Transcendental syntax II: non
deterministic case

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Donner un sens plus pur aux mots de la tribu :
à P.-L., ami et compagnon de route de 30 ans.

Abstract

Using a non deterministic analytics, it is possible to expand the au-
tonomous explanation of logic, a.k.a. transcendental syntax, to full logic,
i.e., first order predicate calculus with equality. This paper concentrates
on additives, intuitionistic negation and on a paradoxically unexplored
territory: predicate calculus. First order terms are handled as proposi-
tions and predicates as connectives, e.g., equality becomes logical equiv-
ance. Transcendental syntax thus achieves a reduction of logic to its
propositional fragment.

1 Introduction

Part\textsuperscript{1} I, i.e., [6], initiates this fully independent approach to logic: no syntax,
no semantics, i.e., no prejudice. Logic is organised along a kantian architecture
constat/performance/usine/usage: the elusive reality is replaced with a dense
knitting analytic/synthetic, a posteriori/a priori.

1.1 Non-determinism

We propose here a relaxation of the performative layer to make the evaluation
non deterministic: this layer can be styled an analytics of non-determinism.

This means, concretely, a sort of type-free approach to non-determinism.
Something of the like has been attempted under the name $\pi$-calculus, $\pi$-calculi
indeed since the notion never quite stabilised. $\pi$-calculus claims to be the non
deterministic counterpart of pure $\lambda$-calculus; but, while the original is strongly
knitted (few primitives, structuring theorems), the imitation is invertebrate

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We propose to reduce the question of non-determinism to that of an adequate relaxation of our previous — completely analytic and densely knitted approach to deterministic computation — stars and constellations [6]: if the rays of a constellation are no longer disjoint, the normal form will no longer be unique. However, such a blunt relaxation can only produce a sort of Alzheimer non-determinism, acceptable in the case of NL — logarithmic space non-determinism. But inadequate for NP: the satisfiability of a classical proposition requires consistent choices true/false for each variable! Since analytics cannot afford the luxury of a memory — lest it lose its knitting —, we must introduce a coordination which takes the form of a coherence relation forbidding certain pluggings, typically that of the choices true and false for the same variable.

The major elements of the analytic knitting rest upon a good condition of strong normalisation; a Church-Rosser property which can only be expected at the level of all possible executions. Once again, the coherence relation will enable us to superpose them without mingling.

The knitting performance/usine, i.e., the correctness condition of proof-nets, poses a problem, that of hidden cuts, see [6]: \( A \otimes \sim A \), i.e., the star \( \prod_A(x) \cdot \overline{y} \overline{p}_A(x) \cdot \overline{y} \) is replaced with unpredictably many cuts \( [A_i \otimes \sim A_i] \), i.e., \( \prod_A(x) \cdot \overline{t}_i \overline{p}_A(x) \cdot \overline{t}_i \). They could be treated as plain cuts provided we can switch the \( A_i \otimes \sim A_i \) independently, but, in the deterministic universe, all switchings are copies of the original switching of \( A \otimes \sim A \). The syntactic treatment by means of proof-nets is treacherous, yet another instance of prejudice: it produces copies of \( A \otimes \sim A \) and switches them independently on its own authority. In order to get rid of this prejudice, the switching must become non-deterministic, typically a sum \( \overline{\gamma}_L + \overline{\gamma}_R \). The problem is that we get in this way a superposition of normal forms that cannot be handled in a correctness criterion: if the normal form is \( n \cdot [p_T(x)] \), how do we know that all independent runs did succeed, typically \( n = 1 \), corresponding to the choice \( \overline{\gamma}_L \), while \( \overline{\gamma}_R \) contributed to nil?

Hence the idea of using positive coefficients: if \( a, b > 0 \) with \( a^2 + b^2 = 1 \) and both \( \overline{\gamma}_L \) and \( \overline{\gamma}_R \) lead to \( [p_T(x)] \), the switch \( a \cdot \overline{\gamma}_L + b \cdot \overline{\gamma}_R \) yields a normal form \( a^2 \cdot [p_T(x)] + b^2 \cdot [p_T(x)] = [p_T(x)] \). This suggests to expand the notion of non-determinism into a notion of probabilistic computation with coefficients in, say, \( \mathbb{Q}^+ \) to make sure that the results are exact.

Coefficients are the natural answer to a logical question, namely the localisation of the \( \&\)-rule, in other terms the building of a sort of proof-nets adapted to additives. The traditional approach used global — i.e., synthetic — structures, boxes, reminiscent of the disjunction elimination of natural deduction. But this does not quite work: there is, so to speak, a problem of « parallel universes ». Take the non deterministic choice between \( A/B \) at work in the additive conjunction: the idea is that, once we reach \( A \& B \), the computation proceeds
within two parallel universes in which only one of $A/B$ survived. We suddenly find ourselves in a sort of story in the style of *The Matrix*: I choose $A$, but how do I know that I am not already in universe $A$? In other terms, the splitting $A/B$ could be anterior to the conjunctive bifurcation, which makes our non-determinism lopsided. Positive real coefficients won’t work since the weights, say $a/1 − a$, of the choice $A/B$ will naturally compensate each other so that the full picture gets the weight 1: this is because these weights are correlated, while they should not be in case the choice $A/B$ has been made before reaching $A & B$. The discussion is between balanced coefficients $a/1 − a$ and unbalanced ones $a/b$; the use of real numbers bringing nothing, we can assume that $a, b$ take values $0, 1$, i.e., are booleans.

There is a similar problem with quantifiers: when we meet $\forall x A$, we must enter a universe with a fresh object $x$. But how do we know that we were not already in universe $x$? Herbrand solved, in his day, the problem by introducing formal dependencies under the form of function letters $x = f(t)$. In analogy to these formal functions, we introduce formal booleans: the choice $a/b$ is indeed a choice $a/\neg a'$ where $a, a'$ stand for something dynamic, namely the « past », what led us to $A & B$. In case we already made our choice $A/B$, the two universes have distinct pasts, hence $a \neq a'$ and $a + \neg a' \neq 1$. Those *Herbrand booleans* are purely formal: they are not supposed to be evaluated, only matched with their negations. In the same way, Herbrand functions $f(t)$ had no value, they were only standing for something unknown to $t$.

I don’t know whether these *Herbrand booleans* are relevant to complexity theory; but they are surely to non-determinism.

Finally, numerical (real, rational) are of no use in logic. However, they correspond to a natural notion of probabilistic computation, and they should at least deserve a try.

### 1.2 Derealism

The monstrous expressive power of logic — as well as its major limitations — starts with second order quantification, typically when we introduce natural numbers through Dedekind’s definition:

$$\text{The smallest set containing } 0 \text{ and closed under } S.$$  

i.e., $n \in \mathbb{N} \iff \forall X ((\forall x (X(x) \Rightarrow X(Sx)) \Rightarrow (X(0) \Rightarrow X(n)))$. Forgetting first order variables, we get $\text{nat} : \forall X ((X \Rightarrow X) \Rightarrow (X \Rightarrow X))$, the familiar definition of natural numbers in system $F$.

The sequent $\text{nat} \vdash (B \Rightarrow B) \Rightarrow (B \Rightarrow B)$ is a way to express *iteration*: if the left hand side is fed with a natural number $n$, the right hand side becomes the functional $\Phi(f) := f \circ f \circ \ldots \circ f$. Natural numbers and iterators can be logically constructed with the help of the second order links:

$$
\frac{A[B/X]}{\exists X A} \quad \frac{A}{\forall X A}
$$
Consider iterators: if \( B^4 := (B \Rightarrow B) \Rightarrow (B \Rightarrow B) \), let \( \pi \) be the proof-net with conclusions \( \sim B^4, B^4 \). Second order existence enables us to pass from \( \sim B^4 \) to \( \exists X \sim X^4 = \sim \text{nat} \), thus yielding a proof-net \( \pi_B \) of conclusions \( \sim \text{nat}, B^4 \). Now, \( \pi \) is correct since it complies with the ordeals associated with \( \sim B^4 \) and \( B^4 \), but what about \( \pi_B' \)?

There is no difference, except that the divide object/subject is no longer respected. This divide was at work in the opposition vehicle/ordeal: on one side, the analytic « reality », on the other side, the synthetic subjectivity. In the case of \( \pi_B \) — more generally of second order existence —, we cannot guess the « witness » \( B \), hence we cannot find any ordeal corresponding to \( \exists X A \). This witness is indeed a gabarit; we can, however, check correctness, provided this ordeal becomes part of the data, i.e., part of the « object ». In other terms, when we deal with second order, we are indeed working with three partners: a vehicle \( V \), an ordeal \( O \) and a subsidiary gabarit \( G \) corresponding to existential « witnesses »: we still require that \( V + O + G \) strongly normalises into \( \llbracket p_f(x) \rrbracket \).

But the opposition is now between \( O \) and \( V + G \), i.e., between the subject and a compound object/subject, an épure\(^2\).

Technically, little has so far changed: there is no essential difference between \( V \) and \( V + G \). But the spirit is different, in particular the relation to l’usage becomes far more complex. Let us explain: in an opposition \( V/O \), everything depends upon the mathematical properties of \( O \), no hypothesis has to be made on \( V \); in an opposition \( V + G/O \), part of l’usage will depend upon properties of \( G \) whose checking belongs somewhere else, is external.

Let us come back to the case of the iterator \( \pi_B \): second order existence asks us for this witness \( B \) which occurs four times in \( \sim B^4 \), respectively negatively, positively, positively, negatively. It thus requires four gabarits, two for \( B \), two for \( \sim B \); the two positive ones should be « the same », ditto for the negative ones. Moreover the positive and negative gabarits should be « complementary », the negation of each other. Complementarity of gabarits \( G, G' \) basically means that we can perform (i.e., eliminate) a cut between them.

In other terms, the choice of \( G \) postulates a reduction usage/usine that does not belong in any decent analyticity. Here lies the very source of foundational doubts: the object, seen as an épure, embodies in itself something beyond justification.

