & C_2 \arrow[l, "\partial_1"'] & C_3 \arrow[l, "\partial_2"'] & \cdots \\
f \downarrow & & & & & \\
D : & D_0 \arrow[from=uu, "f_0"] & D_1 \arrow[l, "\partial_0"] \arrow[from=uu, "f_1"] \arrow[ul, phantom, "="] & D_2 \arrow[l, "\partial_1"] \arrow[from=uu, "f_2"] \arrow[ul, phantom, "="] & D_3 \arrow[l, "\partial_2"] \arrow[from=uu, "f_3"] \arrow[ul, phantom, "="] & \cdots
\end{tikzcd}$$

$$\boxed{f_n \circ \partial_n = \partial_n \circ f_{n+1}}$$

Examples:

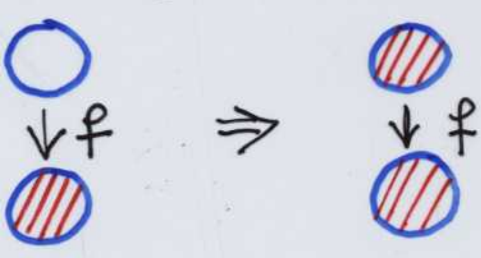




$r \mapsto p$

$c \mapsto 0$

- Homologism: $\begin{cases} \partial_{n-1} x = 0 \\ f_n x = \partial_n y \end{cases} \Rightarrow \exists z \begin{cases} x = \partial_n z \\ f_{n+1} z = y \end{cases}$ $(+\ f_0 \text{ onto})$



Exercise: 1. Prove that each f_n is onto.
2. Prove that each cycle is the image of a cycle.
3. Prove that $H_n(C) \simeq H_n(D)$.

4 HOMOTOPY & HOMOLOGY

$$p \xrightarrow{a} q$$

$$\begin{tikzcd} 0 \arrow[d, "t"] \\ 1 \end{tikzcd} \qquad \begin{tikzcd} f(p) \arrow[r, "f(a)"] \arrow[d, "h(p)"'] \arrow[dr, phantom, "h(a)"] & f(q) \arrow[d, "h(q)"] \\ g(p) \arrow[r, "g(a)"'] & g(q) \end{tikzcd} \qquad f \overset{h}{\sim} g$$

$$\begin{aligned} \partial_1 h_2(a) &= f_1(a) + h_1(q) - g_1(a) - h_1(p) \\ &= (f_1 - g_1)(a) + h_1(q-p) \\ &= (f_1 - g_1)(a) + h_1 \partial_0(a) \end{aligned}$$

$$\partial_1 h_2 - h_1 \partial_0 = f_1 - g_1$$

More generally: $\boxed{\partial_n h_{n+1} - h_n \partial_{n-1} = f_n - g_n}$ $\left(+\partial_0 h_1 = f_0 - g_0\right)$

$$\begin{tikzcd} C: & C_0 \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & C_1 \arrow[l, "\partial_0"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] \arrow[dl, "h_1"'] & C_2 \arrow[l, "\partial_1"'] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "g_2"] \arrow[dl, "h_2"'] & C_3 \arrow[l, "\partial_2"'] \arrow[d, shift right, "f_3"'] \arrow[d, shift left, "g_3"] \arrow[dl, "h_3"'] & \cdots \\ D: & D_0 & D_1 \arrow[l, "\partial_0"] & D_2 \arrow[l, "\partial_1"] & D_3 \arrow[l, "\partial_2"] & \cdots \end{tikzcd}$$

If $f \sim g$, then $H_n(f) = H_n(g)$ for all n.

⑤ RESOLUTIONS & HOMOLOGY OF MONOIDS

- R ring $\to$ complexes of R-modules
- V R-module $\to$ resolution of V by free R-modules

exact complex: $0 \leftarrow V \xleftarrow{\varepsilon} RS_0 \xleftarrow{\partial_0} RS_1 \xleftarrow{\partial_1} RS_2 \cdots$

$\to$ homologism:

$$\begin{tikzcd}
RS_0 \arrow[d, "\varepsilon"] & RS_1 \arrow[l, "\partial_0"'] \arrow[d] & RS_2 \arrow[l, "\partial_1"'] \arrow[d] & RS_3 \arrow[l, "\partial_2"'] \arrow[d] & \cdots \\
V & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots
\end{tikzcd}$$

- existence of resolutions
- uniqueness up to homotopical equivalence.

