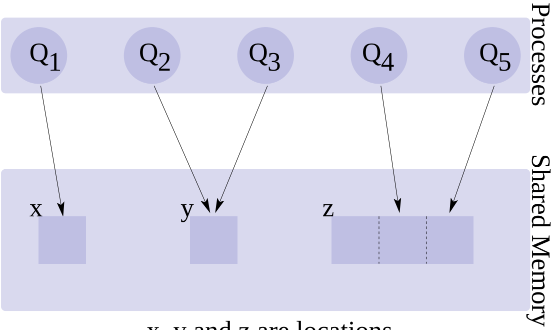
Directed Algebraic Topology and Concurrency

Eric Goubault and Emmanuel Haucourt

GEOCAL 2006 Marseille

Concurrency and Geometry ?

shared memory style

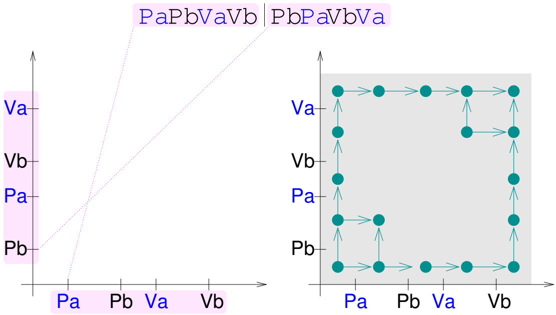


x, y and z are locations

Not sequential programs, bad states, chaotic behavior
$\Longrightarrow$ Need for synchronizations $\Longrightarrow$ Need for locks
$\Longrightarrow$ deadlocks might appear.

First Model

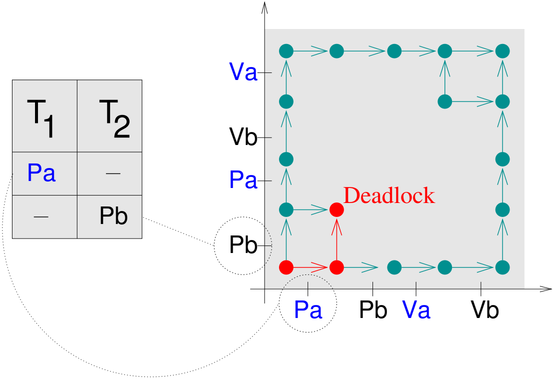
directed graphs of actions



18 states and 20 arrows

A potential execution

program T_1 =PaPbVaVb| T_2 =PbPaVbVa

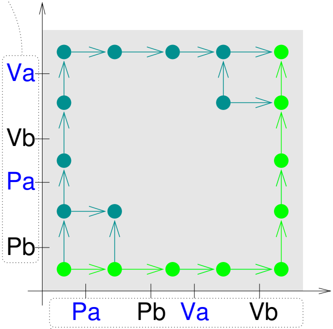


Deadlock

Another potential execution

program T_1 =PaPbVaVb| T_2 =PbPaVbVa

T_1	T_2
Pa	–
Pb	–
Va	–
Vb	–
–	Pb
–	Pa
–	Vb
–	Va



Termination

Notice that...

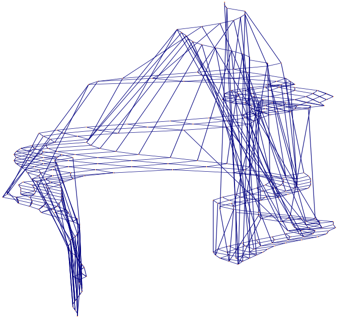
... there are very few "interesting" paths

Suppose $T_1 = Pa(a = a + 1)Pb(b = b + 1)VaVb$, $T_2 = Pb(b = b - 1)Pa(b = 2 * b)VbVa$ and in the beginning $a = 1$ and $b = 2$, we have:

- 1 path "T_2 then T_1" which computes $\underline{b = 3}$ (2*(2-1)+1) and $\underline{a = 2}$.
- 1 path "T_1 then T_2" which computes $\underline{b = 4}$ (2*((2+1)-1)) and $\underline{a = 2}$.
- 2 "equivalent" paths near the diagonal: they do not "terminate" with $a = 2$ and $b = 1$.

Size explosion problem

Dekker's algorithm

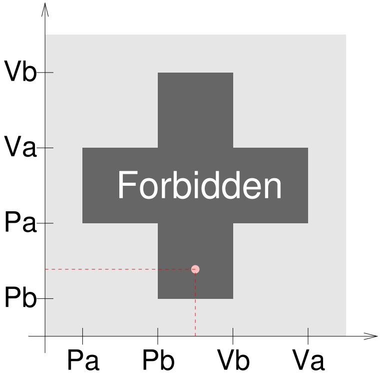


Few lines of C on 2 processes lead to few hundreds of paths, only 2 of which are interesting!

Geometry

"progress graphs" E.W.Dijkstra'68 (later V.Pratt, R. van Glabbeek'91)

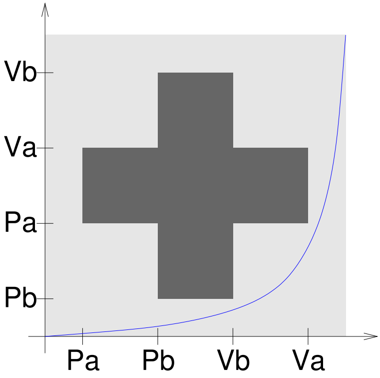
T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



"Continuous model": x_i = local time; dark grey region=forbidden!

Execution paths
are continuous

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb

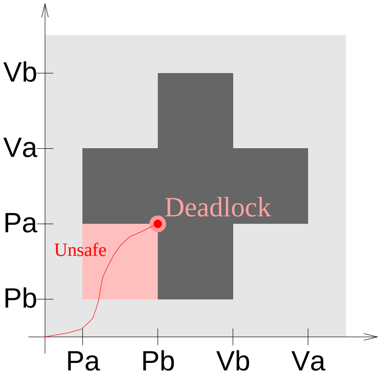


Traces are continuous paths increasing in each coordinate: dipaths.

Deadlocks and Unsafe regions

Swiss flag example

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb

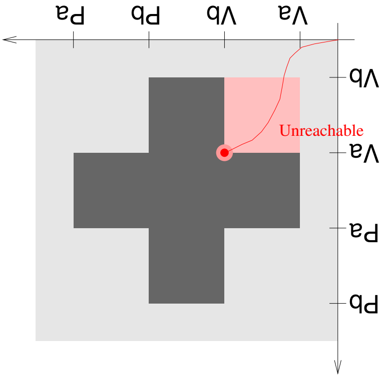


Entering the unsafe region $\Longrightarrow$ finishing to its deadlock.

Unreachable states

Swiss flag example

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



Dual to the previous situation.

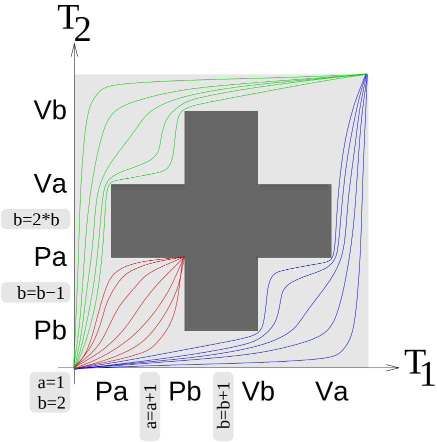
Classes of equivalent dipaths

up to dihomotopy

T1 gets a and b before T2 => a=2 and b=4

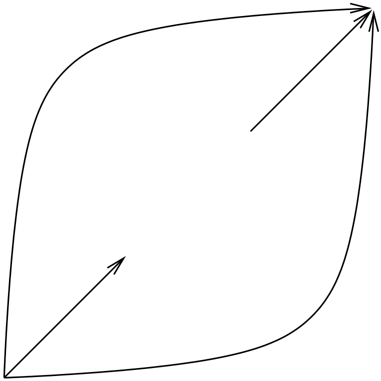
T2 gets b and a before T1 => a=2 and b=3

Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1



Ideally...

not quite true though

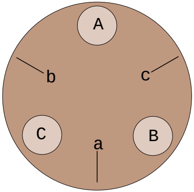


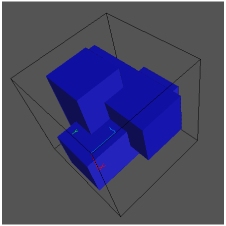
We will get back to this later.

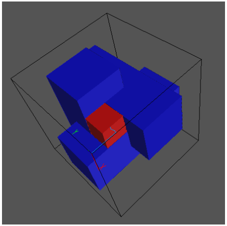
In higher-dimension

philosophers and chopsticks

A=Pb.Pc.Vb.Vc
B=Pc.Pa.Vc.Va
C=Pa.Pb.Va.Vb

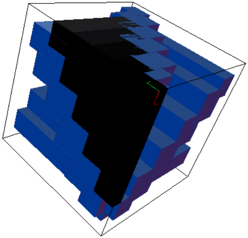




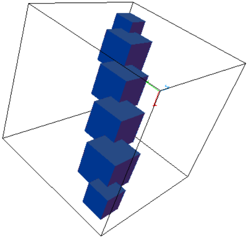


Effect of the level of sharing

A=Pa.Pb.Va.Pc.Vb.Pd.Vc.Pe.Vd.Pf.Ve.Vf
B=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va
C=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va



a,...binary sem.



a,...counting sem.

Correspondences
(almost)

Model [discrete]	combinatorial complex
Model [continuous]	topological space
Relation discrete/continuous	geometric realisation
Parallel composition	product
Action refinement	subdivision
Compositionality	Seifert/van Kampen
Deadlocks/reachability	connected components
Scheduling properties	fundamental group
Observational equivalence	homotopy equivalence (weak/strong)
Computable properties	topological invariants (homology etc.)

Other types of related subjects
and their applications

- Rewriting invariants (Squier like - see talks by Y. Lafont for instance)
- Fault-tolerant distributed systems (realizability and complexity, see M. Herlihy, S. Rajsbaum, N. Shavit etc.)

Models

- Po-spaces, local po-spaces, (pre-)cubical sets (see MFPS'98, with L. Fajstrup and M. Raussen)
- Globular CW-complexes: with P. Gaucher, "Topological Deformation of Higher-Dimensional Automata", *HHA 2003*
- Ω-categories, Category "Flow" (Philippe Gaucher)
- d-spaces (Marco Grandis)
- Higher-Dimensional Transition Systems (Vladimiro Sassone and Gian Luca Cattani, LICS'96)
- ECHIDNA (Richard Buckland and Michael Johnson, AMAST'96)
- Sanjeevi Krishnan's spaces
- *et caetera*

Partially Ordered Spaces

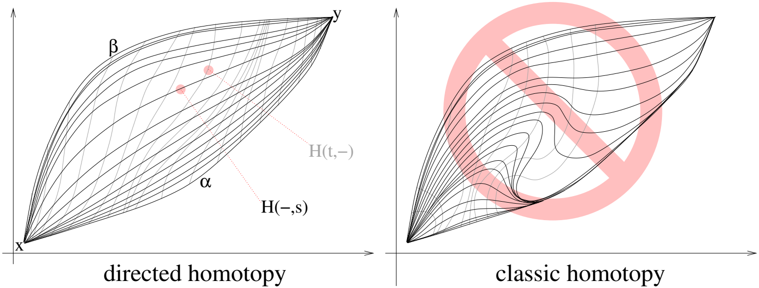
framework for "progress graphs" (one only needs *MFPS'98*)

A topological space X with a (global) closed partial order $\sqsubseteq$

- Morphisms are increasing and continuous maps: dimaps
- (Finite) Traces on $(X, \sqsubseteq)$ are dimaps from $\vec{I} = ([0,1], \leq)$ to $(X, \sqsubseteq)$: dipaths
- Dihomotopies between dipaths α and β with fixed extremities x and y are dimaps $H : \overrightarrow{I} \times \overrightarrow{I} \to X$ such that for all $s \in \overrightarrow{I}$, $t \in \overrightarrow{I}$,
  - $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$
  - $H(0,s) = x$ and $H(1,s) = y$

Deformation of execution paths

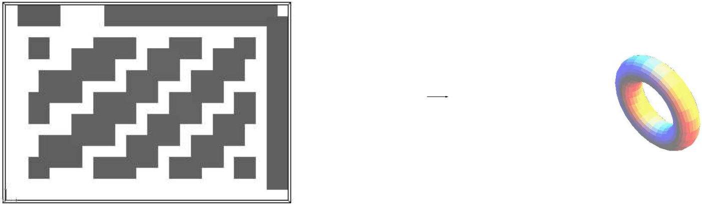
dihomotopy *vs* homotopy



Extension to local po-spaces

Or how to model loops

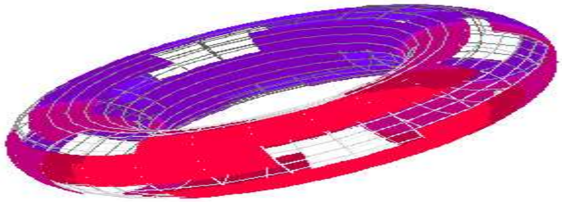
A=PdPa(PbVaVdPcVbPaPdVcPbVaPcVbPaVcPb
VaPcVbPaVc)*VaPeVdVe
B=PePa(PbVaPcVbPaVcPbVaPcVbPaVc)*VaPdVeVd



Gives...

