Directed Algebraic Topology and Concurrency

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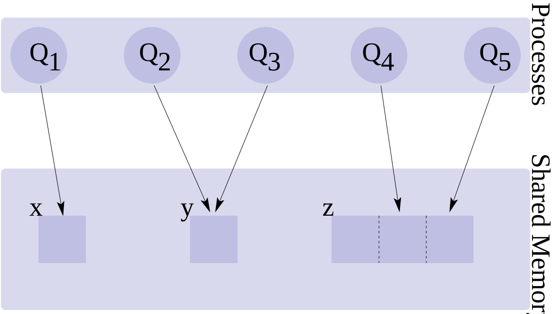
GEOCAL 2006 Marseille

Concurrency and Geometry ?

shared memory style



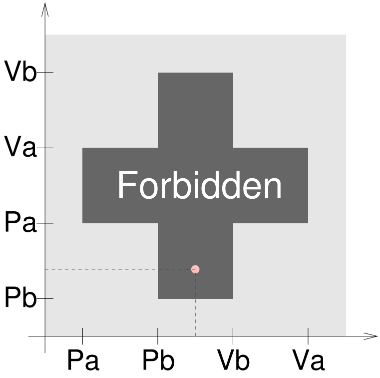
x, y and z are locations

Not sequential programs, bad states, chaotic behavior
$\Longrightarrow$ Need for synchronizations $\Longrightarrow$ Need for locks
$\Longrightarrow$ deadlocks might appear.

Geometry

"progress graphs" E.W.Dijkstra'68 (later V.Pratt, R. van Glabbeek'91)

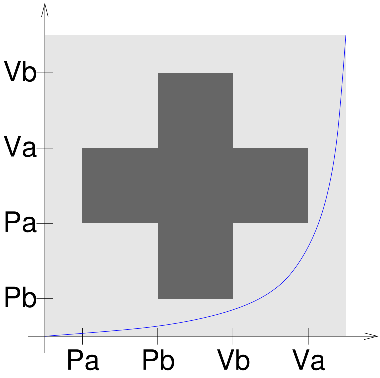
T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



"Continuous model": x_i = local time; dark grey region=forbidden!

Execution paths
are continuous

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



Traces are continuous paths increasing in each coordinate: dipaths.

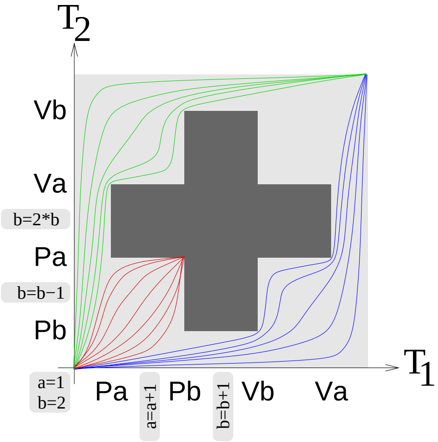
Classes of equivalent dipaths

up to dihomotopy

T1 gets a and b before T2 => a=2 and b=4

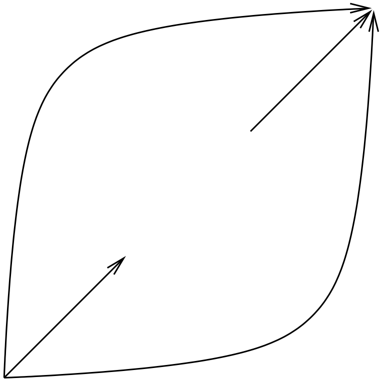
T2 gets b and a before T1 => a=2 and b=3

Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1



Ideally...

not quite true though



We will get back to this later.

A typical object of study

fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$ of a pospace $\overrightarrow{X}$

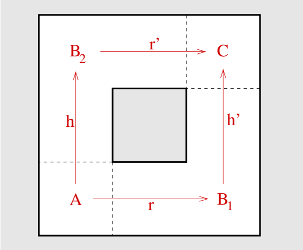
- its objects are the points of X,
- its morphisms are the classes of dipaths up to dihomotopy: a morphism from x to y is a dihomotopy class $[\alpha]$ of a dipath α going from x to y.