The need for this auxiliary part should be obvious from the limitations of the category-theoretic approach to logic. In terms of categories, the absurdity \( 0 \) is an initial object, with the consequence that we cannot distinguish between morphisms into this object. Less pedantically, the absurdity is the empty set, and they are too few functions with values into \( 0 \) to make any useful distinction between them: either \( A = 0 \) and there is only one function from \( A \) to \( 0 \) or \( A \neq 0 \) and there is none. As a consequence, negation \( \neg A := A \Rightarrow 0 \) becomes a bleak operation: from the functional, category-theoretic standpoint, \( \neg A \) is either empty or with a single element.

\(^2\)This word, without satisfactory translation, refers to the representation of an object (our vehicle) through several viewpoints (those of the gabarit).
Now, remember that many major mathematical results or conjectures are negations: « there is no solution to... ». According to the category-theoretic prejudice, the intricate proofs of these results are but constructions of the function from $\emptyset$ to itself... which is quite preposterous! It would be more honest to admit that we reach here one of the major blindspots of logic. This blindspot is sometimes styled as « a proof without algorithmic contents ».

The notion of épure could explain the situation as follows: the vehicle in a proof $V + G$ of a negation may be trivial ($V = 0$). A major part of the real proof is the gabarit $G$ which makes no sense in category-theoretic terms, but which may be very intricate.

As to typing, there are two opposite approaches those of Curry and Church: for Curry, the objects are born untyped, the typing occurs later. To this sort of existentialism, Church opposes an essentialism for which objects are born with their types. The opposition vehicle/ordeal is a sort of implementation of typing à la Curry; such an attempt was necessary for the sake of — say — rationality. Second order shows that Church is not that wrong; not quite that objects are actually born with their types, but that they embody some typing in them through the gabarit $G$. But the épures $V + G$ must be opposed to $O$ to get their type, i.e., they are still untyped. Finally, the derealism at work in épures reconciles the two viewpoints.

1.3 Negation and consistency

The derealistic world of épures is a terra incognita for which we should design specific picks and shovels. Its premature study, in relation to second order quantification, can only lead to catastrophes. I will thus concentrate on one point, namely to show how we can « fill » the absurdity 0 so as to get a huge potentiality of proofs of the negation.

Of course this poses a problem of consistency: filling the absurdity provides, so to speak, « proofs » for all propositions. How do we avoid inconsistency in this case? The answer is involves a derealistic opposition between épures and animæ: an anima is a configuration which cannot be split as a sum $V + G$, in which, so to speak the objective and subjective, analytic and synthetic, components are mingled, and which are therefore not acknowledged as plain proofs.

1.4 First order variables and equality

The possibility of handling negation is due to the fact that the second order translation of the additive truth $\top$ is $\exists X X$, where $X$ occurs only positively: one must not cope with the balance $X/\sim X$.

In the same spirit, we propose an approach to first order terms and variables as specific gabarits, whose shape is simple enough so as to avoid any discussion as to the balance $X/\sim X$. The scandalously neglected logical connective, equality thus becomes a simple equivalence.

However, how do we make sure that the gabarits used for first order terms are the right ones? Some external constraint — here of a limited import — seems
unavoidable. I style those constraints, which are the very heart of quantification of any order, epidictic. The logical challenge is to understand the very nature of epidictism. Typically, do epidictic constraints obey to some form of logic, of deduction? This issue is clearly another story, another programme.

2 Stars and constellations, revisited

2.1 Stars

We work, as in [6], with a denumerable language of terms, based upon infinitely many functions of any arity and infinitely many variables. We specify denumerably many pairs of complementary colours, i.e., unary functions $c_i, \overline{c}_i$: typically, green/magenta, blue/yellow, red/cyan. A star is a finite set consisting of two sorts of edges, linking vertices:

Free edge, a.k.a. ray: $t$, a term, linked to a single vertex; there should be at least one ray.

Captive edge: a match between two complementary colourings of the same term, typically $\overline{t} = \overline{t}$; both $\overline{t}$ and $\overline{t}$ are linked to a vertex, so that each captive edge links two vertices.

The structure must be connected and acyclic, i.e., a tree in the topological acception. Moreover, any variable occurring in one edge occurs in all edges.

The definition is a bit ambiguous in case of repetition of edges; we shall assume for convenience that the rays are pairwise distinct; idem for the captive edges which should be distinct. Don’t worry too much about repetitions: a star with repetitions should anyway be self-incoherent, hence negligible.

We must however keep in mind that what actually matters is the set of rays and that the remainder of the structure, the tree of captive edges is but a convenient labelling of the star enabling one to distinguish between stars with quite the same rays. Typically, we may need to duplicate a star $S$, i.e., introduce another star $T$ which, although considered as distinct, is «the same». This can be easily done by adding to $S$ a useless captive edge $\overline{t}_1 = \overline{t}_1$, linking an edge of $S$ with a fresh edge.

2.2 Constellations I

We must first come back to the notion of substitution: $\theta$ must be given with a domain, i.e., a finite list of variables on which it actually does something. This is the only way in which it can be thought as really finite: $x_1 \rightarrow \theta_1, \ldots, x_n \rightarrow \theta_n$, $y \rightarrow y$ for $y \neq x_1, \ldots, x_n$. Substitutions can be preordered by $\theta_1 \preceq \theta_2$ iff $\theta_2 = \theta_1 \theta'$ for some $\theta'$; the smallest substitution $\iota$ is therefore the idle one, $x_i \rightarrow x_i$. Given any set $\Theta$ of substitutions, its (upwards) closure is defined as $\Theta := \{ \theta'; \exists \theta \in \Theta \theta \preceq \theta' \}$.

A constellation (without parameters) is a finite set of stars equipped with a coherence. The coherence associates to any two stars $S, T$ of the constellation an
upwards closed set $S \uparrow T$ of substitutions of domain $x_1, \ldots, x_k, y_1, \ldots, y_l$, where $X = x_1, \ldots, x_k, Y = y_1, \ldots, y_l$ are the variables, chosen distinct, of $S$ and $T$, that we may thus note $S[X]$ and $T[Y]$. We require that:

1. Up to equivalence, $S \uparrow T$ has finitely many minimal elements.

2. If $s, t$ are edges in $S, T$ and either $S \neq T$ or $S = T$ and $s \neq t$, if $s\theta = t\theta$, then $\theta \in S \uparrow T$.

3. If $\theta \in S[X] \uparrow S[Y]$ is a unifier, i.e., if $\theta x_1 = \theta y_1, \ldots, \theta x_k = \theta y_k$, define $\tau$ of domain $X$ by $\tau x_1 = \theta x_1, \ldots, \tau x_k = \theta x_k$; then $\tau \in S[X] \uparrow T[Z]$.

4. $S \uparrow T = T \uparrow S$.

The coherence corresponds to forbidden substitutions, i.e., « values » that the stars cannot simultaneously take when socialising through diagrams. A unifier $\theta \in S \uparrow S$ is thus a value that $S$ cannot take alone: $S\theta$ cannot therefore be used: this explains condition 3. Condition 1 makes the coherence finitary; of course these minimal elements must be given. Condition 2 is the link to the deterministic case, which corresponds to constellations without coherence, $S \uparrow T = \emptyset$. Indeed, any matcher between the rays $s \in S$ and $t \in T$ should be part of the coherence, except if $s = t$ and $S = T$.

Two stars are incoherent, notation $S \sim T$ when $i \in S \uparrow T$. By condition 3 above, self-incoherence corresponds to the case where the trivial unifier $\theta(x_1) = \theta(y_1) = x_1, \ldots, \theta(x_k) = \theta(y_k) = x_k$ is in $S[X] \uparrow S[Y]$. Self incoherent stars, which include the excluded case of repeated edges, are neglected, identified to 0.

2.3 Normalisation I

A coloured constellation is a constellation in which some rays may be coloured; the colours used for the rays should be distinct from those used in captive edges. This restriction is natural: internal colours correspond to normalisations already done and should not be confounded with future ones, typically in view of the Church-Rosser property.

In a coloured constellations one can form diagrams as trees of stars link through shared edges made of rays of complementary colours, e.g., $\emptyset = \emptyset$. The actualisation of the diagram consists in matching those shared edges, so that $t\theta = u\theta$; as in the deterministic case, most actualisations fail. In case it works, the fact of making a tree of trees makes the actualised diagram a star — provided its edges are distinct, but the coherence will exclude this case.

Between any two actualised diagrams $\mathcal{D}, \mathcal{E}$, we must define a coherence. Select distinct variables $x_1, \ldots, x_k, y_1, \ldots, y_l$ for $S$ occurring in $\mathcal{D}$ and $T$ occurring in $\mathcal{E}$ as $S\theta$ and $T\theta$, and let $\Theta_{ST} := \{\theta'; \theta\theta' \in S \uparrow T\}$. Then $\mathcal{D} \uparrow \mathcal{E} := \bigcup_{\mathcal{D}, \mathcal{E}} \Theta_{ST}$, where $S, T$ vary through all possible choices of occurrences.

An extreme case is that of a self-incoherent actualised diagram $\mathcal{D}$: such a case occurs when $S, T$ occur in $\mathcal{D}$ as $S\theta$ and $T\theta$ and $\theta \in S \uparrow T$.

A constellation is strongly normalising when, for $N$ big enough, all actualised diagrams of size $N$ are self-incoherent; moreover all remaining actualised
diagrams should have at least one free ray. The normal form of the constellation is made of its actualised diagrams equipped with the coherence just defined. Of course, those self-incoherent stars are excluded, or neglected, as you like it: the ultimate purpose of coherence is, through self-incoherence, to forbid certain actualised diagrams.

One should verify that the normal form is actually a constellation, which involves the not-too-exciting verification of conditions 1 – 4 of coherence.

2.4 Church-Rosser

As usual, the Church-Rosser property equates normalisation in one step and normalisation in two steps. There is a problem, already encountered in [6], namely that the two steps version of strong normalisability may be be weaker than the one step version. However, the problem vanishes if we keep track of all actualised diagrams by adding to our constellation the stars $\left[ [x, \text{mg}(x)] \right]$, $\left[ [x, \text{gr}(x)] \right]$, etc. using uncoloured duplicates of the colours bound to be eliminated.