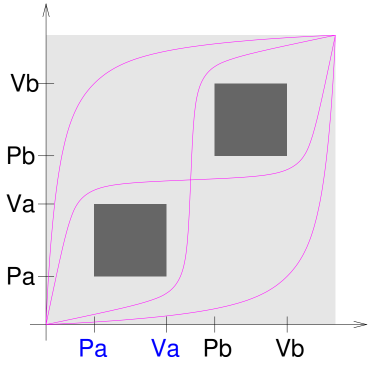
Classical torus knot

$\longrightarrow$ local po-space (*MFPS'98*); manifold like definition, locally a po-space. Extensions of algorithms (see L. Fajstrup, M. Raussen, S. Sokolowski).



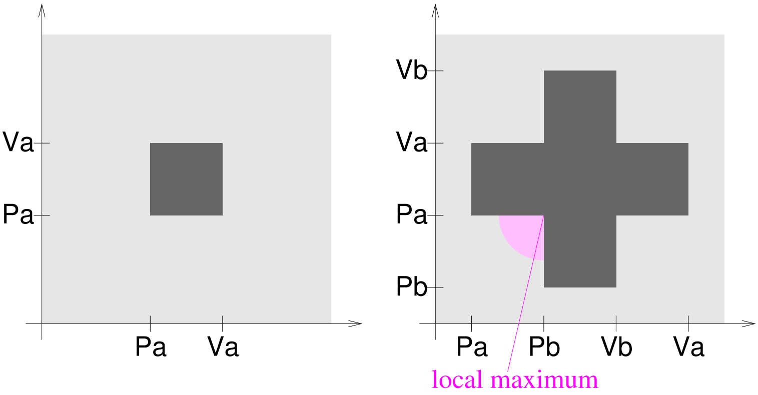
First subtlety

directed homotopy is not classic homotopy



Second subtlety

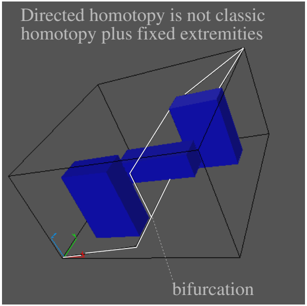
classic homotopy cannot “see” local extrema

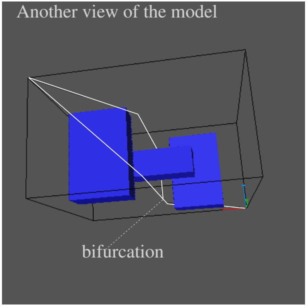


Third subtlety

Floating cube between two pillars

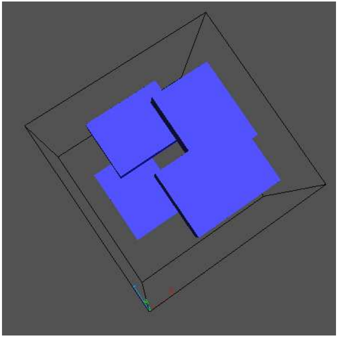
A=Pb.Pc.Vb.Vc
B=Pc.Pa.Vc.Va
C=Pa.Pb.Va.Vb

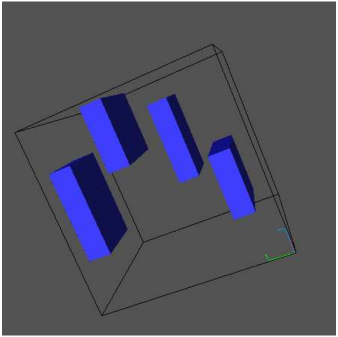




More subtleties

Helix





Correspondences
(almost)

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A typical object of study

fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$ of a pospace $\overrightarrow{X}$

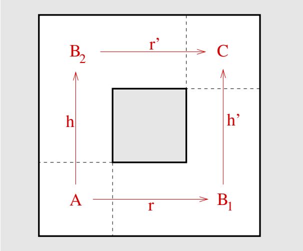
- its objects are the points of X,
- its morphisms are the classes of dipaths up to dihomotopy: a morphism from x to y is a dihomotopy class $[\alpha]$ of a dipath α going from x to y.

A detailed example (1)

square with centered hole

$x \in$	$y \in$	$\overrightarrow{\pi_1}(\overrightarrow{X})[x,y]$
A	A	$\{\sigma_{x,y}\}$
B_1	B_1	$\{\sigma_{x,y}\}$
B_2	B_2	$\{\sigma_{x,y}\}$
C	C	$\{\sigma_{x,y}\}$
A	B_1	$\{r_{x,y}\}$
A	B_2	$\{h_{x,y}\}$
B_1	C	$\{h'_{x,y}\}$
B_2	C	$\{r'_{x,y}\}$
B_1	B_2	$\emptyset$
B_2	B_1	$\emptyset$
A	C	$\{u_{x,y}\ ,\ d_{x,y}\}$

With
$r'_{y,z} \circ h_{x,y} = u_{x,z}$, $h'_{y,z} \circ r_{x,y} = d_{x,z}$
and 3 points x, y, z of the square such that $x \sqsubseteq y \sqsubseteq z$;
if $x \not\sqsubseteq y$ then $\overrightarrow{\pi_1}(\overrightarrow{X}) = \emptyset$.



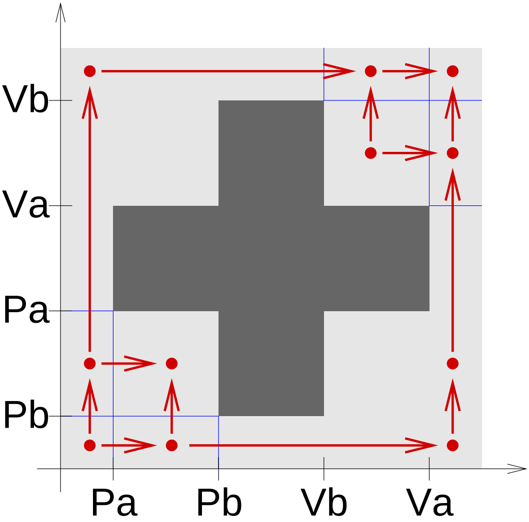
A detailed example (2)

the previous calculation suggests that

- we have a partition A, B_1, B_2, C of the objects of $\overrightarrow{\pi_1}(\overrightarrow{X})$,
- any arrow of $\overrightarrow{\pi_1}(\overrightarrow{X})$ can be given a "type" (σ, h, h', r, r', u or d) according to the components its extremities x and y belong to,
- the type σ is "neutral" in the sense that $\sigma_{y,z} \circ \sigma_{x,y} = \sigma_{x,z}$
- the map which sends
  - any object x of $\overrightarrow{\pi_1}(\overrightarrow{X})$ to its component (A, B_1, B_2 or C)
  - any morphism α to its "type" (σ, h, h', r, r', u or d)

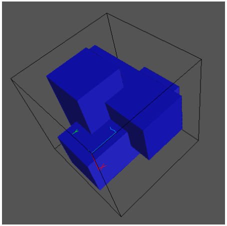
is both an equivalence and a fibration and its codomain is, by definition, the category of components of $\overrightarrow{X}$.

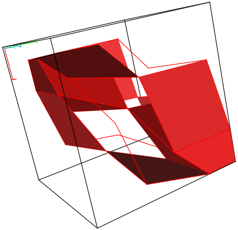
The category of components of the swiss flag



The components category of the 3 philosophers

non-orthogonal representation

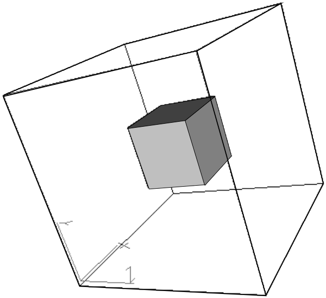




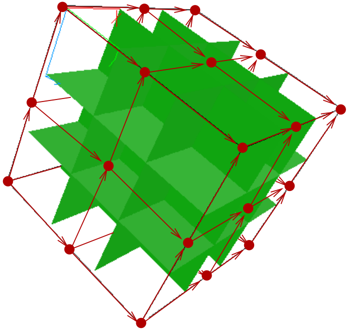
the pospace

its category of components

The components category of a 2-semaphore



the pospace



its category of components

Correspondences
(almost)

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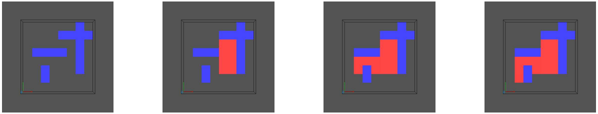
Analogue of connectedness?

Construction of one component

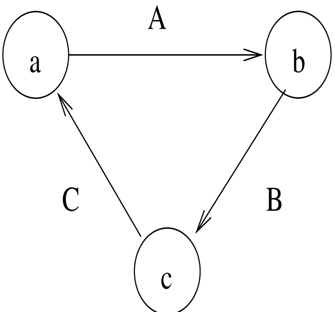
CONCUR'98 paper with M. Raussen and L. Fajstrup

- The intersections J of n (=dimension) forbidden hyperrectangles $R^i = [a_1^i, b_1^i] \times \cdots [a_n^i, b_n^i]$ create deadlocks,
- Forbid the hyperrectangles $[\tilde{x}, x]$, where $x = \min J = (\max_i a_1^i, \cdots \max_i a_n^i)$ and $\tilde{x} = (2\text{nd}\ \max_i a_1^i, \cdots, 2\text{nd}\ \max_i a_n^i)$,



Is it useful?
Compare with request graphs

A sufficient (classical) condition for a parallel system to be deadlock-free is that its request graph be acyclic (geometric condition...).
a, b, c: forks
A, B, C: philosophers



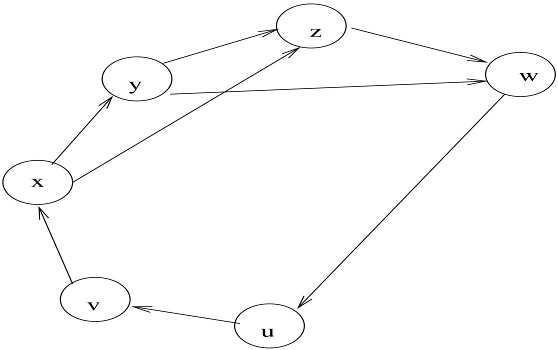
But...not necessary!

Lipsky/Papadimitriou: cycles but no deadlock

A=Px.Py.Pz.Vx.Pw.Vz.Vy.Vw

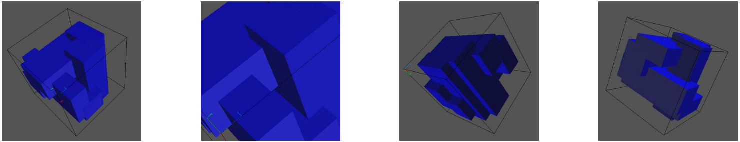
B=Pu.Pv.Px.Vu.Pz.Vv.Vx.Vz

C=Py.Pw.Vy.Pu.Vw.Pv.Vu.Vv



PO-space for [Lipsky/Papadimitriou]

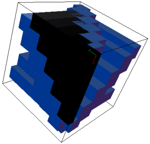
A cube minus an antidiagonal torus



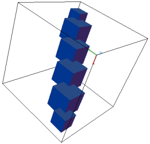
Also: effect of the level of sharing

From 1-semaphores to 2-semaphores

A=Pa.Pb.Va.Pc.Vb.Pd.Vc.Pe.Vd.Pf.Ve.Vf
B=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va
C=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va



a,. . .binary sem.



a,. . .counting sem.