A detailed example (1)

square with centered hole

$x \in$	$y \in$	$\overrightarrow{\pi_1}(\overrightarrow{X})[x,y]$
A	A	$\{\sigma_{x,y}\}$
B_1	B_1	$\{\sigma_{x,y}\}$
B_2	B_2	$\{\sigma_{x,y}\}$
C	C	$\{\sigma_{x,y}\}$
A	B_1	$\{r_{x,y}\}$
A	B_2	$\{h_{x,y}\}$
B_1	C	$\{h'_{x,y}\}$
B_2	C	$\{r'_{x,y}\}$
B_1	B_2	$\emptyset$
B_2	B_1	$\emptyset$
A	C	$\{u_{x,y}\ ,\ d_{x,y}\}$

With
$r'_{y,z} \circ h_{x,y} = u_{x,z}$, $h'_{y,z} \circ r_{x,y} = d_{x,z}$
and 3 points x, y, z of the square such that $x \sqsubseteq y \sqsubseteq z$;
if $x \not\sqsubseteq y$ then $\overrightarrow{\pi_1}(\overrightarrow{X}) = \emptyset$.



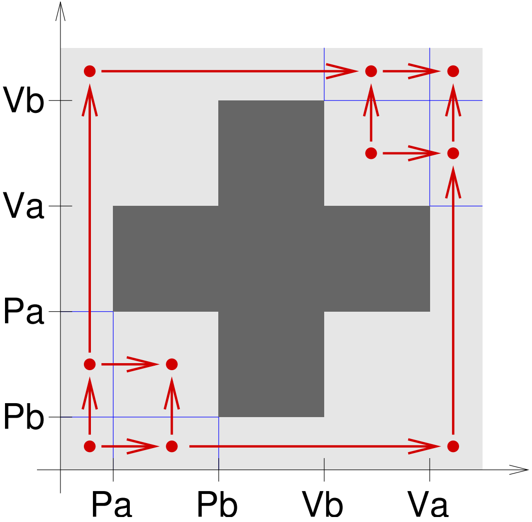
A detailed example (2)

the previous calculation suggests that

- we have a partition A, B_1, B_2, C of the objects of $\overrightarrow{\pi_1}(\overrightarrow{X})$,
- any arrow of $\overrightarrow{\pi_1}(\overrightarrow{X})$ can be given a "type" (σ, h, h', r, r', u or d) according to the components its extremities x and y belong to,
- the type σ is "neutral" in the sense that $\sigma_{y,z} \circ \sigma_{x,y} = \sigma_{x,z}$
- the map which sends
  - any object x of $\overrightarrow{\pi_1}(\overrightarrow{X})$ to its component (A, B_1, B_2 or C)
  - any morphism α to its "type" (σ, h, h', r, r', u or d)

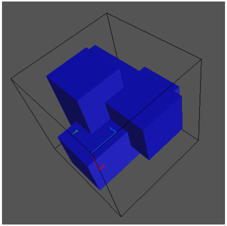
is both an equivalence and a fibration and its codomain is, by definition, the category of components of $\overrightarrow{X}$.

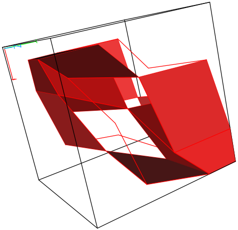
The category of components of the swiss flag



The components category of the 3 philosophers

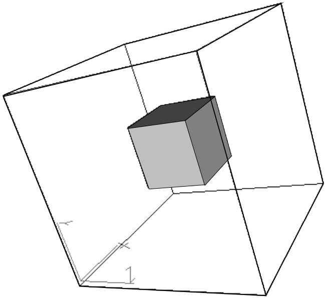
non-orthogonal representation



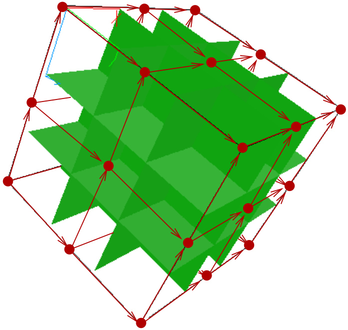


the pospace its category of components

The components category of a 2-semaphore



the pospace

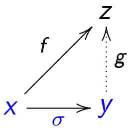


its category of components

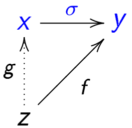
Yoneda morphism
axiomatizing the preservation of the future and the past (1)

Let $\mathcal{C}$ be a small category. A *Yoneda* morphism σ is an element of $\mathcal{C}[x,y]$ such that for all object z of $\mathcal{C}$,

future if $\mathcal{C}[y,z] \neq \emptyset$ then for all $f \in \mathcal{C}[x,z]$, there is a unique $g \in \mathcal{C}[y,z]$ such that



past if $\mathcal{C}[z,x] \neq \emptyset$ then for all $f \in \mathcal{C}[z,y]$, there is a unique $g \in \mathcal{C}[z,x]$ such that