Since $[x]$ matches all green rays, the modified star cannot be deterministic. If $\mathcal{S}$ is in our constellation, the coherence $\mathcal{S} \uparrow \left[ [x, \text{mg}(x)] \right]$ consists of the upwards closure of the substitutions $\theta_t(x) := t$ of domain $x$, for each ray of $\mathcal{S}$ of the form $[x]$ ditto for the coherence $\mathcal{S} \uparrow \left[ [x, \text{gr}(x)] \right]$. Church-Rosser holds in the strict sense for these modified constellations, i.e., for the explicit forms which keep track of the full normalisation procedure. Some caution is thus required when passing to normal forms, i.e., when getting rid of those stars involving the duplicates gr, mg, etc.

2.5 Herbrand booleans

A substar is a star with possibly no ray; the substar $T$ is a substar of star $\mathcal{S}$ when there is a substitution $\theta$ s.t. each ray of $T\theta$ is a ray of $\mathcal{S}$. Two stars are congruent when each is a substar of the other, i.e., when they have the same rays up to renaming of their variables.

To each star $\mathcal{S}$ we associate the boolean algebra $H[\mathcal{S}]$ generated by the formal booleans $\eta_T$, where $T$ is a substar of $\mathcal{S}$. Two congruent stars have therefore the same algebra.

In a constellation, the star $\mathcal{S}$ receives a coefficient which is an element of $H[\mathcal{S}]$. During the process of normalisation, we shall form diagrams that will give rise to stars, as already explained; but they must receive a coefficient which can only be the product of the respective coefficients of the stars involved in the diagram. The problem is that these coefficients dwell in distinct algebras. This is why we must define embeddings $e_{\mathcal{S}, \mathcal{U}}$ when $\mathcal{S}$ occurs in a diagram $\mathcal{D}$ and $\mathcal{D}$ actualises as $\mathcal{U}$. If $\eta_{\mathcal{S}'} \in H[\mathcal{S}]$, let us replace $\mathcal{S}$ with $\mathcal{S}'$ in $\mathcal{D}$; since $\mathcal{S}'$ has fewer rays, some edges disappear; keeping only those stars which are still connected with $\mathcal{S}'$, we get a diagram $\mathcal{D}'$, which actualises as a substar $\mathcal{U}'$; we define $e_{\mathcal{S}, \mathcal{U}}(\eta_{\mathcal{S}}) := \eta_{\mathcal{U}'} \in H[\mathcal{U}]$.

Take the case where $\mathcal{D}$ is made of two stars, $\mathcal{S}$ and $T$ connected through the edge $\equiv \left[ [x] \right]$; if $[x]$ is an instanciation of some ray $x'$ of $\mathcal{S}'$, then $\mathcal{D}'$ is the star
made of $S'$ and $T$ connected through the edge $\mathbb{e} = \mathbb{f}$ and $\mathcal{U}'$ is its actualisation. Otherwise, $\mathcal{U}' = \mathcal{D}' = S'$.

2.6 Constellations II

A constellation is a finite linear combination of stars $\sum \lambda_i S_i$ with $\lambda_i \in H[S_i]$, equipped with a coherence $S_i \uparrow S_j$. We already know how to normalise a constellation in the absence of coefficients. In presence of coefficients, the actualisation $\mathcal{U}$ of diagram $\mathcal{D}$ receives a coefficient which is the product of the coefficients of its constituents, modulo the various embeddings. The same coefficient, say $\eta_S$ may contribute under several embeddings to the product.

2.7 Gluability

We enter a limited form of syntheticity (l’usine) when we start to pass judgements on constellations, typically when we make some identifications. The basic identification is, so to speak, analogous to $\alpha$-conversion, the handling of bound variables: basically only the rays of a star do matter. The internal tree structure, the captive rays, are an elegant way to distinguish between « copies », « occurrences » of the « same » star. So the question is whether to identify two stars of a constellation which are congruent, i.e., with the same rays.

Let $S_i, S_j$ be two distinct congruent stars in a constellation; they basically depend upon the same variables, which can be, as we please, be $X = x_1, \ldots, x_n$ or the distinct $Y = y_1, \ldots, y_n$. $S_i, S_j$ are are glueable when the following hold:

1. $S_i \sim S_j$.
2. $S_i[X] \uparrow S_i[Y] = S_j[X] \uparrow S_j[Y]$.
3. For $k \neq i, j$, $S_i[X] \uparrow S_k[Z] = S_j[X] \uparrow S_k[Z]$.

If $S_i, S_j$ are glueable, there is no obstacle at identifying them, typically at replacing $S_i$ with $S_j$. In which case, $\lambda_i S_i + \lambda_j S_j$ becomes $(\lambda_i + \lambda_j) S_j$.

The typical case is that of a constellation of several congruent stars $S_i$ s.t. :

1. $S_i \sim S_j$.
2. $S_i \uparrow S_i = \emptyset$.

The sum $\sum \lambda_i S_i$ can be replaced with $\langle \sum \lambda_i \rangle [t_1, \ldots, t_n]$ where $t_1, \ldots, t_n$ are the rays of any $S_i$. In this way, a normal form can be viewed as $\lbrack p_T(x) \rbrack$. The delicate point is whether or not the boolean coefficients $\lambda_i$ compensate each other to yield 1. Since Herbrand booleans are purely formal, the only possible compensation comes from identities $\eta_S + \neg \eta_S = 1$.

3 Revisiting part I

Part I made a few allusions to non-determinism. Let us go through them:
3.1 Cancelling ordeals

The switching $\kappa_L'$ has been styled cancelling. We propose to replace $\kappa_L'$ with the sum $\kappa_R + \kappa_L'$, with $\kappa_R \sim \kappa_L'$. Due to the incoherence $\kappa_R \sim \kappa_L'$, no correct diagram can contain both of them, in other terms, the normal form is the sum of the normal forms corresponding to each choice. Now, if we propose a switching between $\kappa_R$ and $\kappa_R + \kappa_L'$ and both yield the normal form $[p_T(x)]$ this can only be because the additional contribution of $\kappa_L'$ is null: in other terms, the cancellation property of [6] holds.

Another sort of cancelling ordeal has been introduced to ensure that the atoms in would-be identity links $[p_A(x),p_{\sim A}(x)]$ actually match. The cancelling ordeal is a sum $S = \sum [ q_A(1-x) q_A(1-x) ]$, where $A$ runs through all occurrences of literals; this sum is added to any plain ordeal $O$. As to coherence, the summands are incoherent with anything else in $O$, moreover, any two summands are coherent, unless they make opposite choices for the same variable: $[ q_A(1-x) q_A(1-x) ] \sim [ q_B(1-x) q_B(1-x) ]$ when $A$ is an occurrence of $X$ and $B$ is an occurrence of $\sim X$. Now, to say that $1 + O + S$ strongly normalises into $[p_T(x)]$ amounts at saying that $1 + O$ strongly normalises into the same $[p_T(x)]$ and that $1 + S$ cancels, whatever switching is chosen for the literals.

The switching $\kappa_R := [ q_B(x) ] + [ q_A(x) \cdot y ] + [ q_A(x') \cdot y' ]$ takes the form $S + T + T'$, where $T, T'$ are congruent and $S \vdash T = T \vdash T' = \{x = x', y = y'\}$, i.e., contains the substitution equating $x$ and $x'$. This means that $T'$ can coexist with $S$ or $T$ only if $x$ and $x'$ assume different values.

3.2 Exponentials

When dealing with exponentials, we first meet a problem: there is a « tensorisation » with a variable $y$ above the premise $A$ of $A \otimes B$. A whole constellation, the ordeal for $A$, has been modified. Quid of the coherences? Let us note $S \otimes y$ the result of the replacement of any ray $t, t$ with $t \cdot y, t \cdot y$. The coherence $(S \otimes y) \vdash (T \otimes y')$ (the variables should be chosen distinct) is defined as $(S \vdash T) \otimes \{y = y'\}$, which means that $S \otimes y$ is coherent with $T \otimes y'$ unless $y = y'$, in case the coherence $S \vdash T$ applies. In other terms, if we see $y$ as a labelling of copies, distinct copies are unrelated.

Let us finally come to the major point missing in [6], namely the consideration of hidden cuts. The problem is due to our refusal of the syntactic prejudice: the duplications at work with exponentials create, so to speak, new formulas and therefore new independent switchings. But this is ad hoc, external. The only reasonable way is to generate these new switchings is duplication: the choice $\mathcal{N}_L / \mathcal{N}_R$, which becomes $\mathcal{N}_L \otimes y / \mathcal{N}_R \otimes y_L$ handled as a non deterministic sum and the coherence $(\mathcal{N}_L \otimes y) \vdash (\mathcal{N}_R \otimes y') = \{y = y'\}$ yields independent switchings of the various copies. The problem is elsewhere: what should be required as to the normal form? The normal form we get in this way is of the form $k[p_T(x)]$, 

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with $k = 2^n$, where $n$ is the number of switchings $N_L \otimes t_i / N_R \otimes t_i$. But how to count these switchings? How do we know, for instance, that one of the parallel runs didn’t cancel? In the preset syntactic universe, we can find the value of this $k$, but this is quite hopeless in our unprejudiced setting.

We propose to use the combination $\eta[p_{AB}(x)] N_L + \neg\eta[p_{AB}(x)] N_R$ when the $N$ is not yet tensorised; the combination becomes $\eta[p_{AB}(x);y] N_L \otimes y + \neg\eta[p_{AB}(x);y] N_R \otimes y$ after tensorisation. The various copies yield independent coefficients $\eta[p_{AB}(x);t_i]$ which will account for the necessary independence of the various switchings.

4 Additives

4.1 The problem

The question of additive proof-nets is a baffling one. The question is indeed ill-posed: wrong constraints leading to an aporia.