Same request graph!

Correspondences
(almost)

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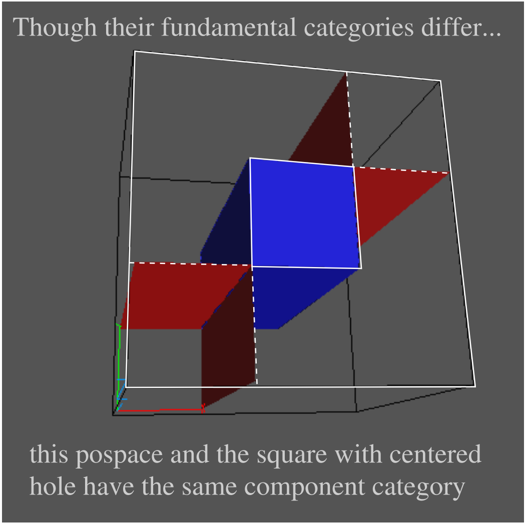
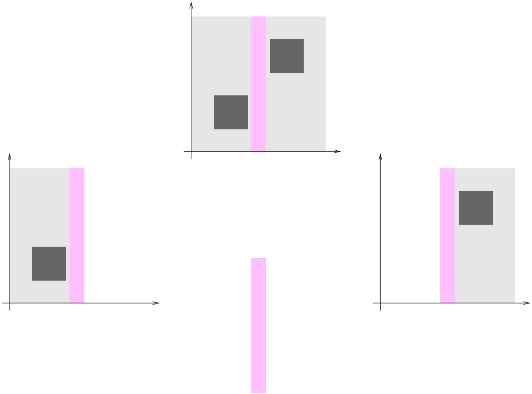
Example of product

parallel "independent" composition



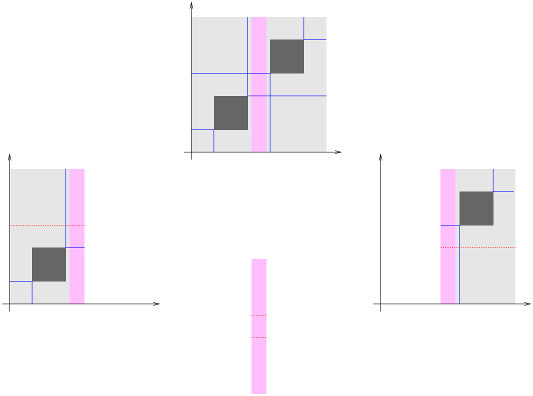
The Seifert/Van Kamen theorem for fundamental category

compositionality



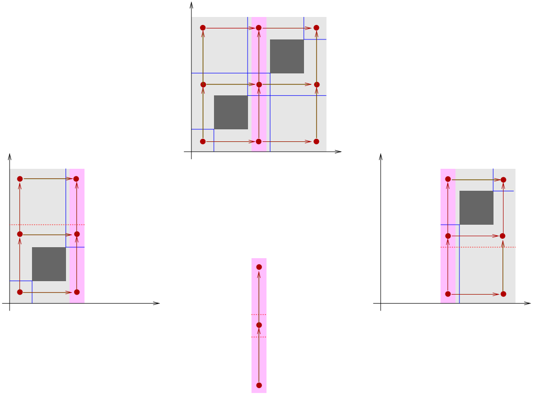
A Seifert/Van Kamen theorem for components category (1)

subdivisions are necessary



A Seifert/Van Kamen theorem for components category (2)

the resulting category of components



The bounding diagrams of the grey squares do not commute.

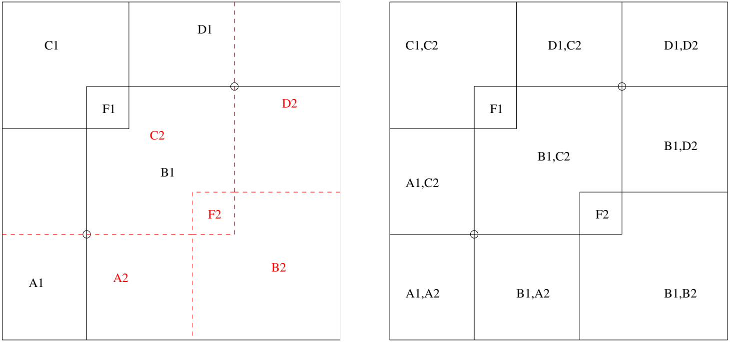
Computations: some theoretical and practical tools
for handling concrete cases

- We have a Seifert/van Kampen for local po-spaces (HHA'03)
- We also have a form of Seifert/van Kampen for components categories, "up to subdivision" (Emmanuel Haucourt), which is of value for practical computations.
- Also, some specific algorithms for mutual exclusion models (M. Raussen in dimension 2, and sub-optimal algorithm by E. Goubault in all dimensions).

Some figures
[Eric Goubault's algorithm]

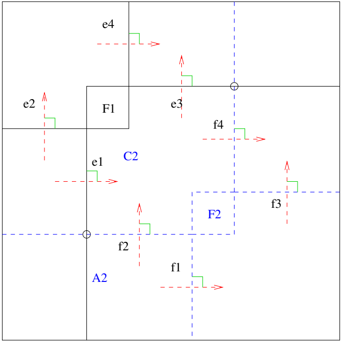
- new3phil.pv: (0.05s) Objects: 27, Morphisms: 48, Relations: 18
- new4phil.pv: (0.07s) Objects: 85, Morphisms: 200, Relations: 132
- new7phil.pv: 147.36s; 81 Mo; (about one million transitions in a standard model) Objects: 2467, Morphisms: 10094, Relations: 15484
- new8phil.pv: 320.02s; 121Mo; (about 10 million transitions in a standard interleaving model) Objects: 3214, Morphisms: 14282, Relations: 24396

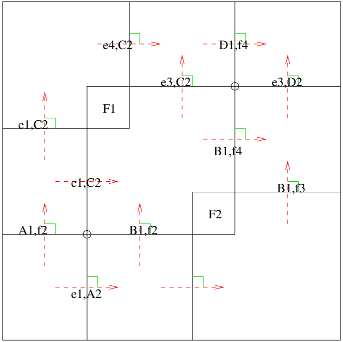
Induction step



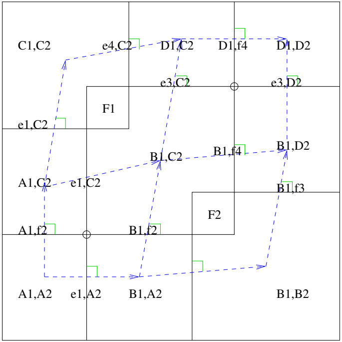
$$(A = Pa.Va.Pb.Vb \mid B = Pb.Vb.Pa.Va)$$

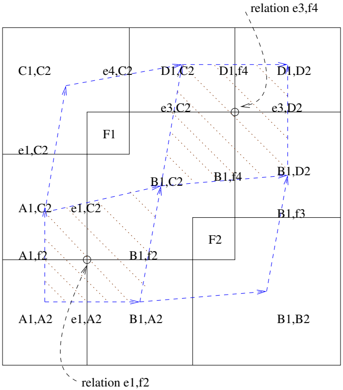
Induction step





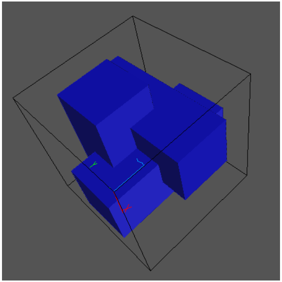
Induction step

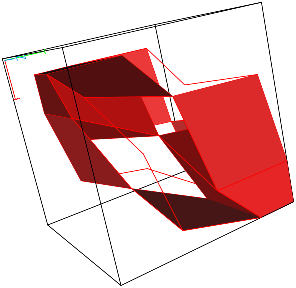




(duality)

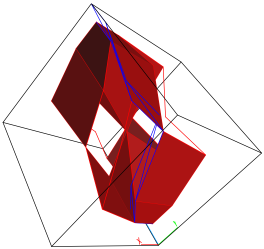
Component category



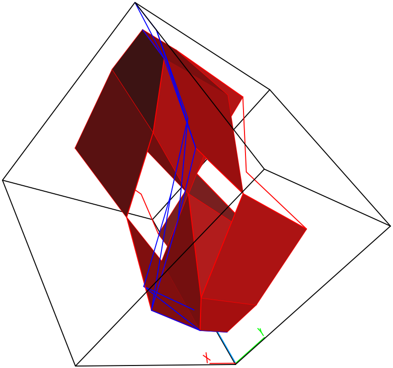


Example

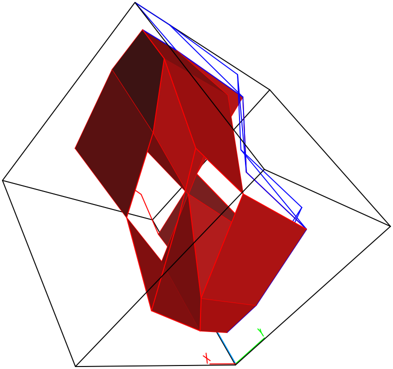
In the case of the 3 philosophers, we find 6 maximal legal paths (all possibles serializations in fact):



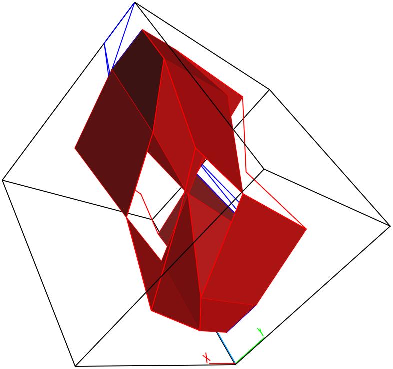
Path 2



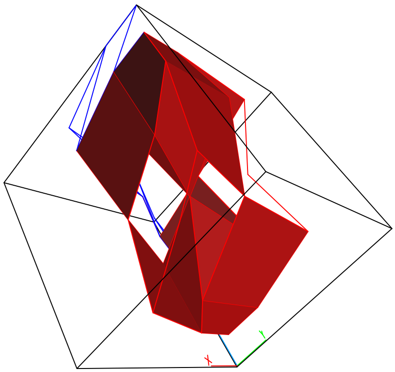
Path 3



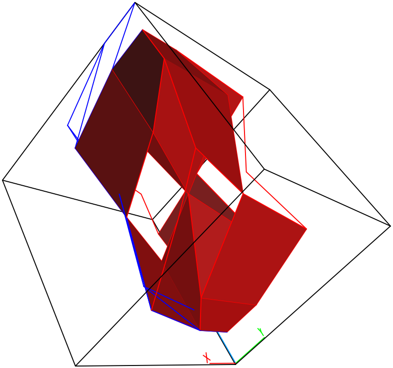
Path 4



Path 5



Path 6



Syntactic representation of a path...

...or how to choose a "good" lift:

- From a continuous path, we want to get back to a "discrete" path
- This "discrete" path is an interleaving path corresponding to this idealized execution
- This can then be analyzed by any standard sequential analyzer

For this...

We remark that:

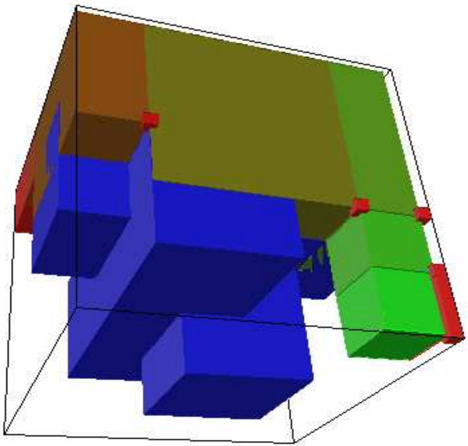
- (1) Every component has a trivial π_1
- (2) There exists a path (unique) from the minimum (or infimum in general) from a component to the minimum of the next component (essentially by the lifting property) (modulo some technicalities I am hiding here...)