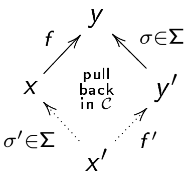


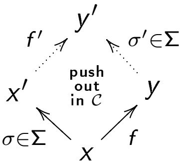
Yoneda system of a small category $\mathcal{C}$

axiomatizing the preservation of the future and the past (2)

A collection Σ of morphisms of $\mathcal{C}$ such that:

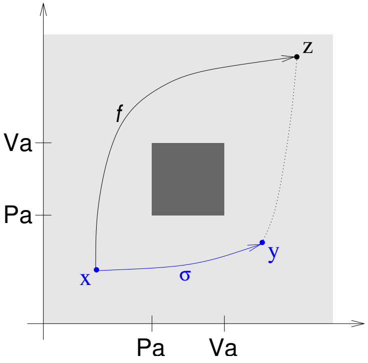
1. Σ is stable under composition,
2. Σ contains all the isomorphisms of $\mathcal{C}$,
3. all the elements of Σ are *Yoneda* morphisms and
4. Σ is stable under change and cochange of base.

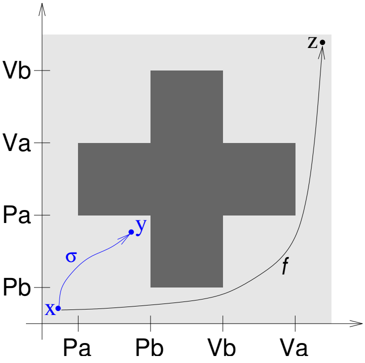




Examples

of morphisms which do not belong to a *Yoneda* system





Lifting properties of the component category

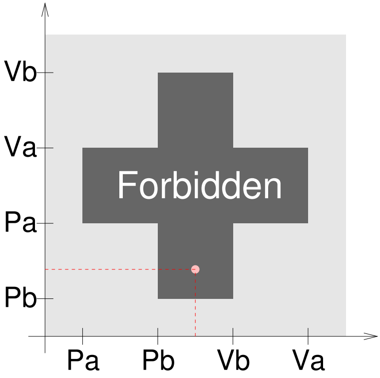
Let $\mathcal{C}$ be a category in which all endomorphisms are identities. Then there is a maximal Yoneda-system Σ in $\mathcal{C}$. Furthermore, let $C_1, C_2 \subset Ob(\mathcal{C})$ denote two components such that the set of morphisms (in $\mathcal{C}/\Sigma$) is *finite*. Then, for every $x_1 \in C_1$ there exists $x_2 \in C_2$ such that the quotient map

$$\mathcal{C}(x_1, x_2) \to \mathcal{C}/\Sigma(C_1, C_2),\ f \mapsto [f]$$

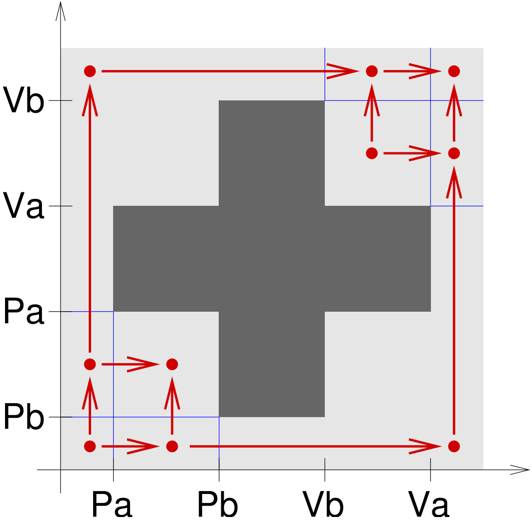
is *bijective*.