The starting point is however clear: everybody agrees that the use of boxes inside proof-nets is a pis-aller. But why are they so bad? The dominant answer — which was mine too — is that boxes are not optimal: they distinguish between things that should be identified. This identification is backed by category-theoretic semantics, i.e., by prejudice. But, right or wrong, we cannot accept a realistic justification. In terms of boxes, identification leads to commutative conversions, one of the most hateful topic in proof-theory: even when Church-Rosser, they do not belong in any decent analytics.

A remedy to commutative conversions is, in terms of proof-nets, the optimal identification between superposed links: if the two choices are the same, count them as a single link. This is analytic in the sense of the constat, but not of the performance: the process of normalisation will produce accidental, non intended, coincidences between portions of proofs, and it is not possible to introduce any identification at this level without destroying the analyticity of the process. By the way, the most elaborated work in this direction is [7]; but the condition can surely not be put in a simple analytic form, that of the normalisation of sum $\langle$ vehicle + ordeal $\rangle$. Optimal identification thus fails on two major knitting issues: performance/usage (the analyticity of normalisation) and performance/useine (the correctness condition).

My attempt [4] is slightly better in the sense that it is more analytic; it relies upon an explicit decomposition of the weights of the various links and is thus a break w.r.t. the semantic prejudice. Along this line, I developed the idea of a dialect, namely of an auxiliary space devoted to the labelling of copies. But this dialect was a bit painful to handle, and I never quite understood why a part of the analytics should be essentialised as « private ». As to correctness, the criterion makes use of jumps. Although interesting and useful for studying proof-nets, see [3], these jumps are sorts of switchings depending upon the vehicle; they thus belong in a more intricate approach in which the separation object/subject is put into question.
Finally, my own ludics [5] provided a satisfactory solution to additive problems. The problem is to determine whether or not it fits in our kantian pattern. Ludics relies on a complete and total identification between blocks 3 and 4, usine and usage. This means that, when ludics sells you a Fiat 500, not only it has tested it, but it has already used it the way you will use it! A bit unrealistic, this means an infinite analytics and a synthetics in which one can be sure of strictly nothing. It was otherwise my finest attempt so far; but it failed since infinite: real transcendental syntax should stay finite.

In the light of the discussions so far led, it seems that we should dump the semantic prejudice: why should we try to find the « objective » contents of a proof, which is, in fine, a subjective artefact? By the way, the same prejudice would compel us into the inelegant $\eta$-expansion which has some minor advantages, but destroys the healthy analyticity of normalisation. $\eta$-expansion belongs in the semantic universe, i.e., the mathematical study of block 4, l’usage: this study, not always irrelevant, should not pollute foundational issues.

Sequentialisation is not a satisfactory benchmark: once we admit that the « same » proof can be written in different ways, there is no reason to suppose that sequent calculus can handle all these ways. If it forgets some of them, where is the loss? Finally we shall require two conditions:

- Normalisation should be done in the standard analytic (plugging + unification) way. The price to pay is the introduction of non-determinism: additive boxes will be replaced by plain superpositions, but incoherent ones to avoid mingling.

- We can establish normalisation, i.e., the reduction of additive cuts, on the sole basis of the correctness criterion: this knitting usine/usage replaces sequentialisation.

4.2 Vehicles

Remember that we are not accessing logic through a translation; if I translate sequent calculus, this should be read the other way around, the sequent calculus proof as an abbreviation for its « translation ». I define, as usual, the subformula from the formula, typically: $p_A(x) := p_{A\&B}(1 \cdot x), p_B(x) := p_{A\&B}(r \cdot x)$; in the same way $p_A(x) := p_{A\oplus B}(1 \cdot x), p_B(x) := p_{A\oplus B}(r \cdot x)$.

$&$-rule: if the proof $\pi$ of $\vdash \Gamma, \Delta, \Delta'$, $A \& B$ comes from proofs $\nu$ of $\vdash \Gamma, \Delta, A$, and $\mu$ of $\vdash \Gamma, \Delta', B$, then $\pi^* := \nu^* + \mu^*$ with $\nu^* \sim \mu^*$, i.e., any star of $\nu^*$ is incoherent with any star of $\mu^*$.

$\oplus_L$-rule: if the proof $\pi$ of $\vdash \Gamma, \Delta, A \oplus B$ comes from a proof $\nu$ of $\vdash \Gamma, \Delta, A$, then $\pi^* := \nu^*$.

$\oplus_R$-rule: if the proof $\pi$ of $\vdash \Gamma, \Delta, A \oplus B$ comes from a proof $\nu$ of $\vdash \Gamma, \Delta, B$, then $\pi^* := \nu^*$. 

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4.3 Gabarits

Define $\&_L := \left[ \frac{q_A(x)}{q_{A\&B}(x)} \right]$, $\&_R := \left[ \frac{q_B(x)}{q_{A\&B}(x)} \right]$, with $\&_L \sim \&_R$; let $\oplus_L := \left[ \frac{q_A(x)}{q_{A\oplus B}(x)} \right]$, $\oplus_R := \left[ \frac{q_B(x)}{q_{A\oplus B}(x)} \right]$, with $\oplus_L \sim \oplus_R$. The following are used to form additive gabarits:

Conjunction: $\eta \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \&_L + \neg \eta \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \&_R$.

Disjunction: $\oplus_L + \oplus_R$.

Both conjunction and disjunction are non deterministic; in the disjunctive case, where a plain sum is used, only one of the summands may contribute to the final normal form $\left[ p_T(x) \right]$. In the conjunctive case, both summands contribute to the normal form $e \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \lambda \left( \eta \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \right) \eta \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \right)$ and is equal to $\left[ p_T(x) \right]$ only if $\mathcal{U}, \mathcal{V}$ can be glued as $\left[ p_T(x) \right]$ and if $e \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \lambda \left( \eta \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \right) = e \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \lambda \left( \eta \left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right] \right)$, i.e., if the sub-stars generated by $\left[ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} \right]$ in $\mathcal{U}$ and $\mathcal{V}$ are the same.

4.4 Cut-elimination

Let us consider an additive cut $[(A \oplus B) \& (\sim A \& \sim B)]$. To make the long story short, the ordeal opposing the vehicle $\mathcal{V}$ can be reduced to the form $\mathcal{O} + S$ with $S = \eta \left[ \frac{q_A(x)}{q_A(x)} \right] \left[ \frac{q_A(x)}{q_A(x)} \right] + \eta \left[ \frac{q_B(x)}{q_B(x)} \right] \left[ \frac{q_B(x)}{q_B(x)} \right] + \neg \eta \left[ \frac{q_A(x)}{q_A(x)} \right] \left[ \frac{q_A(x)}{q_A(x)} \right] + \neg \eta \left[ \frac{q_B(x)}{q_B(x)} \right] \left[ \frac{q_B(x)}{q_B(x)} \right]$.

Let $\lambda_{A,\sim A} \left[ p_T(x) \right]$, $\lambda_{A,\sim B} \left[ p_T(x) \right]$, $\lambda_{B,\sim A} \left[ p_T(x) \right]$, $\lambda_{B,\sim B} \left[ p_T(x) \right]$ be the normal forms of $\mathcal{R} + \mathcal{O} + \left[ \frac{q_A(x)}{q_A(x)} \right]$.

4.5 Algebraic coefficients

Logic requires booleans $\eta_S$, but there is no obstacle at using real positive coefficients; positive rationals if we want exact results.

Instead of $\eta_{\|_{P\&N(x)}} \|_{\mathcal{L}} + \eta_{\|_{P\&N(x)}} \|_{\mathcal{R}}$, we might as well use $\frac{3}{5} \|_{\mathcal{L}} + \frac{2}{5} \|_{\mathcal{R}}$. $\|_{\mathcal{L}}$ being both made of two stars contributes twice to the normal form, ditto for $\|_{\mathcal{R}}$: hence the coefficient $\frac{3^2 + 4^2}{5^2} = 1$.

Why not considering complex numbers, so that our coefficients of $S$ dwell in the commutative $C^*$-algebra generated by $\mathcal{H}[S]$? This might be of some interest in relation to — say — quantum computing.

The only problem is with strong normalisation: how to assert that all diagrams of size $N + 1$ fail? This can only mean something like $\llbracket \sum \text{all actualisations of diagrams of size } N + 1 \text{ is null} \rrbracket$. Thanks to gluability, there is
no problem with getting cancellations, but there is no reason why the cancelling summands should come from diagrams of the same size.

We thus propose the following solution: we take the sum of all actualised diagrams of size at most \( N + 1 \). Those diagrams whose actualisation eventually gets a coefficient 0 due to distributivity are styled cancelled. We require that any uncancelled diagram of size \( N + 1 \) contains a cancelled subdiagram.

5 Negation

5.1 Épures

A specific pair of complementary colours, say \( \text{red}/\text{cyan} \) must be introduced. We distinguish between two categories of constellations, according to the colours they use:

\textbf{Animæ}: an \textit{anima} makes use of the colours \( \text{blue}/\text{red} \).

\textbf{Ordeals}: an \textit{ordeal} makes use of the colours \( \text{yellow}/\text{cyan} \) together with uncoloured rays (for the conclusions).

An épure is an anima which can be split as a sum \( \mathcal{V} + \mathcal{G} \), where \( \mathcal{V} \) is \( \text{blue} \) \( \mathcal{G} \) is \( \text{red} \).

The opposition anima/ordeal is a refinement of our previous opposition between vehicles (in \( \text{blue} \)) and ordeals (in \( \text{yellow} \) and uncoloured).

5.2 The additive truth

The neutral \( \top \top \top \) is defined by means of its unique ordeal: \[ \llbracket \text{R}(x) \mathsf{S}(x) \rrbracket + \llbracket \text{T}(x) \rrbracket. \]

The symbols \( \text{T}(x), \text{R}(x), \mathsf{S}(x), \text{T}(x) \) are shorthands for \( p_{\top}(x), p_{\top}(r \cdot x), p_{\top}(s \cdot x), p_{\top}(t \cdot x) \).

The anima \[ \llbracket \text{R}(x) \rrbracket + \llbracket \text{T}(x) \mathsf{S}(x) \rrbracket, \] which complies with \( \top \top \top \), splits as a sum \( \mathcal{V} + \mathcal{G} \): it is an épure, i.e., a « correct » proof of \( \top \top \top \).