For this...

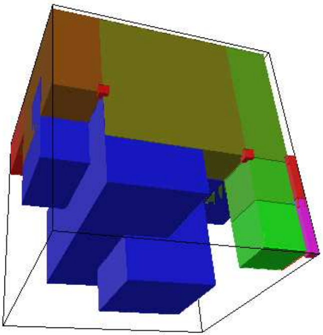
Given the morphisms of the component category, we compute:

- (a) the minimum of the components (i.e. of hyperrectangles minus the forbidden region)
- (b) the program comprising the possible executions between the minimum of a component, and the minimum of the next
- (c) we use the interleaving semantics for finding *1* path in this program (using 1, in a very economical manner)

Discretization of paths



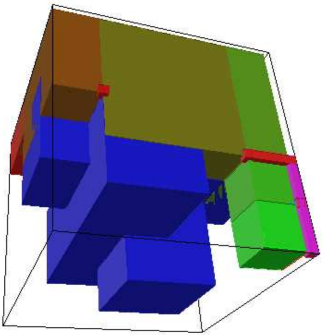
Discretization of paths



0 | 0 | P(a)

In context #sem c 0, #sem b 1, #sem a 1.

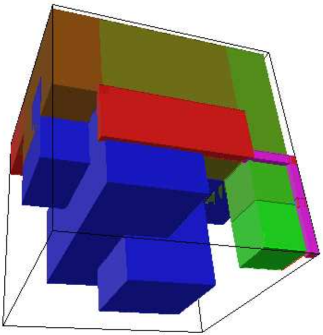
Discretization of paths



0 | P(b) | 0

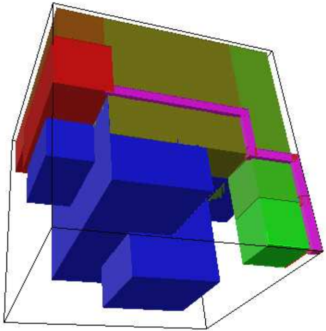
In context #sem c 0, #sem b 1, #sem a 0.

Discretization of paths



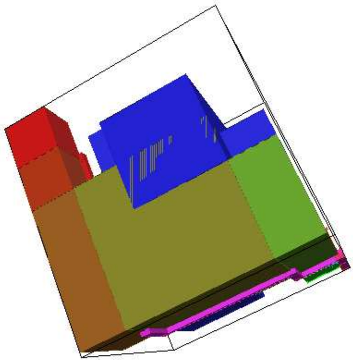
0 | P(c).V(b).V(c) | V(c)
In context #sem c 0, #sem b 0, #sem a 0.

Discretization of paths



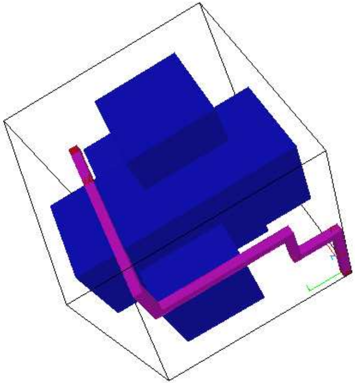
P(a).P(b).V(a) | 0 | V(a)
In context #sem c 1, #sem b 1, #sem a 0.

Discretization of paths



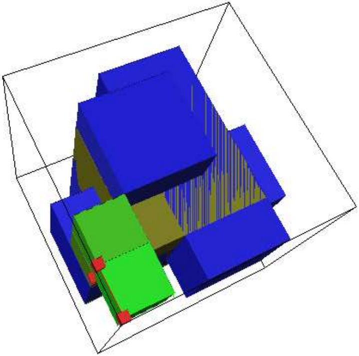
V(b) | 0 | 0
In context #sem c 1, #sem b 0, #sem a 1.

Discretisation of paths



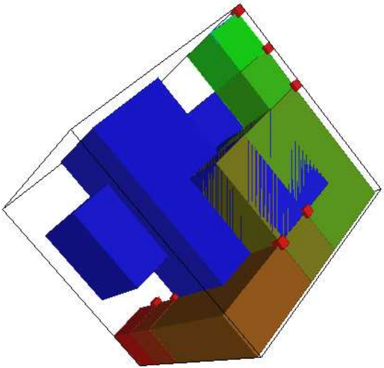
3P(c);3P(a);2P(b);3V(c);2P(c);2V(b);
2V(c);3V(a);1P(a);1P(b);1V(a);1V(b);

Discretisation of paths (2 - deadlock)



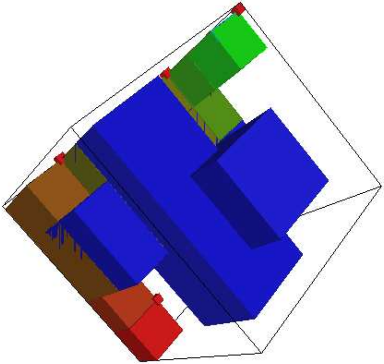
1P(a);3P(c);2P(b);

Discretisation of paths (3)



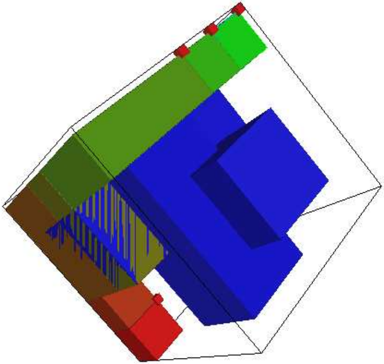
2P(b);2P(c);2V(b);2V(c);3P(c);3P(a);
3V(c);3V(a);1P(a);1P(b);1V(a);1V(b);

Discretisation of paths (4)



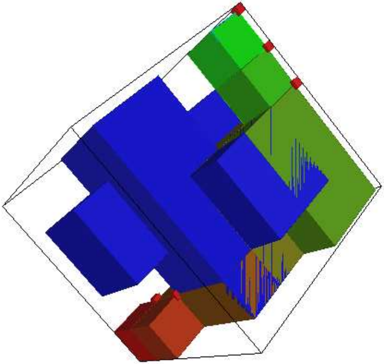
1P(a);1P(b);3P(c);1V(a);3P(a);3V(c);
3V(a);1V(b);2P(b);2P(c);2V(b);2V(c);

Discretisation of paths (5)



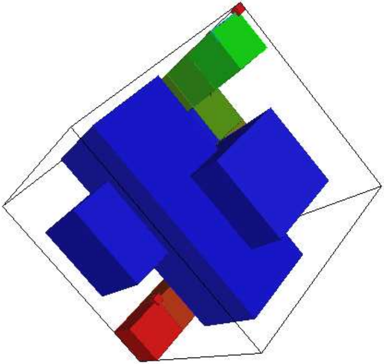
3P(c);3P(a);3V(c);3V(a);1P(a);1P(b);
1V(a);1V(b);2P(b);2P(c);2V(b);2V(c);

Discretisation of paths (6)



2P(b);2P(c);1P(a);2V(b);1P(b);1V(a);
2V(c);3P(c);3P(a);3V(c);3V(a);1V(b);

Discretisation of paths (7)



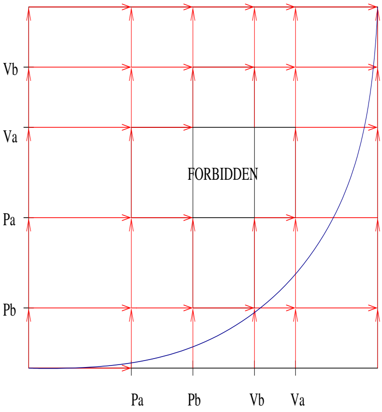
1P(a);1P(b);1V(a);1V(b);2P(b);2P(c);
2V(b);2V(c);3P(c);3P(a);3V(c);3V(a);

Correspondences
(almost)

Model [discrete]	combinatorial complex
Model [continuous]	topological space
Relation discrete/continuous	geometric realisation
Parallel composition	product
Action refinement	subdivision
Compositionality	Seifert/van Kampen
Deadlocks/reachability	connected components
Scheduling properties	fundamental group
Observational equivalence	homotopy equivalence (weak/strong)
Computable properties	topological invariants (homology etc.)

Relation with standard operational semantics?

Towards (pre-)cubical sets



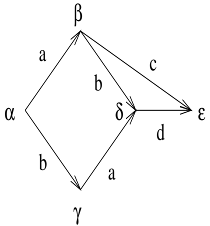
Reminder: Transition systems

A Transition System is a structure

$$(S, i, L, Tran)$$

- S set of states, and $i \in S$ is the initial state,
- L set of labels, and,
- $Tran \subseteq S \times L \times S$ is the transition relation

One classical modeling of computer systems (semantics) through TS.



Categorically
See G. Winskel et al.

- It forms a category with morphisms being simulations (Glynn Winskel et al.).
- A transition system T_1 simulates a transition system T_0 means that if T_0 can fire a in a given context, then T_1 can fire a as well in a similar context.
- A morphism $f : T_0 \to T_1$ defines the way that states and transitions of T_0 are related to the states and transitions of T_1.

Morphisms

...or a first approximation

Let $T_0 = (S_0, i_0, L_0, Tran_0)$ and $T_1 = (S_1, i_1, L_1, Tran_1)$ be two transition systems. A morphism $f : T_0 \rightarrow T_1$ is a pair $f = (\sigma, \lambda)$ where,

- $\sigma : S_0 \rightarrow S_1$,
- $\lambda : L_0 \rightarrow L_1$ is such that $\sigma(i_0) = i_1$ and

$$(s, a, s') \in Tran_0 \Rightarrow (\sigma(s), \lambda(a), \sigma(s')) \in Tran_1$$

Reference: Different definition from that of G. Winskel et al. (no "partial morphisms").

On the discrete side

Precubical sets:

- family of sets $\{K_n / n \geq 0\}$ with face maps $\partial_i^\alpha : K_n \rightarrow K_{n-1}$ $(0 \leq i \leq n-1,\ \alpha = 0,1)$ satisfying the following commutation rules:

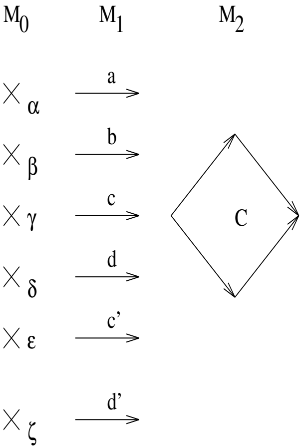
$$\partial_i^\alpha \partial_j^\beta \quad = \quad \partial_{j-1}^\beta \partial_i^\alpha \quad (i < j)$$

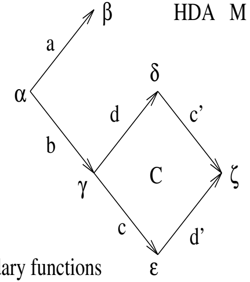
- morphisms: Let K and L be two precubical sets. Then $f = (f_n)_{n \in \mathbb{N}}$ is a morphism of precubical sets from K to L if for all $n \in \mathbb{N}$, f_n is a function from K_n to L_n such that:

$$f_n \circ \partial_i^\alpha = \partial^{\alpha_i} \circ f_{n+1}$$

(for all i, $0 \leq i \leq n$)

Geometrically





Boundary functions

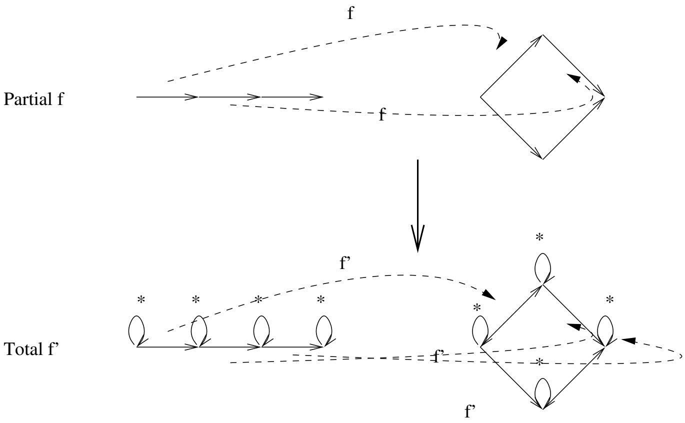
	a	b	c	d	c'	d'	C
d^0_0 =	α	α	γ	γ	δ	ε	c
d^0_1 =	-	-	-	-	-	-	d
d^1_0 =	β	γ	ε	δ	ζ	ζ	c'
d^1_1 =	-	-	-	-	-	-	d'

Partial morphisms

Partial morphisms allow T_1 not to fire any action when T_0 fires one. Let $T_0 = (S_0, i_0, L_0, Tran_0)$ and $T_1 = (S_1, i_1, L_1, Tran_1)$ be two transition systems. A partial morphism $f : T_0 \to T_1$ is a pair $f = (\sigma, \lambda)$ where,

- $\sigma : S_0 \to S_1$,
- $\lambda : L_0 \to L_1$ is a *partial* function. (σ, λ) is such that
  - $\sigma(i_0) = i_1$,
  - $(s, a, s') \in Tran_0$ and $\lambda(a)$ is defined implies $(\sigma(s), \lambda(a), \sigma(s')) \in Tran_1$. Otherwise, if $\lambda(a)$ is not defined, then $\sigma(s) = \sigma(s')$.