Getting back to the swiss flag

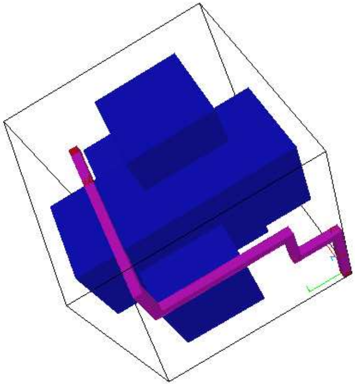
$Pa.Pb.Vb.Va \mid Pb.Pa.Va.Vb$:



Its component category



Discretisation of paths



3P(c);3P(a);2P(b);3V(c);2P(c);2V(b);
2V(c);3V(a);1P(a);1P(b);1V(a);1V(b);

Fundamental theorem (E. Haucourt)

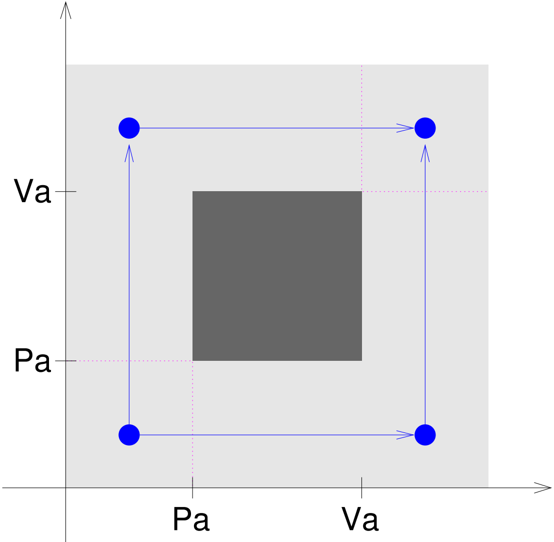
fractions *vs* quotients

Let $\mathcal{C}$ be a small loop-free category and Σ a *Yoneda* system of $\mathcal{C}$, then

1. the collection Σ is pure in $\mathcal{C}$,
2. the small category $\mathcal{C}/_{\Sigma}$ is loop-free,
3. the small categories $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}/_{\Sigma}$ are equivalent and
4. the category $\mathcal{C}[\Sigma^{-1}]$ is fibered over $\mathcal{C}/_{\Sigma}$.
5. Seifert/Van Kampen on component categories

extension and improvement of *Components of the Fundamental Category* - APCS 04

Example



Components of compact pospaces

statement

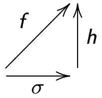
- If $\overrightarrow{K}$ is a compact pospace such that any pair of element of K has an upper/lower bound, then $\overrightarrow{K}$ has a greatest/least element.
- If $\overrightarrow{K}$ is a compact pospace, then any component of $\overrightarrow{\pi_1}(\overrightarrow{K})$ has both a greatest lower bound and an least upper bound in $(|K|, \sqsubseteq)$.

Future components

Or how to distinguish states by their future (up to dihomotopy)

Let $\mathcal{C}$ be a small category, $\Sigma \subseteq Mo(\mathcal{C})$ is a *Yoneda*-f-system iff (by definition) Σ is stable under composition (of $\mathcal{C}$) and satisfies

- all σ in Σ are epis in $\mathcal{C}$
- Σ is stable under pushout (with any morphim in $\mathcal{C}$)
- If there is $u : \beta \to \gamma$ in $\mathcal{C}$, then for all $\sigma : \alpha \to \beta$ in Σ, and all $f : \alpha \to \gamma$ in $\mathcal{C}$, f factors through σ, that is, there exists $h : \beta \to \gamma$ such that the following diagram commutes

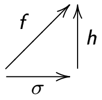


Past components

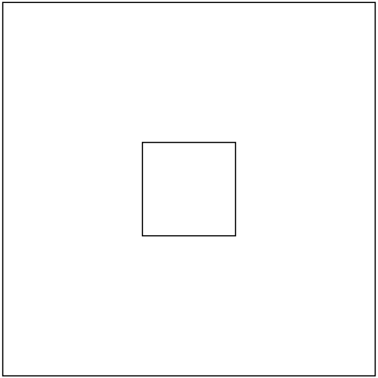
Or how to distinguish state up by their past (up to dihomotopy)

Let $\mathcal{C}$ be a small category, $\Sigma \subseteq Mo(\mathcal{C})$ is a *Yoneda*-p-system iff (by definition) Σ is stable under composition (of $\mathcal{C}$) and satisfies

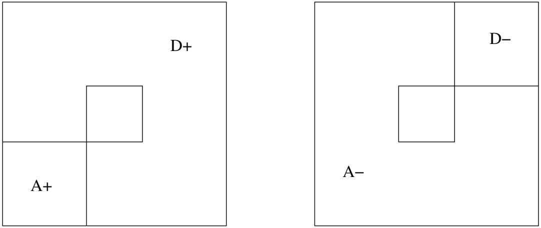
- all σ in Σ are monos in $\mathcal{C}$
- If there is $u : \beta \to \gamma$ in $\mathcal{C}$, then for all $\sigma : \alpha \to \beta$ in Σ, and all $f : \alpha \to \gamma$ in $\mathcal{C}$, f factors through σ, that is, there exists $h : \beta \to \gamma$ such that the following diagram commutes



Example



Its future and past components



An extra condition on components (?)