More generally, the rule

\[
\begin{align*}
\vdash \Gamma, \neg A & \\
\vdash \Gamma, \top & 
\end{align*}
\]

of sequent calculus, which corresponds to the definition \( \top := \exists X X \) makes sense in terms of épures: if \( \mathcal{V} + \mathcal{G} \) is an épure with conclusions \( \Gamma, \neg A \), select an ordeal \( \mathcal{O} \) for \( \neg A \) and replace in \( \mathcal{O} \) the \( \llbracket \text{S}(x) \rrbracket \) and \( p_{\neg A}(x) \) with \( \llbracket \text{T}(x) \rrbracket \) so as to get \( \mathcal{O}' \). Then \( \mathcal{V} + (\mathcal{G} + \mathcal{O}') \) is an épure of conclusions \( \Gamma, \top \), corresponding to a proof of this sequent.

5.3 Consistency

Transcendental syntax defines propositions by means of the associated ordeals and \textit{preproofs} as any animæ complying with those ordeals; a proof is a preproof.
which is actually an épure. Preprogs which are not épures are sort of animist artefacts, flawed constructions mingling objects and subject\(^3\).

Consistency is therefore the existence of propositions without proofs. The typcal example of such a proposition is the additive neutral \(0\) which is defined by its three ordeals: \(\lfloor r(x) \rfloor \lor \lfloor s(x) \rfloor \lor \lfloor t(x) \rfloor\) and \(\lfloor u(x) \rfloor \lor \lfloor v(x) \rfloor\). The symbols \(0(x), r(x), s(x), t(x)\) are shorthands for \(p_0(x), p_0(r \cdot x), p_0(s \cdot x), p_0(t \cdot x)\).

\(0\) admits as a preproof the anima \(\lfloor \ell(x) \rfloor + \lfloor \ell(x) \rfloor\) complies with the ordeals for \(0\). However, there is no proof of \(0\), i.e., no épure \(\forall + \forall G\) complying with these ordeals. Indeed, any anima complying with them should use the rays \(\lfloor \ell(x) \rfloor, \lfloor n(x) \rfloor, \lfloor u(x) \rfloor\), since it can contain neither \(\lfloor \ell(x) \rfloor\) nor \(\lfloor \ell(x) \rfloor\). it must contain \(\lfloor \ell(x) \rfloor\) and also \(\lfloor \ell(x) \rfloor\), which mix \(\text{blue}\) and \(\text{cyan}\) and is thus animist.

This establishes the consistency of transcendental syntax: the absurdity has no « proof », i.e., harbours no épure.

By the way, the épure \(\lfloor R(x) \rfloor \lor \lfloor G(x) \rfloor \lor \lfloor \ell(x) \rfloor\) constitutes a « proof » of the sequent \(\vdash 0, 1\).

### 5.4 Cut-elimination

The knitting usage, i.e., cut-elimination, is an opportunity to see how normalisation works for épures in a simple case. Consider a cut \(\lfloor C \rfloor := \lfloor T \otimes 0 \rfloor\); the cut is normalised as usual, by painting \(\lfloor p_T(x) \rfloor\) and \(\lfloor p_0(x) \rfloor\) in \(\text{green}\) and adding the feedback \(\lfloor p_T(x) p_0(x) \rfloor\). There is only one delicate point: épures use two pairs of colours: the part of the vehicle \(\forall\) making use of \(p_T, p_0\) becomes \(\text{green}\) but the part of the gabarit \(\forall G\) making use of them must use two other complementary colours. Since a fourth pair of complementary colours is beyond my typographic inventiveness, I will use \(\text{blue}, \text{yellow}, \text{red}, \text{cyan}\) to mean indeed \(\text{green}, \text{magenta}\) and the two extra colours dedicated to gabarits.

The feedback \(\lfloor p_T(x) p_0(x) \rfloor\) can be replaced with its specialisation, its « \(\eta\)-expansion » on the three sublocations \(r \cdot x, s \cdot x, t \cdot x\), i.e., with the sum \(T_1 + T_2 + T_3\) with \(T_1 := \lfloor R(x) p_T(x) \rfloor\), \(T_2 := \lfloor S(x) s(x) \rfloor\) and \(T_3 := \lfloor t(x) T(x) \rfloor\).

An épure \(\forall + \forall G\) with conclusions \(\forall, \lfloor C \rfloor\) must comply with the \(\forall + \forall_i\) \(i = 1, \ldots, 3\), where \(\forall_i := \lfloor R(x) \rfloor + \lfloor S(x) \rfloor + \lfloor T(x) \rfloor\) and

\(\text{The daimon of ludics [5] proposed also a sort of animist preproof; its existence was dependent upon a polarisation of logic (negative/positive behaviours) a bit painful to handle. The approach through épures offers a simpler and better knitting.}\)
\( \mathcal{O}_2 := \left[ \frac{R(x)}{S(x)} \right] + \left[ \frac{P_C(x) \cdot \mathcal{I}(x) \cdot \mathcal{J}(x)}{\mathcal{I}(x)} \right] \) and

\( \mathcal{O}_3 := \left[ \frac{R(x)}{S(x)} \right] + \left[ \frac{P_C(x) \cdot \mathcal{I}(x)}{\mathcal{I}(x)} \right]. \)

If \( \mathcal{V} + \mathcal{G} \) complies with the \( \mathcal{O} + \mathcal{O}_1 \), then \( \mathcal{V} + \mathcal{G} + \mathcal{O} \) strongly normalises into a constellation \( \mathcal{C} \) s.t. \( \mathcal{C} + \mathcal{O}_1 \) strongly normalises into \( \mathcal{p}_T + \mathcal{C}(x) \) for \( i = 1, 2, 3 \). The free rays of \( \mathcal{C} \) are \( \mathcal{I}(x) \), \( \mathcal{J}(x) \), \( \mathcal{K}(x) \), and \( \mathcal{T}(x) \), \( \mathcal{r}(x) \), \( \mathcal{s}(x) \) and \( \mathcal{R}(u_1) \), \( \mathcal{S}(u_1) \), \( \ldots \), \( \mathcal{R}(u_n) \), \( \mathcal{S}(u_n) \) together with the \( \mathcal{p}_T(x) \). These rays are dispatched in various stars; considering the \( \mathcal{O}_i \), we see that \( \mathcal{T}(x) \), \( \mathcal{r}(x) \) are in distinct stars \( \mathcal{S}_1, \mathcal{S}_2 \), ditto for \( \mathcal{R}(u_k) \) and \( \mathcal{S}(u_k) \) for \( k = 1, \ldots, n \); \( \mathcal{s}(x) \) are the sole rays of the same star \( \mathcal{S}_3 \), so that \( \mathcal{C} \) can be written \( \mathcal{D} + \mathcal{S}_3 \).

\( \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \) normalises into a star \( \mathcal{U} \) with the sole rays \( \mathcal{R}(x), \mathcal{S}(x) \). In terms of normalisation, \( \mathcal{U} \) and \( \left[ \frac{\mathcal{I}(x)}{\mathcal{I}(x)} \right] \) play the same role as the stars \( \left[ \frac{\mathcal{R}(x)}{\mathcal{S}(x)} \right] \) and \( \left[ \frac{\mathcal{P}_C(x) \cdot \mathcal{I}(x)}{\mathcal{I}(x)} \right] \) of \( \mathcal{O}_3 \); the difference is that \( p_C(x) \) is replaced with part of the \( p_T(x) \). The combination \( \mathcal{C} + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \) thus normalises into \( \left[ p_T(x) \right] \).

Like in ludics, cut-elimination works for general animæ; however, in the case of an épure, the subjective colours (cyan, red and their two plugged analogues) stand alone; hence they still stand alone after normalisation. In other terms, the normal form of an épure is still an épure. This means that our notion of consistency is a real, deductive one, not a paraconsistent, non deductive doohickey. In fact, we could define, like in ludics, truth as the existence of a an épure complying with the ordeals, which is more exciting than the Tarskian approach « truth is the fact of being true ». The fact that épures are preserved by normalisation makes this notion compatible with deduction, as long as normalisation converges, of course.

### A First order quantifiers and equality

This part is so new, the notions are so fresh, that can hardly be more than a blueprint; I am not yet familiar with the notions introduced, which may account for the sloppiness of notations. Hence the inclusion of this section in annex. I beg for the leniency of the reader in view of the absolute novelty of the ideas: the reduction of predicates to propositions!

#### A.1 First order variables

First order variables and related operations are the poor relatives of logic. Whereas we have a rather good understanding of what is a proposition, when it becomes a predicate, i.e., starts to « depend » upon a first order parameter, we are lost. Nobody knows what the parameters of predicate calculus stand for: they belong to the ideal semantic realm, i.e., to the kingdom of prejudice.
Let us give a simple, albeit informative, example: $\forall x. A \Rightarrow A[y/x]$, a familiar principle of predicate calculus, (roughly) correct and easy to formulate in the axiomatic style typical of XIX\textsuperscript{th} century logic. And also (slightly) wrong: from $\forall x. A \Rightarrow A[y/x]$ and the dual $A[y/x] \Rightarrow \exists x. A$, we get $\forall x. A \Rightarrow \exists x. A$ whose proof rests upon an irrelevant variable $y$. In the light of XX\textsuperscript{th} century logic, e.g., cut-elimination, the handling of such a proof requires the distinction between two kinds of variables: the variables proper, a.k.a. *eigenvariables*, those corresponding to $\vdash \forall$ and $\exists \vdash$ rules, and junk variables, those devoted to the case just considered and which behave like constants. Two kinds of first order variables, what an unknitting for this trifle! Especially since $\forall y(\forall x. A \Rightarrow \exists x. A)$ in which $y$ becomes proper, does not need this artificial category. We can indeed formulate sequent calculus in a pedantic way and indicate by a superscript $\vdash\{x,y,z,\ldots\}$ which free variables are available, so that $\forall x. A \vdash_{x} A[y/x]$ is valid only when $y \in X$. The rules $\vdash \forall$ and $\exists \vdash$ modify this stock of variables, to the effet that each variable is the *eigenvariable* of one of those rules:

$\begin{array}{ll}
\Gamma \vdash_{x} A, \Delta & \Gamma, A \vdash_{x} A, \Delta \\
\Gamma \vdash_{x} \forall x. A, \Delta & \Gamma, \exists x. A \vdash_{x} \Delta
\end{array}$

We just proposed a minor modification of predicate calculus reknitting first order variables. Now, we could consult semantics about the dubious principle $\forall x. A \Rightarrow A[y/x]$. Its answer is typical: the principle is valid since models are non empty. But why should models be non empty? This ukase does not follow from any deep consideration: this is just the conjunction of two features, the axiomatic style which makes $\forall x. A \Rightarrow A[t/x]$ easier to formulate and the obvious lack of interest of empty models. On the other hand, it unknits logic for no serious reason.