Adding up idle transitions
Example



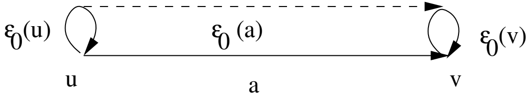
Cubical sets

Adding up more general degeneracies

A cubical set K is a family of sets $\{K_n / n \geq 0\}$ with face maps $\partial_i^\alpha : K_n \to K_{n-1}$ $(0 \leq i \leq n-1,\ \alpha = 0, 1)$ and degeneracy maps $\epsilon_i : K_{n-1} \to K_n$ $(0 \leq i \leq n-1)$ satisfying the cubical relations:

$$
\begin{array}{lcll}
\partial_i^\alpha \partial_j^\beta & = & \partial_{j-1}^\beta \partial_i^\alpha & (i < j) \\
\epsilon_i \epsilon_j & = & \epsilon_{j+1} \epsilon_i & (i \leq j) \\
\partial_i^\alpha \epsilon_j & = & \left\{ \begin{array}{ll} \epsilon_{j-1} \partial_i^\alpha & (i < j) \\ \epsilon_j \partial_{i-1}^\alpha & (i > j) \\ Id & (i = j) \end{array} \right. &
\end{array}
$$

Example of degeneracy
in dimension 1 and 2



Morphisms

Let K and L be two cubical sets. Then f is a morphism of cubical sets from K to L if it maps n-cubes to n-cubes, and,

$$f_{n-1} \circ \partial_j^k = \partial_j^k \circ f_n$$

$$f_{n+1} \circ \epsilon_j = \epsilon_j \circ f_n$$

(for all $n \in \mathbb{N}$, $0 \leq i \leq n$).

$\longrightarrow$ Presheaf category $\Upsilon = \square^{op} Set$
$\longrightarrow$ Degeneracies are like idle transitions (but somehow they are "freely generated")!
$\longrightarrow$ Nice adjunctions (like the ones between models of concurrency by G. Winskel et al.) with transition systems, transition systems with independence etc.

Relationship with continuous models

Let $\square_n$ be the standard cube in $\mathbb{R}^n$ $(n \geq 0)$,

$$\square_n = \{(t_1, \ldots, t_n) | \forall i, 0 \leq t_i \leq 1\}$$

$$\square_0 = \{0\}$$

and let $\delta_i^k : \square_{n-1} \to \square_n$, $1 \leq i \leq n$, $k = 1, 2$, be the continuous functions $(n \geq 1)$,

$$\begin{array}{ccc} \square_n & \xleftarrow{\delta_i^0} & \square_{n-1} \\ \uparrow \delta_i^1 & & \\ \square_{n-1} & & \end{array}$$

$$\delta_i^k(t_1, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, k, t_i, \ldots, t_{n-1})$$

Geometric realization

- $\mathbf{R}(M) = \coprod_n M_n \times \square_n$. The sets M_n have the discrete topology and $\square_n$ is topologized as a subset of $\mathbb{R}^n$ with the standard topology thus $\mathbf{R}(M)$ is a topological space with the disjoint sum topology.
- Let $\equiv$ be the equivalence relation induced by the identities:

$$\forall k, i, n, \forall x \in M_{n+1}, \forall t \in \square_n, n \geq 0, (\partial_i^k(x), t) \equiv (x, \delta_i^k(t))$$

Geometric realization

- Let $\mid M \mid = \mathbf{R}(M)/\equiv$ have the quotient topology.
- Let $p \in \mid M \mid$, then there is a minimal cube in the subdivision of $\mid M \mid$ containing p, namely the unique cube $x \times \square_k$ which has p in the interior. We call x the *carrier* of p.

Geometric realization

The open star of a point $p \in \mid M \mid$ with respect to the subdivision M is

$$St(p, M) = \{q \in \mid M \mid \, | carrier(p) \text{ is a face of } carrier(q)\}$$

For a cube $x \in M_n$ we define the open star

$$St(x, M) = \{y \in M | \exists (k_1, l_1), \dots, (k_i, l_i), \partial_{l_1}^{k_1} \dots \partial_{l_i}^{k_i}(y) = x\}$$

Local partial order

- for p in $\mid M \mid$: representative (x,t) where $x = carrier(p)$ and $t \in int(\square)_n$ (for some n).
- First, we can partially order any $(y,u) \in U^x$ (interior of $St(x,M)$) with any (x,t). $x = \partial^{k_1}_{l_1} \ldots \partial^{k_i}_{l_i}(y)$ because $y \in St(x,M)$, so (x,t) is identified with $(y, \delta^{k_i}_{l_i} \ldots \delta^{k_1}_{l_1}(t))$.
- We set,

$$(x,t) \leq_{U^x} (y,u) \text{ if } \delta^{k_i}_{l_i} \ldots \delta^{k_1}_{l_1}(t) \leq u \text{ in } \square_{n+i}$$

$$(y,u) \leq_{U^x} (x,t) \text{ if } \delta^{k_i}_{l_i} \ldots \delta^{k_1}_{l_1}(t) \geq u \text{ in } \square_{n+i}$$

New local partial order

- Let x be a vertex of M and let (z,v) be a point in U^x with carrier z.
- We say $(z,v) \leq_x (y,u)$ if there exists b in the star of x and t such that
$$(z,v) \leq_{U^b} (b,t) \leq_{U^b} (y,u)$$

Theorem

The geometric realization of a non self-linked cubical complex M defines a locally po-space with,

- covering being $\{St(x,M)/x \in M_0\}$ and,
- local partial order $\leq_x$ on $St(x,M)$.

Examples

Let M be the cubical complex

$$M_2 = A, B, C, D, M_1 = a, b, c, d, e, f, g, h, M_0 = p, q, r, s$$

$$d_1^0 A = d_1^0 C = a,\ d_2^0 A = d_2^0 B = b,\ d_1^0 B = d_1^0 D = c,\ d_2^0 D = d_2^0 C = d$$
$$d_1^1 A = d_1^1 D = e,\ d_2^1 A = d_2^1 D = f,\ d_1^1 B = d_1^1 C = g,\ d_2^1 B = d_2^1 C = h$$

$$d_1^0 a = d_1^0 b = d_1^0 c = d_1^0 d = p,\ d_1^1 a = d_1^1 c = d_1^0 f = d_1^0 h = q,$$
$$d_1^1 b = d_1^1 d = d_1^0 e = d_1^0 g = r,\ d_1^1 e = d_1^1 f = d_1^1 g = d_1^1 h = s$$

Then $\mid M \mid$ is the projective plane, and one can give cubical models for projective spaces of all dimensions in the same way.

Singular cube functor

- Standard n-cube is $\square_n$ with partial order $\leq_{\square_n}$ pointwise ordering in $\mathbb{R}^n$.
- For $n \in \mathbb{N}$, $S(M)_n$ is the set of dimaps $\square_n \to M$ together with the operators ∂_I^k such that $\partial_I^k(f) = f \circ \delta_I^k$. This gives $S(M)$ the structure of a precubical complex.
- S is a functor, right-adjoint to the geometric realization functor.

Preservation of homotopic properties

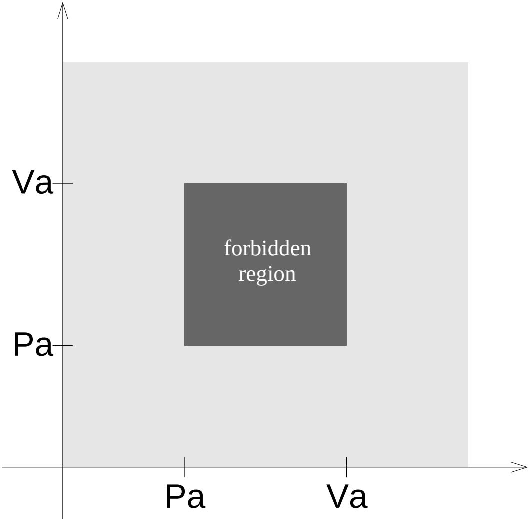
- All dipaths in the geometric realization of a finite precubical set are homotopic to some dipath on the 1-skeleton (i.e. to the realization of a combinatorial dipath)
- Two dipaths in the geometric realization are homotopic iff any of their “1-skeleton” representative are (combinatorially) homotopic (Lisbeth Fajstrup).

If we want to go further

- π_0!
- (Strong) dihomotopy equivalence
- Higher-dimensional fundamental categories (difficult)
- Model-category theoretic explanation of dihomotopy (P. Gaucher, or K. Hess/K. Worytkiewicz/P. Bubenik, T. Kahl, S. Krishnan) or "algebraic homotopical explanation" (see M. Grandis)

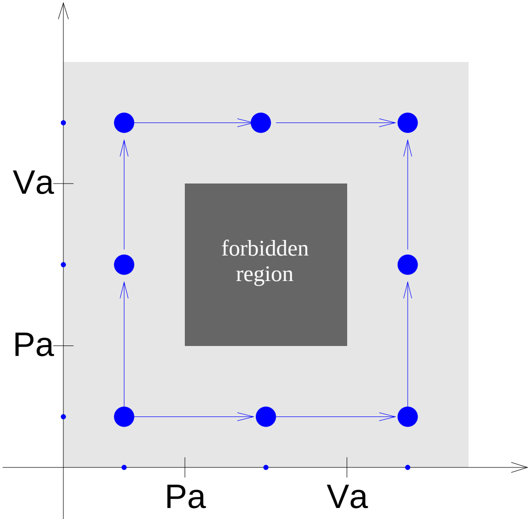
PaVa|PaVa

Dijkstra 68, *Pratt/van Glabbeek* 91, *Goubault* 92



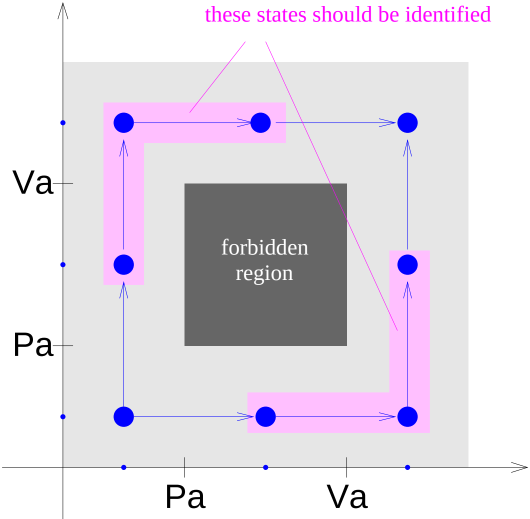
PaVa|PaVa

classical discrete model



PaVa|PaVa

reduction



Pospace $\overrightarrow{X}$

Eilenberg 41, *Nachbin* 48 65, *Johnstone* 82

1. a topological space X,
2. a partial order $\sqsubseteq$ over $|X|$ whose graph is closed in $X \times X$.