(Conjecture): automatically true in the PV case

- Let $\mathcal{D}$ be the category whose objects are X, X_0, X_1, ..., X_n, ..., and whose only morphisms are of the form $X \rightarrow X_i$ $(i \geq 0)$.
- Let F be a functor from $\mathcal{D}$ to a category $\mathcal{C}$.
- We call infinite pushout the colimit of $F(\mathcal{D})$ in $\mathcal{C}$, when it exists.

Ask for future (resp. past) components to have infinite pushouts (resp. pullbacks).

Extension of the lifting property

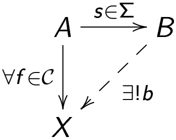
With this extra property, we have both for past and future components:

- the lifting property holds
- even if the set of morphisms (in $\mathcal{C}/\Sigma$) between two objects is not finite.

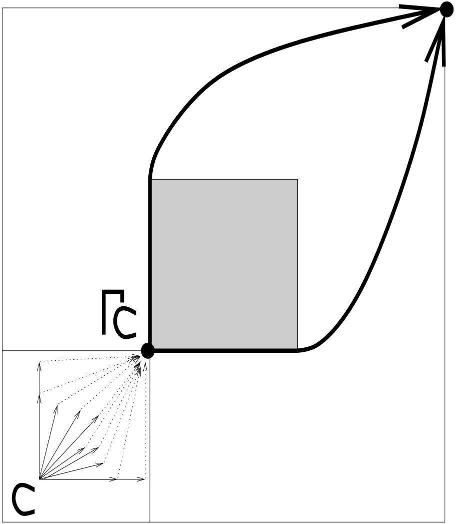
Orthogonal subcategories

See e.g. Borceux

Let $\mathcal{C}$ be a category and Σ a class of morphisms of $\mathcal{C}$. By the orthogonal subcategory of $\mathcal{C}$ determined by Σ, we mean the full subcategory $\mathcal{C}_\Sigma$ of $\mathcal{C}$, whose objects are those $X \in \mathcal{C}$ such that $s \perp X$ for every $s \in \Sigma$, i.e., such that for every $s : A \to B \in \Sigma$, for every morphism $f : A \to X$, there exists a unique morphism $b : B \to X$ such that $b \circ s = f$.



The orthogonal subcategory of Σ_+ is reflective



Theorem

Let Σ be the inessential morphisms in the future, in the category $\mathcal{C} = \overrightarrow{\pi_1}(\overrightarrow{X})$ for some local po-space X.

Suppose that Σ has infinite pushouts then

$\mathcal{C}_\Sigma$ is reflective in $\overrightarrow{\pi_1}(\overrightarrow{X})$.

Sketch of proof

By definition of the orthogonal subcategory:

- we have an obvious inclusion functor I from $\mathcal{C}_\Sigma$ to $\mathcal{C}$.
- To prove that we have a reflective subcategory, we need to construct the left adjoint to $\Gamma \dashv I$.

Sketch of proof

- Let $C \in \mathcal{C}$. For every pair (s, f) where $s : S \rightarrow T \in \Sigma$ and $f : S \rightarrow C \in \mathcal{C}$,
- by the properties of Σ, we know we have a pushout diagram (we call it a (s, f) pushout square):

$$\begin{array}{ccc} S & \xrightarrow{s \in \Sigma} & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g_{sf}} \\ C & \xrightarrow[t_{sf} \in \Sigma]{} & P_{sf} \end{array}$$

where $t_{sf} \in \Sigma$.

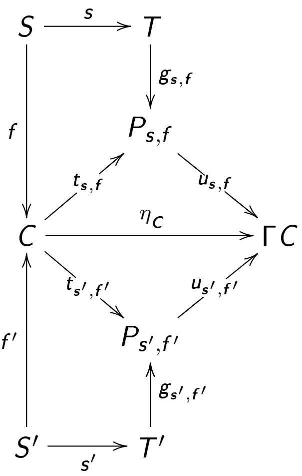
Sketch of proof

Consider now the diagram composed of all t_{sf} for all pairs (s, f) which is small since Σ is a set, thus its colimit $\left(\Gamma C, (u_{sf})_{sf}\right)$ exists in Σ and provide $\Gamma C \in \Sigma$.