Crucial property of free R-modules
(projectivity)

$$\begin{tikzcd}
 & U \arrow[d, two heads] \\
RS \arrow[r] \arrow[ur] & V
\end{tikzcd}$$

- M monoid $\to$ ring of M: $\mathbb{Z}M$
  $\to$ $\mathbb{Z}$ seen as a $\mathbb{Z}M$-module (trivial action)

$\to$ resolution of $\mathbb{Z}$ by free $\mathbb{Z}M$-modules

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}MS_0 \xleftarrow{\partial_0} \mathbb{Z}MS_1 \xleftarrow{\partial_1} \mathbb{Z}MS_2 \cdots$$

$\to$ complex $\mathbb{Z}S$: $\mathbb{Z}S_0 \xleftarrow{\tilde{\partial}_0} \mathbb{Z}S_1 \xleftarrow{\tilde{\partial}_1} \mathbb{Z}S_2 \cdots$

Definition: $H_n(M) = H_n(\mathbb{Z}S)$.

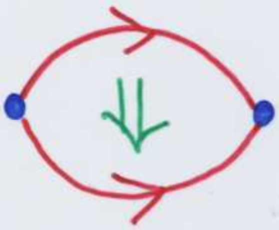
6 NON ABELIAN COMPLEXES

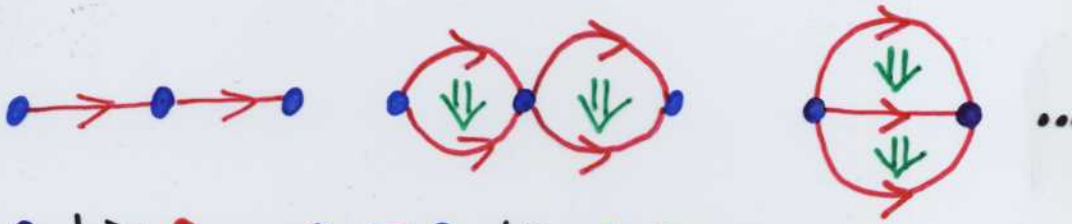
$$C:\ C_0 \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} C_1 \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} C_2 \underset{\tau_2}{\overset{\sigma_2}{\leftleftarrows}} C_3 \cdots$$

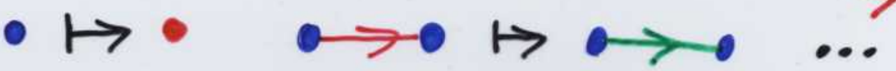
(sets)

$$\begin{cases} \sigma_0 \circ \sigma_1 = \sigma_0 \circ \tau_1 \\ \tau_0 \circ \sigma_1 = \tau_0 \circ \tau_1 \\ \vdots \end{cases}$$

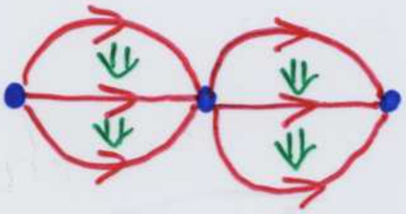
$$\boxed{\begin{cases} \sigma_{n-1} \circ \sigma_n = \sigma_{n-1} \circ \tau_n \\ \tau_{n-1} \circ \sigma_n = \tau_{n-1} \circ \tau_n \end{cases}}$$



+ compositions  ...

+ identities  ...

+ properties of associativity, left & right unit, exchange



C_0 is a set

$C_0 \leftleftarrows C_1$ is a category

$C_0 \leftleftarrows C_1 \leftleftarrows C_2$ is a 2-category

$\vdots$

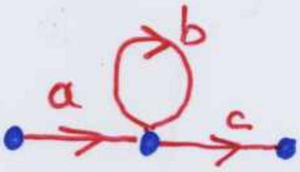
C is a (strict) ∞-category.