Lemma: for any $x \in \overrightarrow{X}$, $\{y \in X | x \sqsubseteq y\}$ (denoted $\uparrow x$) is closed in X.

Morphisms of pospaces from $\overrightarrow{X}$ to $\overrightarrow{Y}$

A map $f : |X| \longrightarrow |Y|$ inducing:

1. a continuous map from X to Y and
2. an increasing map from $(|X|, \sqsubseteq_X)$ to $(|Y|, \sqsubseteq_Y)$.

Hence the category of pospaces denoted: PoSpc.

Usual Pospaces

some common examples

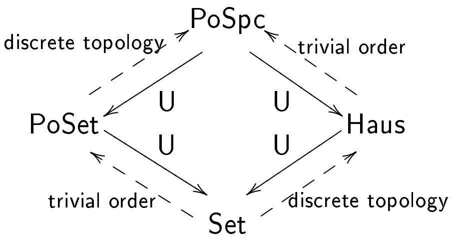
1. directed real line $\mathbb{R}$ with its classical topology and order ($\overrightarrow{\mathbb{R}}$),
2. directed unit segment $[0,1]$ with the structure induced by $\overrightarrow{\mathbb{R}}$ ($\overrightarrow{[0,1]}$),
3. any morphism of PoSpc from $\overrightarrow{[0,1]}$ to $\overrightarrow{X}$ is called a directed path on $\overrightarrow{X}$. Formelly, the set of directed paths on $\overrightarrow{X}$ is $\mathrm{PoSpc}\left[\overrightarrow{[0,1]}, \overrightarrow{X}\right]$, also denoted $d\overrightarrow{X}$.

The underlying topological space of a pospace

is separated in the senes of *Hausdorff*

A general fact of topology is that X is *Hausdorff* separated iff $\Delta_X := \{(x,x)|x \in |X|\}$ is closed in $X \times X$ which is the case since $\Delta_X = \sqsubseteq \cap \sqsupseteq$

Forgetful functors PoSpc



Categorical properties of PoSpc

analogy between Top and PoSpc

1. complete and cocomplete,
2. symmetric monoidal closed,
3. compact pospaces is complete, cocomplete and admits $\overrightarrow{[0,1]}$ as a cogenerator,
4. the full sub-category of compactly generated pospaces is reflective in PoSpc and cartesian closed.

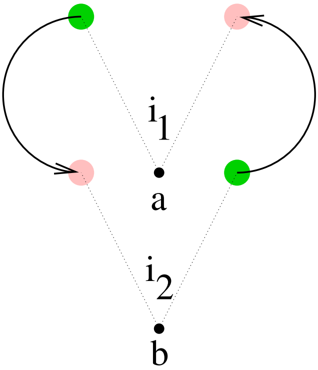
Cocompleteness of PoSpc

sketch of proof

1. Prove that the category RSpc admits quotients.
2. Use quotients of RSpc to prove its cocompletness.
3. Use quotients of RSpc to construct the reflect of any object of RSpc in PoSpc.
4. It is a general fact that any reflective subcategory of a cocomplete category is cocomplete, hence PoSpc is cocomplete.

A pushout in PoSpc (1)

the directed circle in PoSpc squashed to a point



a and b are not ordered

A pushout in PoSpc (2)

the directed circle in PoSpc squashed to a point

$i_1 : \{a, b\} \to \overrightarrow{[0, 1]}$; $i_1(a) = 0$; $i_1(b) = 1$
$i_1 : \{a, b\} \to \overrightarrow{[0, 1]}$; $i_2(a) = 1$; $i_2(b) = 0$
suppose $f, g : \{a, b\} \to \overrightarrow{[0, 1]}$ with $f \circ i_1 = g \circ i_2$,
we have $f(i_1(a)) = g(i_2(a))$ i.e. $f(0) = g(1)$ and the same way $g(0) = f(1)$.
Hence $f(0) \sqsubseteq f(1) = g(0) \sqsubseteq g(1) = f(0) \Longrightarrow$
$f(0) = f(1) = g(0) = g(1)$
and then f and g are constant and equal.

Directed homotopy on $\overrightarrow{X}$ from α to β

Grandis 01, *Fajstrup/Raussen/Goubault* 98...

A morphism h defined on $\overrightarrow{[0,1]} \times \overrightarrow{[0,1]}$ with values in $\overrightarrow{X}$ such that $U(h)$ be a classic homotopy from $U(\alpha)$ to $U(\beta)$.
We denote $\sim_{\overrightarrow{X}}$ the symmetric and transitive closure of

$$\left\{(\alpha,\beta) \in d\overrightarrow{X} \times d\overrightarrow{X} \middle| \text{il existe une homotopie dirigée de } \alpha \text{ vers } \beta\right\}.$$

Two dipaths α and β are said dihomotopic when $\alpha \sim_{\overrightarrow{X}} \beta$.

Directed Homotopy *vs* classic homotopy

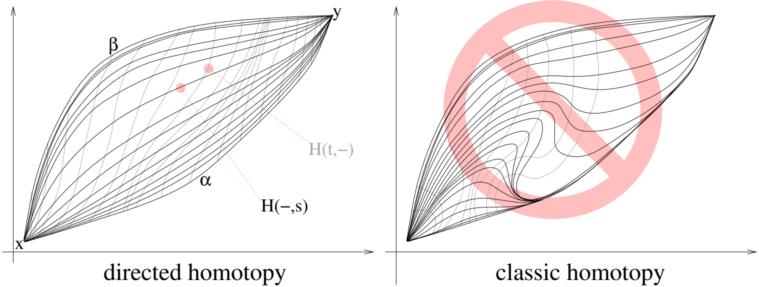


Image of a dipath

Singular facts about pospaces

1. The image of a dipath α on a pospace $\overrightarrow{X}$ is either isomorphic (in PoSpc) to $\overrightarrow{[0,1]}$ or $\{\bullet\}$.
2. Two dipaths sharing the same image are dihomotopic.
3. There is no directed *Peano* curve.

Fundamental category of a pospace $\overrightarrow{X}$

denoted $\overrightarrow{\pi_1}(\overrightarrow{X})$

1. objects: the elements de $|X|$,
2. morphisms from x to y: the set of $\sim_{\overrightarrow{X}}$-equivalence classes of

$$\left\{\alpha \in d\overrightarrow{X} \middle| \alpha(0) = x \text{ et } \alpha(1) = y\right\}$$

Loop-free categories
play the role of the groupoids

A (small) category $\mathcal{C}$ such that for any objects x and y of $\mathcal{C}$, if $\mathcal{C}[x,y] \neq \emptyset$ and $\mathcal{C}[y,x] \neq \emptyset$ then $x = y$ and $\mathcal{C}[x,x] = \{id_x\}$.
We denote LfCat the full subcategory of Cat whose objects are the small loop-free category.

1. LfCat is cartesian closed and reflective in Cat.
2. The fundamental category of a pospace is loop-free, hence the functor

$$\mathsf{PoSpc} \xrightarrow{\vec{\pi}_1} \mathsf{LfCat}$$

$\overrightarrow{\pi_1}(\overrightarrow{X})$ is loop-free

proof

A morphism of $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the $\sim_{\overrightarrow{X}}$-equivalence class of some dipath α from x to y, hence $x \sqsubseteq y$; suppose that $\overrightarrow{\pi_1}(\overrightarrow{X})[y,x] \neq \emptyset$, then we also have $y \sqsubseteq x$ and then $x = y$. Further, if α is a dipath from x to x, then for any $t \in [0,1]$, we have
$x = \alpha(0) \sqsubseteq \alpha(t) \sqsubseteq \alpha(1) = x$ i.e. α is constant.

Pureness
of a collection of morphisms

A collection Σ of morphisms of a category $\mathcal{C}$ is said pure in $\mathcal{C}$ when for all morphisms f_2, f_1 of $\mathcal{C}$, if $f_2 \circ f_1 \in \Sigma$ then f_2, $f_1 \in \Sigma$.

Endomorphisms, sections and retractions

of a loop-free category $\mathcal{C}$

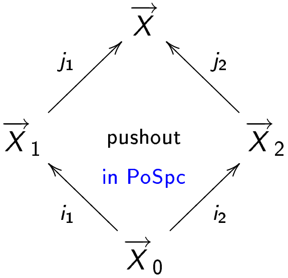
Suppose that $f_2 \circ f_1 = id_x$, the source and the target of f_1 and f_2 is x and then $f_1 = f_2 = id_x$. Hence the only isomorphisms of $\mathcal{C}$ are its identities and the collection of identities of $\mathcal{C}$ is pure in $\mathcal{C}$. In particular, the limits and colimits in a loop-free category are strictly unique and not only up to isomorphism.

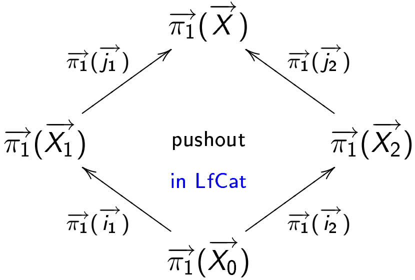
Suppose that the source and the target of f_1 and f_2 is x, then $f_1 = f_2 = id_x$. Hence the collection of endomorphisms of $\mathcal{C}$ is its collection of identities.

Directed *Van Kampen* Theorem

Grandis 01, *Goubault* 01

$$X := \mathring{X}_1 \cup \mathring{X}_2 \text{ and } X_0 := \mathring{X}_1 \cap \mathring{X}_2$$



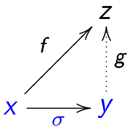


Yoneda morphism

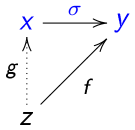
axiomatizing the preservation of the future and the past (1)

Let $\mathcal{C}$ be a small category. A *Yoneda* morphism σ is an element of $\mathcal{C}[x,y]$ such that for all object z of $\mathcal{C}$,

future if $\mathcal{C}[y,z] \neq \emptyset$ then for all $f \in \mathcal{C}[x,z]$, there is a unique $g \in \mathcal{C}[y,z]$ such that



past if $\mathcal{C}[z,x] \neq \emptyset$ then for all $f \in \mathcal{C}[z,y]$, there is a unique $g \in \mathcal{C}[z,x]$ such that



Some properties of *Yoneda* morphisms

statements

- *Yoneda* morphisms compose
- if $\mathcal{C}$ is loop-free and $\sigma \in \mathcal{C}[x,y]$ is a *Yoneda* morphism, then $\mathcal{C}[x,y] = \{\sigma\}$
- any *Yoneda* morphism is a monomorphism and an epimorphism

Some properties of *Yoneda* morphisms

proofs

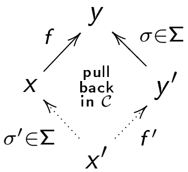
- *Yoneda* morphisms compose since injective maps compose as well as surjective ones.
- If σ is a *Yoneda* morphism, then the map $\gamma \in \mathcal{C}[y,y] \mapsto \gamma \circ \sigma \in \mathcal{C}[x,y]$ is a bijection; since $\mathcal{C}$ is loop-free, $\mathcal{C}[y,y] = \{id_y\}$, hence the result.
- A *Yoneda* morphism σ is an epimorphism since $\gamma \in \mathcal{C}[y,z] \mapsto \gamma \circ \sigma \in \mathcal{C}[x,z]$ is a bijection as soon as $\mathcal{C}[y,z] \neq \emptyset$, the same we prove that σ is a monomorphism.

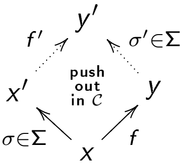
Yoneda system of a small category $\mathcal{C}$

axiomatizing the preservation of the future and the past (2)

A collection Σ of morphisms of $\mathcal{C}$ such that:

1. Σ is stable under composition,
2. Σ contains all the isomorphisms of $\mathcal{C}$,
3. all the elements of Σ are *Yoneda* morphisms and
4. Σ is stable under change and cochange of base.