We have defined the object part of Γ. Now we construct a family of morphisms of $\mathcal{C}$, denoted $(\gamma_C)_{C \in Ob\mathcal{C}}$, that will be the unit of the adjunction.

Sketch of a proof

Let us determine $\gamma_C : C \longrightarrow \Gamma C$. Given two (s, f)-pushout squares, by definition of a colimit, we have $u_{s,f} \circ t_{s,f} = u_{s',f'} \circ t_{s',f'}$, hence we can set $\gamma_C := u_{s,f} \circ t_{s,f}$ since it does not depend on the (s, f)-pushout square we have chosen. $\gamma_C \in \Sigma$ for it is given by the composite of two morphisms of Σ.



Sketch of proof

We determine the morphism part of Γ, this construction will implicitly prove that γ is an adjunction from $Id_{\mathcal{C}}$ to $I \circ \Gamma$. Let $h : C^1 \longrightarrow C^2$. For each pushout square

$$\begin{array}{ccc} S & \xrightarrow{s} & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g^1_{s,f}} \\ C^1 & \xrightarrow[t^1_{s,f}]{} & P^1_{s,f} \end{array}$$

Sketch of proof

We have the pushout diagram (since $t^1_{s,f} \in \Sigma$):

$$\begin{array}{ccc} C^1 & \xrightarrow{\;t^1_{s,f}\;} & P^1_{s,f} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g^2_{t^1_{s,f},h}} \\ C^2 & \xrightarrow[\;t^2_{t^1_{s,f},h}\;]{} & P^2_{t^1_{s,f},h} \end{array}$$

Sketch of proof

Then, using the factorisation of the unit (on C^1 and C^2) we have:

$$\begin{array}{ccccc}
C^1 & \xrightarrow{t^1_{s,f}} & P^1_{s,f} & \xrightarrow{u^1_{s,f}} & \Gamma(C^1) \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g^2_{t^1_{s,f},h}} & & \\
C^2 & \xrightarrow[t^2_{t^1_{s,f},h}]{} & P^2_{t^1_{s,f},h} & \xrightarrow[u^2_{t^1_{s,f},h}]{} & \Gamma(C^2)
\end{array}$$

Sketch of proof

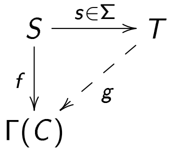
Since $u^1_{s,f}$ is in Σ, it is invertible in the future (and there is a map from $\Gamma(C^1)$ to $\Gamma(C^2)$ because of the universal property of the colimit for $\Gamma(C^2)$), giving the map $\Gamma(h)$ making the following diagram commute:

$$
\begin{array}{ccccc}
C^1 & \xrightarrow{t^1_{s,f}} & P^1_{s,f} & \xrightarrow{u^1_{s,f}} & \Gamma(C^1) \\
\downarrow h & & \downarrow g^2_{t^1_{s,f},h} & & \downarrow \Gamma(h) \\
C^2 & \xrightarrow[t^2_{t^1_{s,f},h}]{} & P^2_{t^1_{s,f},h} & \xrightarrow[u^2_{t^1_{s,f},h}]{} & \Gamma(C^2)
\end{array}
$$

Sketch of proof

Now we prove that $\forall C \in \mathcal{C},\ \Gamma(C) \in \mathcal{C}_\Sigma$.

Given (s, f) with $s : S \to T \in \Sigma$, $src(f) = S$ and $tgt(f) = \Gamma(C)$, we have a g making the right side diagram commutative. Precisely, $g := u_{sf} \circ g_{sf}$, indeed, $u_{sf} \circ g_{sf} \circ s = u_{sf} \circ t_{sf} \circ f = \gamma_C \circ f$.

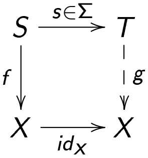


Sketch of proof

- The uniqueness of g is due to the bijectivity of $\gamma \in \mathcal{C}[S, T] \longrightarrow \gamma \circ s \in \mathcal{C}[S, \Gamma(C)]$, because s is inessential in the future.
- Thus, $s \perp \Gamma(C)$ and $\Gamma(C)$ is in the orthogonal subcategory determined by Σ.