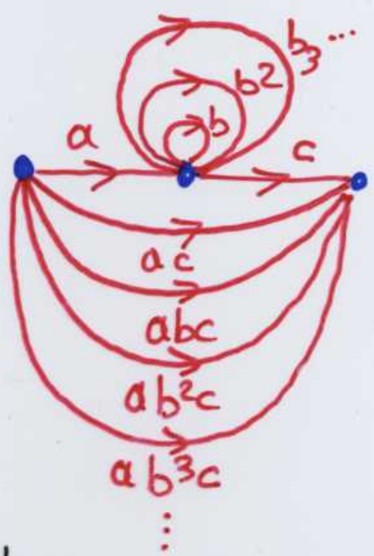
Special case: M is a monoid (or a group)

$$M:\ 1 \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} M \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} M \underset{\tau_2}{\overset{\sigma_2}{\leftleftarrows}} M \cdots$$

⑦ FREE COMPLEXES OR POLYGRAPHS

- Set $S_0 \longmapsto$ set $S_0^* = S_0$
- Graph $S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1 \longmapsto$ Category $S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^*$

Example: 

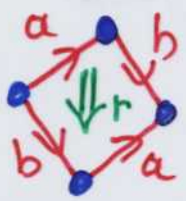
$\longmapsto$ 

Special case: 

$\longmapsto$ free monoid $\{a,b\}^*$

- 2-Graph $S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^* \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} S_2 \longmapsto$ 2-Category $S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^* \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} S_2^*$

Special case: 

\+ 

$\longmapsto$ 2-monoid of reductions $u \longrightarrow^* v$

- 3-Graph $S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^* \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} S_2^* \underset{\tau_2}{\overset{\sigma_2}{\leftleftarrows}} S_3 \longmapsto$ 3-Category $S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^* \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} S_2^* \underset{\tau_2}{\overset{\sigma_2}{\leftleftarrows}} S_3^*$

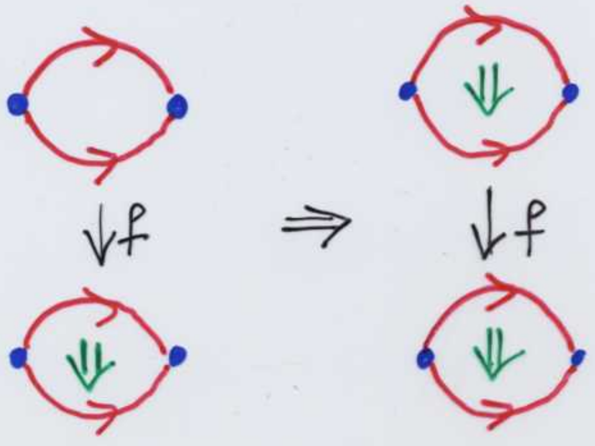
Free complex: $S: S_0^* \underset{\tau_0}{\overset{\sigma_0}{\leftleftarrows}} S_1^* \underset{\tau_1}{\overset{\sigma_1}{\leftleftarrows}} S_2^* \underset{\tau_2}{\overset{\sigma_2}{\leftleftarrows}} S_3^* \dots$

3) TRIVIAL FIBRATIONS & RESOLUTIONS

<u>morphism</u>:
$$\begin{tikzcd}
C_0 & C_1 \arrow[l, shift right, "\sigma_0"'] \arrow[l, shift left, "\tau_0"] & C_2 \arrow[l, shift right, "\sigma_1"'] \arrow[l, shift left, "\tau_1"] & C_3 \arrow[l, shift right, "\sigma_2"'] \arrow[l, shift left, "\tau_2"] \\
D_0 \arrow[from=u, "f_0"] & D_1 \arrow[l, shift right, "\sigma_0"'] \arrow[l, shift left, "\tau_0"] \arrow[from=u, "f_1"] & D_2 \arrow[l, shift right, "\sigma_1"'] \arrow[l, shift left, "\tau_1"] \arrow[from=u, "f_2"] & D_3 \arrow[l, shift right, "\sigma_2"'] \arrow[l, shift left, "\tau_2"] \arrow[from=u, "f_3"]
\end{tikzcd}$$

f preserves source, target, composition, identity.

lifting property:



$(+\ f_0$ onto$)$

In that case, f is called a <u>trivial fibration</u>.

Special case: D is the singleton $1 \leftleftarrows 1 \leftleftarrows 1 \leftleftarrows 1 \cdots$ (terminal object)

$\rightarrow$ filling property:

<u>resolution of C</u>: trivial fibration $p : S^* \rightarrow C$

- existence
- uniqueness up to homotopical equivalence

<u>homology of C</u>: $H_n(C) = H_n(\mathbb{Z}S)$ where $\mathbb{Z}S$ is the <u>abelianization</u> of S^*.