Pureness of *Yoneda* system Σ

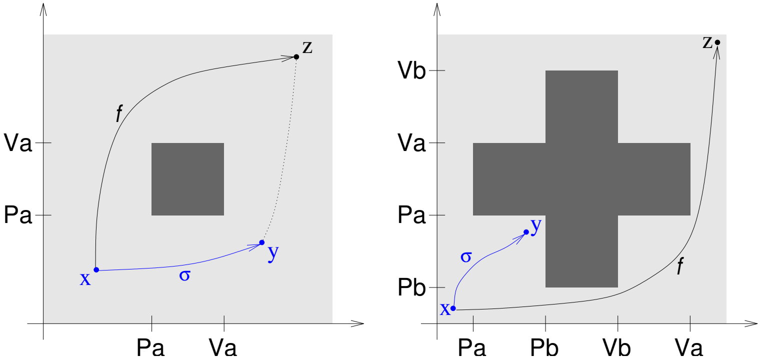
of a loop-free category $\mathcal{C}$

Suppose $f_2 \circ f_1 = \sigma \in \Sigma$, by cochange of base, we have the left hand side pushout below.

Still, f_1 in an epimorphism since so is σ; it follows that the right hand square above is also a pushout. By the strict uniqueness of the colimits of a loop-free category, f_1' is an identity and $f_2 = \sigma' \in \Sigma$. Using change of base, we prove that $f_1 \in \Sigma$ too.

Examples

of morphisms which do not belong to a *Yoneda* system



Locale of *Yoneda* systems

pointless topology on a small loop-free category

The collection of *Yoneda* systems of a small loop-free category, ordered by inclusion, forms a locale whose greatest and least elements are respectively denoted $\Sigma_\top$ and $\Sigma_\bot$. Besides $\Sigma_\bot$ is the collection of identities of $\mathcal{C}$.

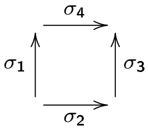
Σ-zigzags and Σ-components
of a loop-free category $\mathcal{C}$

A Σ-zigzag between two objects x and y of $\mathcal{C}$ is a finite sequence $(\sigma_n, \ldots, \sigma_0)$ $(n \in \mathbb{N})$ of morphisms of Σ such that there is a finite sequence $(z_0, \ldots, z_{n+1})$ of objects of $\mathcal{C}$ such that $z_0 = x$, $z_{n+1} = y$ and for all $k \in \{0, \ldots, n\}$, $\sigma_k \in \mathcal{C}[x_k, x_{k+1}] \cup \mathcal{C}[x_{k+1}, x_k]$.
Then we say that x and y are Σ related: thus we have an equivalence relation since Σ contains identities and is stable under composition.
The equivalence classes of this relation are called the Σ-components.

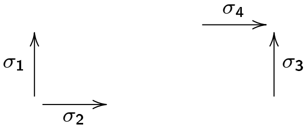
Fundamental theorem of the Σ-components

$\mathcal{C}$ loop-free and Σ *Yoneda* system of $\mathcal{C}$

Any Σ-component X of $\mathcal{C}$ ordered by $x \sqsubseteq y$ when $\mathcal{C}[x,y] \neq \emptyset$ is a lattice. Further given x, $y \in X$, $\mathcal{C}[x,y]$ is a singleton whose only element belongs to Σ. Finally, any square of arrows of Σ



is both at once the pushout of the left hand side diagram and the pullback of the right hand side diagram below



lattice = the l.u.b. and the g.l.b. of any pair of elements of X exists

Components of compact pospaces

statement

- If $\overrightarrow{K}$ is a compact pospace such that any pair of element of K has an upper/lower bound, then $\overrightarrow{K}$ has a greatest/least element.
- If $\overrightarrow{K}$ is a compact pospace, then any component of $\overrightarrow{\pi_1}(\overrightarrow{K})$ has both a greatest lower bound and an least upper bound in $(|K|, \sqsubseteq)$.

Components of compact pospaces

proof

- Suppose $\overrightarrow{K}$ does not have a greatest element, then $K = \bigcup_{x \in K} \left(\uparrow x \right)^c$. Still, for K is compact and $\left(\uparrow x \right)^c$ is open, we have $K = \bigcup_{x \in F} \left(\uparrow x \right)^c$ for some finite $F \subseteq K$, but F has an upper bound $\top$ in K and thus $K = \left(\uparrow \top \right)^c$, which is a contradiction.
- C is a lattice, then, for K is compact we know [*Nachbin*] that the topological closure of $\downarrow C$ is a $\vee$-lattice and a compact subset of K so we can apply the first point.

Category of components

play the role of the collection of arcwise connected components

The category of components of a small loop-free category $\mathcal{C}$ is then quotient category $\mathcal{C}/_{\Sigma_\top}$.

Fundamental theorem

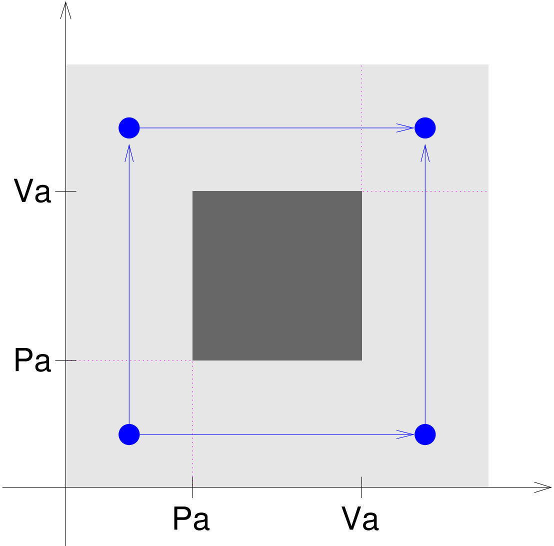
fractions *vs* quotients

Let $\mathcal{C}$ be a small loop-free category and Σ a *Yoneda* system of $\mathcal{C}$, then

1. the collection Σ is pure in $\mathcal{C}$,
2. the small category $\mathcal{C}/_\Sigma$ is loop-free,
3. the small categories $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}/_\Sigma$ are equivalent and
4. the category $\mathcal{C}[\Sigma^{-1}]$ is fibered over $\mathcal{C}/_\Sigma$.

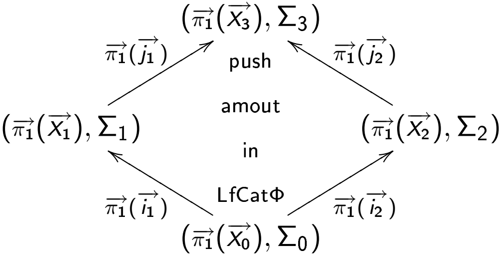
extension and improvement of *Components of the Fundamental Category* - APCS 04

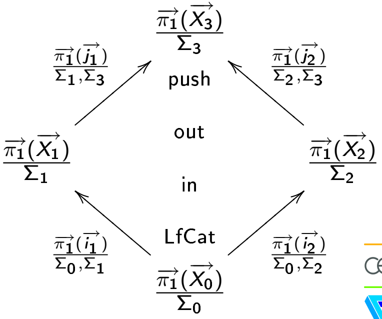
Example



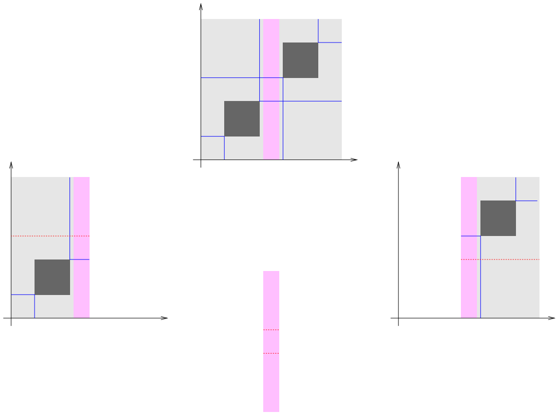
Van Kampen theorem
for categories of components (1)

Let Σ_1 and Σ_2 be two *Yoneda* systems of $\overrightarrow{\pi_1}(\overrightarrow{X_1})$ and $\overrightarrow{\pi_1}(\overrightarrow{X_2})$.
Suppose that $\Sigma_3 := \overrightarrow{\pi_1}(\overrightarrow{j_1})(\Sigma_1) \uplus \overrightarrow{\pi_1}(\overrightarrow{j_2})(\Sigma_2)$ is a *Yoneda* system of $\overrightarrow{\pi_1}(\overrightarrow{X_3})$ and that $\overrightarrow{\pi_1}(\overrightarrow{i_1})(\Sigma_0) \subseteq \Sigma_1$ et $\overrightarrow{\pi_1}(\overrightarrow{i_2})(\Sigma_0) \subseteq \Sigma_2$, then



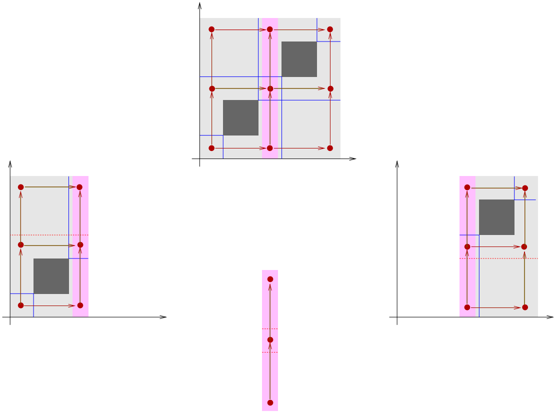


Van Kampen theorem
for categories of components: subdivisions (2)



Van Kampen theorem

for categories of components: subdivisions (3)



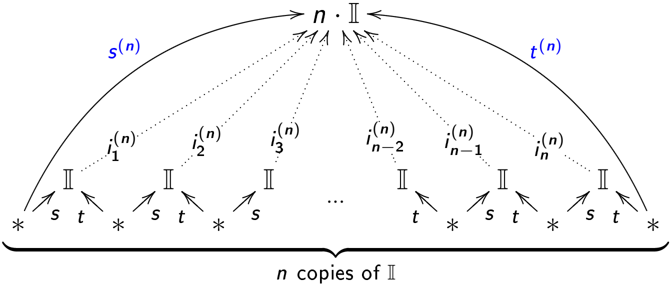
Generic segment of C
axiomatizing the notion of *Moore* paths (1)

A generic segment of C is a triple $(\mathbb{I}, s, t)$ where $\mathbb{I}$ is an object of C and s, t two points of $\mathbb{I}$ such that:

1. for any automorphism ϕ of $\mathbb{I}$ we have

$$\{\phi \circ s, \phi \circ t\} = \{s, t\}$$

2. and for any $n \in \mathbb{N}$ we have the colimit



Directed generic segment

axiomatization of the notion of direction

- A generic segment $(\mathbb{I}, s, t)$ is said directed when for any automorphism ϕ of $\mathbb{I}$, we have $\phi \circ s = s$ and $\phi \circ t = t$.
- Any automorphism ϕ of $\mathbb{I}$ such that $\phi \circ s = t$ and $\phi \circ t = s$ is called an inversion of (the) time (flow)
- In PoSpc, the generic segment $\overrightarrow{[0,1]}$ is directed while the generic segment $([0,1], =)$ does not.

the map $t \mapsto 1 - t$ is an inversion of time

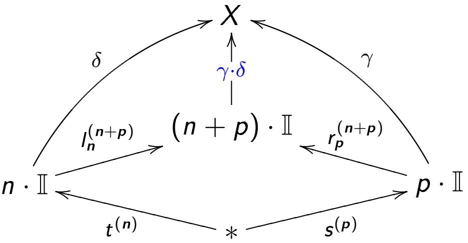
Category of paths on an object X of C

axiomatization of the notion of *Moore* path (2)

The objects of this category, denoted $\Gamma(X)$, are the points of X and its morphisms, called the paths on X, are the elements of

$$\bigcup_{n\in\mathbb{N}} \mathsf{C}[n\cdot\mathbb{I}, X],$$

the source and the target of $\gamma \in \mathsf{C}[n\cdot\mathbb{I}, X]$ are $\gamma\circ s^{(n)}$ and $\gamma\circ t^{(n)}$; the concatenation being given by the push-out:



Homotopic congruence over C

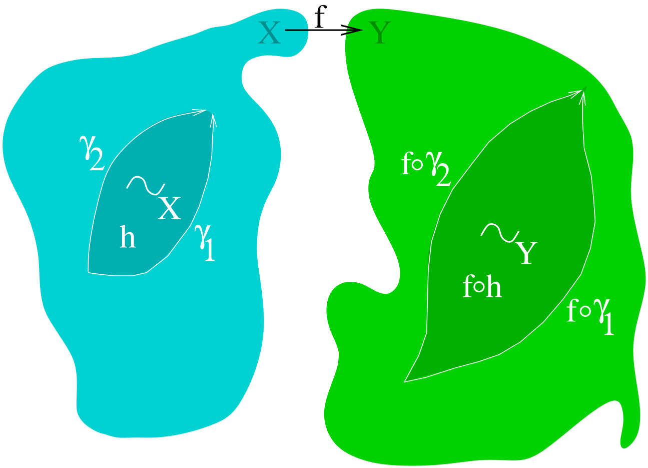
axiomatization of the notion of (di)homotopic (di)paths

A path $\gamma \in \mathcal{C}\left[n \cdot \mathbb{I}, X\right]$ is said constant when it can be written $\gamma = p \circ \mu$ where p is a point of X, it is the value of γ.
A homotopic congruence on C is defined by, for each object X of C, a congruence $\sim_X$ on the category of paths on X, such that for all paths γ_1 and γ_2 on X,

1. if γ_1 and γ_2 are constant with the same value, then $\gamma_1 \sim_X \gamma_2$,
2. if $\gamma_1 \sim_X \gamma_2$, then
   1. γ_1 and γ_2 share the same extremities and
   2. for all morphism f of C from X to Y we have $f \circ \gamma_1 \sim_Y f \circ \gamma_2$.