This example shows that, any time semantics is subpoenaed to elucidate a foundational point, it turns to be a bribed witness: it thus justifies the existence of objects by requiring domains to be non empty or explains equality by equality.

Another delicate issue is that of dependency. If a predicate is a function depending upon parameters, how can such a dependency make sense? Again, if we subpoena semantics on this issue, we end with a function from an elusive domain of parameters to propositions, i.e., with an infinite contraption, in contrast to mathematical experience: we can actually deal with predicates, what would be impossible if they were actually infinite. In this respect, the absolute zero is obtained with the idea of the universal quantifier as an infinite product: a proof of $\forall x. A$ should examine all possible cases, one by one!

### A.2 Equality

The treatment of equality is so bleak that, after 300 years, Leibniz’ definition

$$a = b \iff \forall X (X(a) \Rightarrow X(b))$$

is still of interest; despite obvious drawbacks. First, it refers to all possible properties $X(\cdot)$ thus making equality a most complex, super-synthetic contraption; while it should be simple, close to analytic. Worse, the definition is almost
empty: typically, is the the left « n » in « meaning » equal to the right one? Since one is to the right of the other, they can be distinguished by a property \( P(\cdot) \). The question thus reduces to the relevance of \( P(\cdot) \): Leibniz' equality supposes a preformatting which decretes what we can/cannot consider as a legitimate property of an object. Eventually, « legitimate properties » turn out to be those compatible with equality!

On this issue, semantics plays its usual role of bribed witness: the two « n » are equal when they refer to the same ideal object, i.e., when we decide to identify them!

Indeed, the basic mistake seems to be the recognition of those objects: why should there be objects? The only justification lies between convenience — the expression of algebraic structures — and pure conservatism: any discussion of the relevance of « objects » is a sort of blasphem against Frege. But no serious logical argument, as we shall now prove by reducing them to specific propositions.

If terms are indeed propositions, the aporia concerning the two n disappears: we are not relating properties of two objects, we are just relating two properties, period; the question of the relevance of \( P(\cdot) \) thus disappears. By the way, if we remove all problematic elements from Leibniz' definition: the second order quantifiers, hence getting \( X(a) \Rightarrow X(b) \), the objects, hence getting \( a \Rightarrow b \) and the classical setting, thus eventually getting \( a \rightarrow b \), we obtain something which still retains the deepest part of the definition: equality handled as a linear implication.

How do we turn terms into properties; Leibniz basically handles the « object » \( a \) through its properties \( P(\cdot) \), i.e., those \( P \) s.t. \( P(a) \). A proposition \( P \) with a hole inside makes sense in terms of épures: it can be seen as an ordeal waiting for a gabarit \( a \) and a vehicle (the proof of \( P(a) \)). This leaves us with an original idea, the only one since a very long time in this terra incognita: first-order variables refer to gabarits. Due to the problems of epidicticity (the matching \( X/\sim X \)), one must stick to a simple class: the simplest one is undoubtedly that of multiplicative gabarits.

A.3 Objections

It is natural to investigate a few objections against the identification terms = propositions (= gabarits). The main point is that equality becomes logical equivalence \( = \). Classical logic cannot offer us enough terms: the tautology \( a = b \vee b = c \vee c = a \) shows that there can be at most two different terms. Ditto for intuitionistic logic, up to double negation. \( \neg\neg(a = b \vee b = c \vee c = a) \).

The situation is quite different for linear logic. Here we shall use phase semantics\(^4\) to show that infinitely distinct objects can coexist: consider as phase space the additive group \( \mathbb{Z} \) with \( \perp := \mathbb{Z} \setminus \{1\} \). If \( X \subset \mathbb{Z} \), then \( \sim X = \{1 - x; x \notin X\} \), so that all subsets of \( \mathbb{Z} \) are facts. Among them, \( \emptyset \), the empty set. (Intuitionistic) negation, \( \neg X := \{!X \rightarrow \emptyset \} \) is equal to \( X \cap \{0\} \rightarrow \emptyset \). It is

\(^4\)This show that semantics can be useful in menial tasks!
equal to 0 if \(0 \in X\), to \(\top = \mathbb{Z}\) otherwise. If \(X, Y\) are distinct, then 0 cannot belong to both \(X \rightarrow Y\) and \(Y \rightarrow X\), hence, \(\neg(X \rightarrow Y \& Y \rightarrow X)\) takes the value \(\top\). All propositions of the space are therefore logically distinct, in the strong sense that their inequivalence takes the value \(\top\).

Another objection, or difficulty comes from the constructive contents of equality: should there be a proof of \(a = b\)? If we say that there is a unique proof (with no contents) when \(a = b\), nothing otherwise, we face a situation hard to handle. Fortunately, if equality is equivalence, we can expect non trivial proofs of equality. But we must face another problem: equivalence splits into two halves, \(a \rightarrow b\) and \(b \rightarrow a\). Now, let us refine our request and admit that bi-implication is not enough, we would like a sort of isomorphism. But it is almost impossible to coordinate the two halves. This is why we shall stick to multiplicative gabarits (and non negated literals): in this case, even if we cannot coordinate \(a \rightarrow b\) and \(b \rightarrow a\), the existence of the second half forces the first half to be isomorphic. A proof of equality is thus made of two unrelated isomorphisms! By the way, the possibility of several proofs of \(a = b\) shows that equality is distinct from identity.

### A.4 Connectives as partitions

Multiplicative gabarits can be approached through a calculus of partitions, whose lineaments are due to Danos & Regnier [2]. Let \(n > 0\); a \(n\)-ary (multiplicative) connective is a set \(C\) of partitions of \(\{1, \ldots, n\}\) subject to certain constraints.

If \(E, F\) are two partitions of \(\{1, \ldots, n\}\), consider the graph with vertices the disjoint sum \(E + F\) with \(n\) edges \(i = 1, \ldots, n\) linking \(e \in E\) and \(f \in F\) exactly \(i \in e \cap f\). \(E, F\) are orthogonal, \(E \perp F\), when the induced graph is a topological tree, i.e., is connected and acyclic. If \(p, q\) are the respective cardinalities of \(E, F\), Euler-Poincaré yields \(p + q - n = 1\).

A \(n\)-ary connective is a set \(C\) of partitions of \(\{1, \ldots, n\}\) equal to its bi-orthogonal; it must be non-trivial, i.e., neither empty nor full. In which case its orthogonal \(C^\perp\) is in turn a \(n\)-ary connective. A \(n\)-ary connective receives a weight, namely the common cardinality of all its partitions; if \(C\) has weight \(w\), then \(C^\perp\) has weight \(n + 1 - w\).

The typical example is the binary connective \(\land\) consisting of the sole partition \(\{\{1\}, \{2\}\}\) whose orthogonal, \(\lor\) consists of the sole partition \(\{\{1, 2\}\}\). All connectives cannot be constructed from those two, typically \(\forall\) (4-ary) which consists of \(\{\{1, 2\}, \{3, 4\}\}\) and \(\{\{2, 3\}, \{4, 1\}\}\); \(\exists\) consists of \(\{\{1, 3\}, \{2\}, \{4\}\}\) and \(\{\{2, 4\}, \{1\}, \{3\}\}\). The connective \(\forall, \exists\), for which proof-nets can be constructed, cannot be expressed in sequent calculus: they are not sequential.

\(n\)-ary connectives can be compared: \(C \subset D\) means that there is a bijection \(\varphi\) of \(\{1, \ldots, n\}\) s.t. \(\{\varphi(1), \ldots, \varphi(n)\} \perp C \forall D^\perp\). If \(C \subset D\), their weights \(w, w'\) are such that \(n + w + n + 1 - w' - 2n = 1\), hence \(w = w'\). Indeed, if \(C \subset D\), then \(\varphi(C) \in D\). Another numerical invariant, the number of partitions, a.k.a. size, of \(C\) is of interest: if \(C \subset D\), then \(D\) has a greater size. As a corollary, if
$C \subset D$ and $D \subset C$, then both have the same size. Then the bijection $\varphi$ which « proves » the inclusion $C \subset D$ is indeed an isomorphism.

### A.5 Connectives as gabarits

In order to turn a connective-as-partition into a gabarit, we must basically select $n+1$ disjoint terms $q_1(x) := q(1 \cdot x), \ldots, q_n(x) := q(n \cdot x), p(x)$ accounting for the $n$ variables of the connective and the output. Let thus $C$ be $n$-ary, and let $C \in C, c \in C$; the ordeal $\mathcal{O}_{Cc}$ consists of the $\mathbb{Z}[\hat{q}_d(x)]$ for $c \neq d \in C$ and $\mathbb{Z}[\hat{q}_d(x)/\hat{p}(x)]$.

Of course, if — say — $d = \{i, j, k\}$, $\mathbb{Z}[\hat{q}_d(x)]$ is short for $\mathbb{Z}[\hat{q}_i(x) \hat{q}_j(x) \hat{q}_k(x)]$.

The gabarit associated to $C$ is defined as the set of all $\mathcal{O}_{Cc}$, for $c \in C \in C$.