Sketch of proof

Conversely, suppose that $X \in \mathcal{C}_\Sigma$. Given (s, f) with $s : S \to T \in \Sigma$, $src(f) = S$ and $tgt(f) = X$, we have a unique g making the right side diagram commutative, which is in fact necessarily the pushout square given by the definition of Σ.



Sketch of proof

With the notation introduced at the beginning of the proof,

- $g_{s,f} = g$ and $t_{s,f} = id_X$.
- But then the colimit of such pushout squares, defining $(\Gamma(X), u_{s,f})$ is the colimit of the family $(id_X)_{\{(s,f) \text{ with } s:S \rightarrow T \in \Sigma, \ src(f)=S \text{ and } tgt(f)=X\}}$.
- Hence $u_{s,f} \cong id_X$ for all such pairs (s, f) and $\Gamma(X) \cong X$ in $\mathcal{C}$.

Sketch of proof

The last part of the proof consists in seeing that

$$\alpha \in \mathcal{C}_\Sigma[\Gamma(C), D] \longmapsto I(\alpha) \circ \gamma_c \in \mathcal{C}[C, I(D)]$$

is a bijection, where $\gamma_C : C \to \Gamma C$ is the canonical morphism given by the colimit.

Sketch of proof

Given $D \in \mathcal{C}_\Sigma$ and $m : C \to D$, we must find a unique $n : \Gamma C \to D$ such that $n \circ \gamma_C = m$.

- As $D \in \mathcal{C}_\Sigma$, it is orthogonal to all morphisms $s \in \Sigma$.
- In particular, for all pairs (s, f) as above, there exists a unique b_{sf} such that $b_{sf} \circ s = m \circ f$.
- By the pushout property defining P_{sf} we deduce that there is a unique morphism $a_{sf} : P_{sf} \to D$ such that $a_{sf} \circ t_{sf} = m$ and $a_{sf} \circ g_{sf} = b_{sf}$.

Sketch of proof

- This is done for all pairs (s, f).
- Hence by the colimit property defining ΓC, we find a unique morphism $n : \Gamma_C \to D$ such that $n \circ u_{sf} = \alpha_{sf}$.
- Hence

$$\begin{aligned} n \circ \gamma_C &= n \circ u_{sf} \circ t_{sf} \\ &= \alpha_{sf} \circ t_{sf} \\ &= m \end{aligned}$$

which ends the proof.

Relationship with Marco Grandis work

- Equivalences in the past and in the future
- When two categories are (resp. past) future equivalent, what we have seen provides a common (co-) reflective category

Generic segment of C

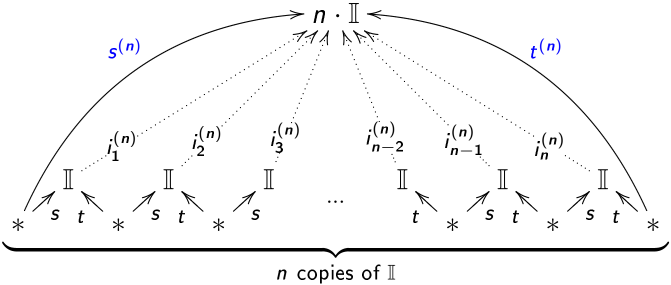
axiomatizing the notion of *Moore* paths (1)

A generic segment of C is a triple $(\mathbb{I}, s, t)$ where $\mathbb{I}$ is an object of C and s, t two points of $\mathbb{I}$ such that:

1. for any automorphism ϕ of $\mathbb{I}$ we have

$$\{\phi \circ s, \phi \circ t\} = \{s, t\}$$

2. and for any $n \in \mathbb{N}$ we have the colimit



Directed generic segment

axiomatization of the notion of direction

- A generic segment $(\mathbb{I}, s, t)$ is said directed when for any automorphism ϕ of $\mathbb{I}$, we have $\phi \circ s = s$ and $\phi \circ t = t$.
- Any automorphism ϕ of $\mathbb{I}$ such that $\phi \circ s = t$ and $\phi \circ t = s$ is called an inversion of (the) time (flow)
- In PoSpc, the generic segment $\overrightarrow{[0,1]}$ is directed while the generic segment $([0,1], =)$ does not.

the map $t \mapsto 1 - t$ is an inversion of time

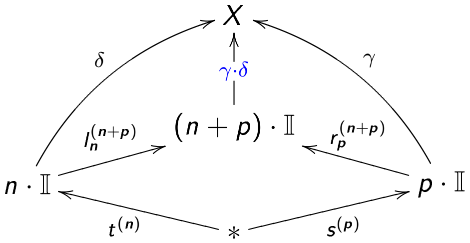
Category of paths on an object X of C

axiomatization of the notion of *Moore* path (2)