Homotopic congruence

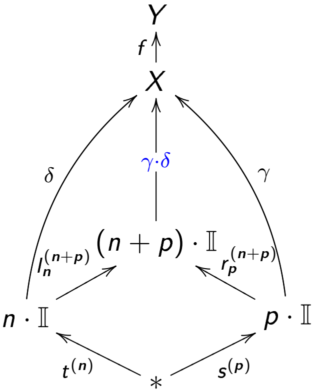
in picture



Think of $\sim_X$ as "there exists a classic homotopy h from the paths γ_1 to γ_2"

Generalized fundamental category

We set $\overrightarrow{\pi_1}(\overrightarrow{X}) := \Gamma(X)/\sim_X$ and we have a functor $\overrightarrow{\pi_1} : \mathsf{C} \longrightarrow \mathsf{Cat}$.



Since $\gamma_1 \sim_X \gamma_2$ implies $f \circ \gamma_1 \sim_Y f \circ \gamma_2$, we can define $\overrightarrow{\pi_1}(\overrightarrow{f})[\gamma]_{\sim_X} := [f \circ \gamma]_{\sim_Y}$, moreover, the left hand side diagram shows that we have $f \circ (\gamma \cdot \delta) = (f \circ \gamma) \cdot (f \circ \delta)$ whence the functoriality of $\overrightarrow{\pi_1}(\overrightarrow{f})$ from $\overrightarrow{\pi_1}(\overrightarrow{X})$ to $\overrightarrow{\pi_1}(\overrightarrow{Y})$.

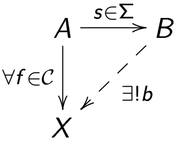
directed *vs* undirected generic segment

in the framework of PoSpc

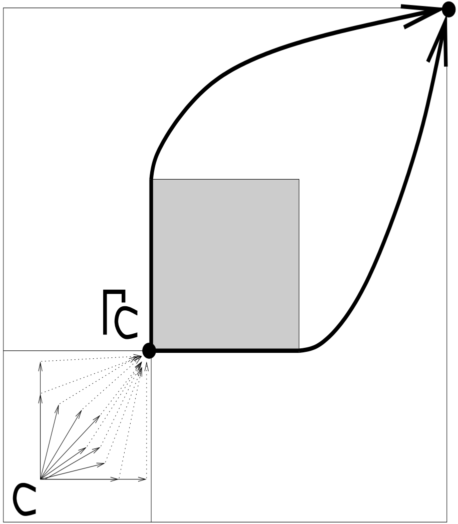
- With the generic segment $([0,1],=)$ over PoSpc, for any pospace $\overrightarrow{X}$, $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the fundamental groupoid of X.
- With the generic segment $([0,1],\leq)$ over PoSpc, for any pospace $\overrightarrow{X}$, $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the fundamental category of $\overrightarrow{X}$.

Orthogonal subcategories

Let $\mathcal{C}$ be a category and Σ a class of morphisms of $\mathcal{C}$. By the orthogonal subcategory of $\mathcal{C}$ determined by Σ, we mean the full subcategory $\mathcal{C}_\Sigma$ of $\mathcal{C}$, whose objects are those $X \in \mathcal{C}$ such that $s \perp X$ for every $s \in \Sigma$, i.e., such that for every $s : A \to B \in \Sigma$, for every morphism $f : A \to X$, there exists a unique morphism $b : B \to X$ such that $b \circ s = f$.



The orthogonal subcategory of Σ_+ is reflective



Theorem

Let Σ be the inessential morphisms in the future, in the category $\mathcal{C} = \overrightarrow{\pi_1}(\overrightarrow{X})$ for some local po-space X.

Suppose that Σ has infinite pushouts (it is automatically the case in mutual exclusion models, i.e. I^n minus isothetic hyperrectangles), then $\mathcal{C}_\Sigma$ is reflective in $\overrightarrow{\pi_1}(\overrightarrow{X})$.

Sketch of proof

By definition of the orthogonal subcategory, we have an obvious inclusion functor I from $\mathcal{C}_\Sigma$ to $\mathcal{C}$. To prove that we have a reflective subcategory, we need to construct the left adjoint to $\Gamma \dashv I$. Let $C \in \mathcal{C}$. For every pair (s, f) where $s : S \rightarrow T \in \Sigma$ and $f : S \rightarrow C \in \mathcal{C}$, by the properties of Σ, we know we have a pushout diagram (we call it a (s, f) pushout square):

$$\begin{array}{ccc} S & \xrightarrow{s \in \Sigma} & T \\ {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle g_{sf}} \\ C & \underset{t_{sf} \in \Sigma}{\dashrightarrow} & P_{sf} \end{array}$$

where $t_{sf} \in \Sigma$.

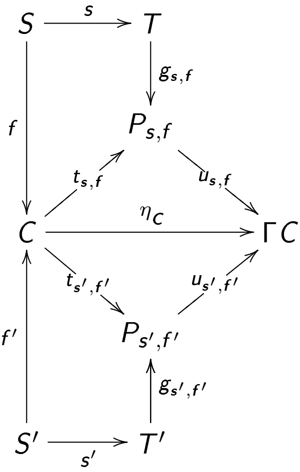
Sketch of proof

Consider now the diagram composed of all t_{sf} for all pairs (s, f) which is small since Σ is a set, thus its colimit $\left(\Gamma C, (u_{sf})_{sf}\right)$ exists in Σ and provide $\Gamma C \in \Sigma$.

We have defined the object part of Γ. Now we construct a family of morphisms of $\mathcal{C}$, denoted $(\gamma_C)_{C \in Ob\mathcal{C}}$, that will be the unit of the adjunction.

Sketch of a proof

Let us determine $\gamma_C : C \longrightarrow \Gamma C$. Given two (s, f)-pushout squares, by definition of a colimit, we have $u_{s,f} \circ t_{s,f} = u_{s',f'} \circ t_{s',f'}$, hence we can set $\gamma_C := u_{s,f} \circ t_{s,f}$ since it does not depend on the (s, f)-pushout square we have chosen. $\gamma_C \in \Sigma$ for it is given by the composite of two morphisms of Σ.



Sketch of proof

We determine the morphism part of Γ, this construction will implicitly prove that γ is an adjunction from $Id_{\mathcal{C}}$ to $I \circ \Gamma$. Let $h : C^1 \longrightarrow C^2$. For each pushout square

$$\begin{array}{ccc} S & \xrightarrow{s} & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g^1_{s,f}} \\ C^1 & \xrightarrow[t^1_{s,f}]{} & P^1_{s,f} \end{array}$$

Sketch of proof

we have by hypothesis on Σ and since $t_{s,f} \in \Sigma$, we have the commutative diagram:

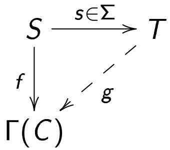
$$\begin{array}{ccccc}
C^1 & \xrightarrow{t^1_{s,f}} & P^1_{s,f} & \xrightarrow{u^1_{s,f}} & \Gamma(C^1) \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle g^2_{t^1_{s,f},h}} & & \Big\downarrow{\scriptstyle \Gamma(h)} \\
C^2 & \xrightarrow[t^2_{t^1_{s,f},h}]{} & P^2_{t^1_{s,f},h} & \xrightarrow[u^2_{t^1_{s,f},h}]{} & \Gamma(C^2)
\end{array}$$

where the left square is a pushout square as in the beginning of the proof with $C := C^2$, $s := t^1_{s,f}$ and $f := h$.

Sketch of proof

Now we prove that $\forall C \in \mathcal{C},\ \Gamma(C) \in \mathcal{C}_\Sigma$.

Given (s, f) with $s : S \to T \in \Sigma$, $src(f) = S$ and $tgt(f) = \Gamma(C)$, we have a unique g making the right side diagram commutative. Precisely, $g := u_{sf} \circ g_{sf}$, indeed, $u_{sf} \circ g_{sf} \circ s = u_{sf} \circ t_{sf} \circ f = \gamma_C \circ f$. The uniqueness is due to the bijectivity of $\gamma \in \mathcal{C}[S, T] \longrightarrow \gamma \circ s \in \mathcal{C}[S, \Gamma(C)]$, because s is inessential in the future. Thus, $s \perp \Gamma(C)$ and $\Gamma(C)$ is in the orthogonal subcategory determined by Σ.



Sketch of proof

Conversely, suppose that $X \in \mathcal{C}_\Sigma$. Given (s, f) with $s : S \to T \in \Sigma$, $src(f) = S$ and $tgt(f) = X$, we have a unique g making the right side diagram commutative, which is in fact a pushout square. With the notation introduced at the beginning of the proof, $g_{s,f} = g$ and $t_{s,f} = id_X$. But then the colimit $(\Gamma(X), u_{s,f})$ is the colimit of the family $(id_X)_{\{(s,f) \text{ with } s:S\to T\in\Sigma,\ src(f)=S \text{ and } tgt(f)=X\}}$. Hence $u_{s,f} \cong id_X$ for all such pairs (s, f) and $\Gamma(X) \cong X$ in $\mathcal{C}$.

$$\begin{array}{ccc} S & \xrightarrow{s\in\Sigma} & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[id_X]{} & X \end{array}$$

Sketch of proof

The last part of the proof consists of seeing that

$$\alpha \in \mathcal{C}_\Sigma[\Gamma(C), D] \longmapsto I(\alpha) \circ \gamma_c \in \mathcal{C}[C, I(D)]$$

is a bijection. Now, let us consider the canonical morphism $\gamma_C : C \to \Gamma C$ of the colimit. Given $D \in \mathcal{C}_\Sigma$ and $m : C \to D$, we must find a unique $n : \Gamma C \to D$ such that $n \circ \gamma_C = m$.

Sketch of proof

As $D \in \mathcal{C}_\Sigma$, it is orthogonal to all morphisms $s \in \Sigma$. In particular, for all pairs (s, f) as above, there exists a unique b_{sf} such that $b_{sf} \circ s = m \circ f$. By the pushout property defining P_{sf} we deduce that there is a unique morphism $a_{sf} : P_{sf} \longrightarrow D$ such that $a_{sf} \circ t_{sf} = m$ and $a_{sf} \circ g_{sf} = b_{sf}$. This is done for all pairs (s, f). Hence by the colimit property defining ΓC, we find a unique morphism $n : \Gamma_C \longrightarrow D$ such that $n \circ u_{sf} = \alpha_{sf}$. Hence

$$\begin{aligned} n \circ \gamma_C &= n \circ u_{sf} \circ t_{sf} \\ &= \alpha_{sf} \circ t_{sf} \\ &= m \end{aligned}$$

which ends the proof.