Gabarits can be delocated in the usual way: typically, by replacing $p(x)$ and the $q_i(x)$ with $r(p(x))$ and $r(q_i(x))$; the result is noted $r(C)$. Using delocation, orthogonality reduces to negation: delocate $C, D$ of the same arity, i.e., using the same $q_i(x), p(x)$ to the disjoint $r(x), s(x)$. To make the long story short, repaint the atoms $r(q_i(x)), s(q_i(x))$ in red and unpaint $r(p(x)), s(p(x))$, then the folklore on multiplicatives [2] establishes that the following conditions:

1. $\mathbb{Z}[\hat{r}(p(x))] \mathbb{Z}[\hat{s}(p(x))]$ complies with the gabarit $r(C) + s(D)$.

2. The cut $[r(p(x)) \otimes s(p(x))]$ can be replaced with $n$ cuts $[r(q_i(x)) \otimes s(q_i(x))]$, which are equivalent to the fact that $C^\perp = D$. More precisely: the first condition says that, whenever $C \in C, D \in D$, then $C \perp D$. The second condition says that, whenever $C \in C^\perp, D \in D^\perp$, then $C \perp D$. Together, they say that $C^\perp = D^\perp$, hence $C^\perp = D$.

An important operation is composition, best explained in terms of gabarits: let $C$ be a $n$-ary connective, using $q_1(x), \ldots, q_n(x), p(x)$ and let $D_1, \ldots, D_n$ be connectives of arities $k_1, \ldots, k_n$ using $s_1(x), \ldots, s_n(x), r(x)$. Delocate $D_1, \ldots, D_n$ into $q_1(D_1), \ldots, q_n(D_n)$. These $k+1$ connectives, in magenta and green, put together, induce, after normalisation, a gabarit $\mathcal{C}(D_1, \ldots, D_n)$ whose constellations use $p(x)$ as well as the $k_1 + \ldots + k_n$ rays $q_i(s_j(x))$ ($i = 1, \ldots, n; j = 1, \ldots, k_i$).

Composition can also be described in terms of partitions: partitions $D_i \in D_i$ are welded by means of a partition $C \in C$ and of a selection of a $d_i$ in each $D_i$; the resulting partition consists of the $\bigcup_{i \in C} d_i$ for each $c \in C$, together with the unselected $d' \in D$, which stay as they were.

As a set of partitions, $\mathcal{C}(D_1, \ldots, D_n)$ is not equal to its biorthogonal; hence let us define $\mathcal{C}[D_1, \ldots, D_n] := \mathcal{C}(D_1, \ldots, D_n)^{\bot \perp}$. The main result about composition is that it commutes with negation (or orthogonality):

$$\mathcal{C}[D_1, \ldots, D_n] = \mathcal{C}[D_1^\perp, \ldots, D_n^\perp]$$

Indeed, if orthogonality corresponds to the identity group of logic (axiom and cut), then this commutation is nothing but $\eta$-expansion of identity links and the reduction of a cut on a compound connective.
\( \land \) and \( \otimes \) define, through composition, what we could style *series/parallel* connectives. The question is thus to determine whether all connectives are of this form; as expected, the answer is negative, the simplest example being the circular connective \( \bowtie \). Indeed, if a connective is series/parallel, it (or its negation) can be written as \( C \rightarrow D \). Now, it easy to see that the partitions of \( C \oplus D \) are exactly the unions \( C \cup D \), with \( C \in C, D \in D \) (no welding). Due to its circular nature, neither \( \bowtie \) nor \( \bowtie \perp \) is of this form, hence the connective is not series/parallel. The connectives \( \bowtie, \bowtie \perp \) are not *sequential* in the sense that they admit no sequent calculus formulation: no system of cut-free rules will establish \( a \rightarrow a', b \rightarrow b', c \rightarrow c', d \rightarrow d', \bowtie(a, b, c, d) \vdash \bowtie(a', b', c', d') \). \( \bowtie, \bowtie \perp \) are, however, perfectly legitimate logical operations, which can be handled by means of, say, proof-nets.

### A.6 Terms and equality

Predicate calculus supposes a language of terms, with abstract function letters, what transcendentally syntax can hardly provide. However, connectives (represented by their gabarits) form perfectly swell functions, typically \( \bowtie \) which is 4-ary and circular. Connectives being of non zero arity, there is no room for constants: hence no hope of proving \( \forall v A \Rightarrow \exists v A \). This limitation is not that important: should I need a constant \( c \), I simply delegate the task to a specific variable \( v_0 \), with the convenant that \( v_0 \) is implicitly universally quantified; hence we can get \( \forall v A \Rightarrow \exists v A \) if life to hard to endure without this principle!

The next thing is linked with equality; the vague — but on the whole, real — message of Leibniz’ equality is that objects are (simple) gabarits and equality is (linear) equivalence. Equivalence being anything but a primitive connective, it must be split into a conjunction, \( v \subset w \land v \supset w \); equality is replaced with \( \subset \). Now, how to handle \( \subset \)? The obvious candidate is the linear implication \( v \rightarrow w \), i.e., \( \sim v \perp w \); this means that a variable \( v \) may occur negatively or positively and that these two sorts of occurrences hardly mingle. Indeed:

1. Variables come by pairs \( v, \sim v, w, \sim w, \ldots \), used only positively, e.g., \( v \perp w \).
2. Terms come also by pairs, but they are homegeneous, they do not mix \( v, w, \ldots \) with \( \sim v, \sim w, \ldots \). Typical terms are thus \( \bowtie(v, w, w, v) \) and its negation \( \bowtie((\sim v, \sim w, \sim w, \sim v)) \); but \( v \perp \sim v \) is not a legitimate term.

The strict schizophrenia \( v/\sim v, T/T \) is the only way to coerce implication into something like inclusion: \( T \subset U \) being \( \top \perp \top \), the fact that \( \top \) uses only the \( \top \) and \( T \) only the \( v \) forces a proof to be a bijection between the occurrences of variables in \( T \) and in \( U \). If \( U \) were allowed to mix \( v, \sim v \), then the proof could make use of a link between those two, what would definitely lead astray from inclusion.

In this very first approach, the role of a formula depending upon variables \( v, w, \ldots \) is rendered by lacunary ordeals: the ordeal contains zones that should be filled with \( v, \sim v, w, \sim w, \ldots \). These zones are individised by the colour \textcolor{cyan}{cyan-} if \( f(v) \).

---

[^5]: And \( \supset \), but \( v \supset w \) is nothing but a delocation of \( w \subset v \).
waits for one of our variables, and $G$ is a gabarit corresponding to a connective, then we can plug the delocation $r(G)$ by means of the colouring $\llbracket r(G) \rrbracket$.

In this way, we start to see how dependency might work. Let us try an example, the simplest of all: the ordeal $\llbracket \frac{r(p(x))}{r(x)} \rrbracket + \llbracket \frac{s(p(x))}{s(x)} \rrbracket + \llbracket \frac{r(q(x))}{r(q(x))} \rrbracket + \llbracket \frac{s(q(x))}{s(q(x))} \rrbracket$ can be fed with various connectives (in red) making use of $p(x)$ and various $q_i(x) := q(i \cdot x)$. This ordeal can be seen as the inclusion $v \subset w$, but $v, w$ only make sense when we use them. If we feed the « hole » with opposite terms corresponding to the connectives $\oplus, \otimes$ then the result becomes the inclusion $G \subset G$, which has a proof, namely $\llbracket \frac{r(q(x))}{s(q(x))} \rrbracket$.

A.7 Function letters

A last objection to our approach: predicate calculus makes use of function letters. These function letters should be injective: typically, we shouldn’t prove $f(x, y) = f(y, x)$. The question is therefore the following: do we have enough injective terms? The logic folklore tells us that an injective binary term, a « pairing function », is enough. Indeed, $(v \ Y u) \otimes (v \ Y v \ Y w)$ provides such a pairing function, loosely inspired from that of set theory $\{ \{x\}, \{x, y\}\}$.

Its injectivity comes from an easy lemma: a term can be split as the $\gamma$ of several terms that cannot in turn be split as $\gamma$; the splitting is unique up to permutation. The same holds, dually, for $\otimes$. $(t \ Y u) \otimes (t \ Y t \ Y u)$ can uniquely be split as a tensor $T \otimes U$. Both of $T, U$ can in turn be split as a $\gamma$ of prime components $t_i (i = 1, \ldots, k)$; this decomposition is unique if we consider the $t_i$ up to equality and introduce their multiplicities $m_i$ and $n_i$ in $T, U$; $T, U$ can be distinguished by the requirement $m_i \leq n_i (i = 1, \ldots, k)$. We can recover $t$ as the $\gamma$ of the $t_i$ each of them with multiplicity $n_i - m_i$ and $u$ as the $\gamma$ of the same $t_i$, each of them with multiplicity $2m_i - n_i$.

A.8 Quantification

Equality (indeed inclusion) provides us with (at last!) one predicate constant! It will remain, in the next section, to see how a general predicate $X(v, u)$ can work, but we are no longer stuck with the question of dependency: we know how to make a gabarit depend upon a first order variable!

The next step is naturally that of quantification; whereas the dependency upon $v$ could be handled manually, by deciding what to plug/not to plug on the cyan zone, something new occurs: indeed, the variable $v$ eventually reaches its transcendental status of variable through quantification. Consider a quantification $B := \forall v A$ or $B := \exists v A$, located at $p_B(x)$. Then three coloured sublocations should be introduced, $p_v(x), p_y(x), p_x(x)$. Each time a « hole » for $v$, say $r(x)$ occurs, we « fill » it by means of some $\llbracket p_v(x \cdot i) \rrbracket$, ditto for the holes for $y, \ldots$. This operation is that the holes
are centralised in $p_v(x), p_q(x)$, the indices $i$ being used to distinguish between them in case of repetitions. Those centralised holes are no longer waiting for connectives $G, G^\perp$: they expect their tensorised versions $G \cdot y, G^\perp \cdot y$.