The objects of this category, denoted $\Gamma(X)$, are the points of X and its morphisms, called the paths on X, are the elements of

$$\bigcup_{n\in\mathbb{N}} \mathsf{C}[n\cdot\mathbb{I}, X],$$

the source and the target of $\gamma \in \mathsf{C}[n\cdot\mathbb{I}, X]$ are $\gamma\circ s^{(n)}$ and $\gamma\circ t^{(n)}$; the concatenation being given by the push-out:



Homotopic congruence over C

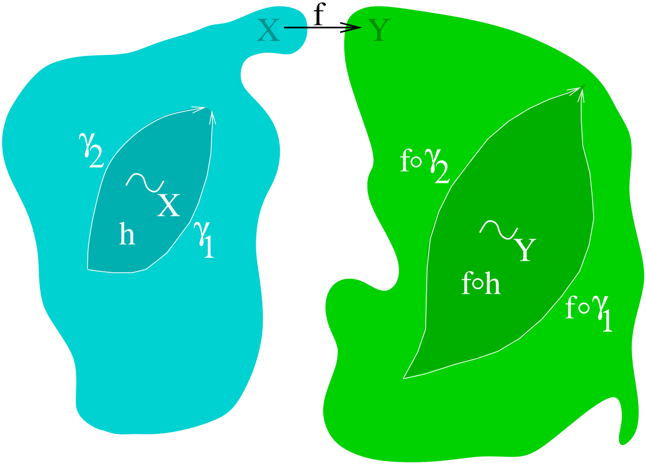
axiomatization of the notion of (di)homotopic (di)paths

A path $\gamma \in \mathcal{C}\left[n \cdot \mathbb{I}, X\right]$ is said constant when it can be written $\gamma = p \circ \mu$ where p is a point of X, it is the value of γ.
A homotopic congruence on C is defined by, for each object X of C, a congruence $\sim_X$ on the category of paths on X, such that for all paths γ_1 and γ_2 on X,

1. if γ_1 and γ_2 are constant with the same value, then $\gamma_1 \sim_X \gamma_2$,
2. if $\gamma_1 \sim_X \gamma_2$, then
   1. γ_1 and γ_2 share the same extremities and
   2. for all morphism f of C from X to Y we have $f \circ \gamma_1 \sim_Y f \circ \gamma_2$.

Homotopic congruence

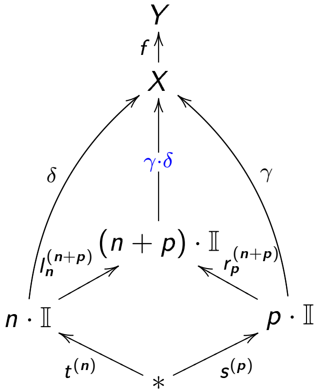
in picture



Think of $\sim_X$ as "there exists a classic homotopy h from the paths γ_1 to γ_2"

Generalized fundamental category

We set $\overrightarrow{\pi_1}(\overrightarrow{X}) := \Gamma(X)/\sim_X$ and we have a functor $\overrightarrow{\pi_1} : \mathsf{C} \longrightarrow \mathsf{Cat}$.



Since $\gamma_1 \sim_X \gamma_2$ implies $f\circ\gamma_1 \sim_Y f\circ\gamma_2$, we can define $\overrightarrow{\pi_1}(\overrightarrow{f})[\gamma]_{\sim_X} := [f\circ\gamma]_{\sim_Y}$, moreover, the left hand side diagram shows that we have $f\circ(\gamma\cdot\delta) = (f\circ\gamma)\cdot(f\circ\delta)$ whence the functoriality of $\overrightarrow{\pi_1}(\overrightarrow{f})$ from $\overrightarrow{\pi_1}(\overrightarrow{X})$ to $\overrightarrow{\pi_1}(\overrightarrow{Y})$.

directed *vs* undirected generic segment
in the framework of PoSpc

- With the generic segment $([0,1],=)$ over PoSpc, for any pospace $\overrightarrow{X}$, $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the fundamental groupoid of X.
- With the generic segment $([0,1],\leq)$ over PoSpc, for any pospace $\overrightarrow{X}$, $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the fundamental category of $\overrightarrow{X}$.