Let us now consider the existential quantifier: $B := \exists v A$. What is expected is purely derealist, i.e., a gabarit filling the holes. In standard logical terms, this gabarit is styled an *existential witness*. We therefore require that an épure for $\exists v A$ consists of tensorised red gabarits $p_v(G) \cdot y, p_v(G^\perp) \cdot y$, together with a vehicle $p(A[y])$. The delocation of variables has the performative value of an ordeal for $p_k k_k$. The delocation of variables has the performative value of an

The only thing to observe is that, since $G, G^\perp$ depend upon other variables $w, \vartheta, \ldots$, this means that $G$ has in turn « holes ».

And that these holes should be eventually delocated; so let us do it immediately.

The only thing to observe is that, since $G, G^\perp$ are tensorised with $y$, the holes should be delocated at something like $p_v(x \cdot (i \cdot y)), p_q(x \cdot (i \cdot y)).$

The second thing is really more serious: the two gabarits $G, G^\perp$ should be complementary. This is the first major instance of an epidictic constraint, something not quite justifiable on analytic grounds. Of course, by *ad hoc* identity links, it is possible to make sure that those are not too big; but this does not exclude the case $G := v \ Y (v \otimes v)$, with, in the role of $G^\perp$, $H := (\vartheta \otimes \vartheta) \ Y \vartheta$: a cut between $G, H$ need not normalise.

Two things should be observed. First, since there is no closed term, $G, G^\perp$ depend upon other variables $w, \vartheta, \ldots$. This means that $G$ has in turn « holes ».

And that these holes should be eventually delocated; so let us do it immediately.

The only thing to observe is that, since $G, G^\perp$ are tensorised with $y$, the holes should be delocated at something like $p_v(x \cdot (i \cdot y)), p_q(x \cdot (i \cdot y)).$

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The issue of complementarity of gabarits turns out to be the foundational blindspot which cannot be fixed in general: otherwise no doubt would remain about the safety of logical inference. Indeed, the distinction épure/anima is of epidictic nature and the restriction of terms to special gabarits as well. A simple argument shows that complementarity of gabarits cannot be expressed by means of ordeals: if $G, G^\perp$ do comply with an ordeal, so do any subgabarits of them: typically, in the previous counter-example, $H$ is a subgabarit of $G^\perp$.

Let us now turn our attention towards the universal quantifier: it is a sort of intuitionistic implication and should be switched in a way to make sure that $v$ is unknown « below » $B$. But this is routine. One must also provide values for $v$, i.e., specific gabarits. The idea being that $v$ only relates to $\vartheta$. For this, we shall consider three cases:

1. $v$ is the trivial (unary) connective $\text{Id}$, and so is $\vartheta$.
2. $v := \text{Id} \ Y \text{Id}$ and $\vartheta := \text{Id} \otimes \text{Id}$.
3. $v := \text{Id} \otimes \text{Id}$ and $\vartheta := \text{Id} \ Y \text{Id}$.

Items 2 and 3 make sure that $v$ is not plugged with anything but $\vartheta$; we must however exclude a joke, that of a plugging restricted to the sole sublocations involved in cases 2 and 3. This is the reason for item 1; this full connective should be written without index, i.e., $[\vartheta(x), p(x)]$ (and not $[\vartheta(1 \cdot x), p(x)]$).

The example of an inclusion $v \subset w$ given last section can be developed so as to become $\forall v \subset v$. In a first step (basically delocation of the variables) our
ordeal (supposedly located in \(x\), with the two sublocations \(r(x), s(x)\) becomes either: 
\[
\begin{align*}
[ p_2(p(x)) ] & + [ p_2(q(x)) ] + [ p_2(q(x)) ] + [ p_2(q(x)) ] \\
[ p_2(p(x)) ] & + [ p_2(q(x)) ] + [ p_2(q(x)) ] + [ p_2(q(x)) ] \\
[ p_2(p(x)) ] & + [ p_2(q(x)) ] + [ p_2(q(x)) ] + [ p_2(q(x)) ] \\
[ p_2(p(x)) ] & + [ p_2(q(x)) ] + [ p_2(q(x)) ] + [ p_2(q(x)) ] \\
[ p_2(p(x)) ] & + [ p_2(q(x)) ] + [ p_2(q(x)) ] + [ p_2(q(x)) ]
\end{align*}
\]
This first step basically changes nothing but the location of the missing gabarit and the conclusions (from \(r(x), s(x)\) to the sole \(x\), which involves a duplication). The universal quantification adds one of the following to any of these two ordeals one of the following substitutive gabarits:

1. 
\[
[ p_2(p(x)) ] + [ p_2(q(x)) ] + [ p_2(q(x)) ] + [ p_2(q(x)) ]
\]
2. 
\[
[ p_2(p(x)) ] + [ p_2(q(1 \cdot x)) ] + [ p_2(q(2 \cdot x)) ] + [ p_2(q(2 \cdot x)) ]
\]
3. 
\[
[ p_2(p(x)) ] + [ p_2(q(1 \cdot x)) ] + [ p_2(q(1 \cdot x)) ] + [ p_2(q(2 \cdot x)) ]
\]
4. 
\[
[ p_2(p(x)) ] + [ p_2(q(1 \cdot x)) ] + [ p_2(q(1 \cdot x)) ] + [ p_2(q(2 \cdot x)) ]
\]
5. 
\[
[ p_2(p(x)) ] + [ p_2(q(2 \cdot x)) ] + [ p_2(q(1 \cdot x)) ] + [ p_2(q(2 \cdot x)) ]
\]

The proof of the formula is, as expected, the vehicle 
\[
J[q(x)] J[q(x)]
\]

A.9 Predicate variables

As we explained in [6], generic propositional constants are a mistake — of the same sort as improper variables; ditto for generic predicate constants.

Let us first consider linear predicates: a binary predicate \(P(\cdot, \cdot, \cdot)\) is basically a lacunary structure with four « holes » to be filled with the two arguments \(t, u\) and their negations \(\bar{t}, \bar{u}\): \(P(t, \bar{t}, u, \bar{u})\), as well as a « conclusion »: the four holes are disjoint sublocations of the conclusion. This requires a lot of pedantic definitions; but the problem basically amounts at finding the right ordeals for predicates. The question is made complex by the heavy amount of red tape needed to make this precise; but the only thing that should be eventually ensured is the restriction of identity links between \(\sim P(\bar{t}, t, u, \bar{u}), P(t', \bar{t}', u', \bar{u}')\) to vehicles \(V\) linking \(\sim P, P\) (the respective conclusions; they are sort of propositions, hence by [6], matchings other than \(\sim P/P\) are excluded) and establishing the equalities \(t = t', u = u'.\) These equalities are obtained by showing that \(V\) splits into four parts, linking respectively \(\bar{t}/t', t/\bar{t}', u'/u', u/\bar{u}'\). The general handling of « first order » variables, makes sure that \(V\) links pairs of opposite variables \(v/\bar{v}, u/\bar{u}\), etc.

Our switching of the predicate \(P\) consists in selecting a 4-ary multiplicative connective \(C\) which will be used to connect the four holes with the conclusion. Of course, \(\sim P\) will simultaneously receive the dual switching \(C^\perp\). Now, what are the possible matchings? Matchings between, say, \(t'\) and \(t\) are excluded: their variables cannot be connected. Now, \(V\) cannot link \(t'\) with \(\bar{t}'\) or \(u':\) consider
\( \mathcal{C} := \{1, 2, 3, 4\} \): \( \mathcal{V} \) can only link \( t' \) with \( \bar{t} \) or \( \bar{u} \), possibly both. By considering \( \mathcal{C}(x, x', y, y') := (x \otimes x') \not\supseteq (y \otimes y') \), we see that \( \mathcal{V} \) relates \( t, \bar{t} \) either with the sole \( t', \bar{t}' \) or with the sole \( u', \bar{u}' \). With \( \mathcal{C}(x, x', y, y') := x \not\supseteq (x' \otimes y \otimes y') \), we see that \( \mathcal{V} \) actually relates \( t \) with the sole \( \bar{t}' \), etc. Eventually, \( \mathcal{V} \) split into the four matches \( \bar{t}/t', t/\bar{t}', \bar{u}/u', u/\bar{u}' \).

Unfortunately, we are not quite done: our approach to predicates has been linear: in the unary case, it proves \( t = u \not\supseteq (P(t) \not\supseteq P(u)) \), i.e., the linearity of \( P \). But not all predicates can be linear: a general predicate should only enjoy the weaker \( t = u \Rightarrow (P(t) \not\supseteq P(u)) \). We thus propose to modify our pattern by replacing \( t, \bar{t}, \ldots \) with their « tensorisations » with a fresh variable \( y \), in the spirit of our treatment of the connective \( \otimes \) expounded in [6]. Remember that this extra variable has its own switching, either \( y = x \) or \( y = 1 \); all « occurrences » of \( P/\sim P \) should be switched in the same way, either \( x \) or \( 1 \).

### A.10 Discussion: dependency

We are done with predicate calculus; since we dumped the semantic prejudice, it might be of interest to ask how the issue of linearity of predicates fits in. The answer lies in a chapter yet to be written concerning second order existential quantification which rests upon an explicit substitution of an abstraction term — i.e., a gabarit « depending » upon first order parameters — for a predicate variable. Our constraints, which ensure the implicit exponentiated \(~!{(t = t'),!(u = u')}\), ensure the soundness of this specific usage.

The issue of « dependency » of a proposition over a parameter is a difficult one. Mainly because it has been blurred by semantics which tends to confuse \emph{usine} and \emph{usage}: a dependent formula thus appears as a function from parameters to propositions. This stumbles on many problems; the first one being that a function, a badly infinite object, cannot be directly handled by logic. Even if we succeed in circumventing this problem, we are left with the problem of determining the shape of the « function ».

For a while, I put my fishing net in the waters of Martin-Löf’s type theory [8] and its rather remarkable notion of dependent type inspired from De Bruijn. How can I express the dependency of a type over a parameter? \( \Sigma x \in A B(x) \) seems the most natural, but its negative use will coerce me into something like \( \Pi x \in A \sim B(x) \): there are, at least, two dual forms of dependency and no way to reconcile them.

Now, let us forget semantics and just observe that we are able to manage those dependencies without knowing their actual shapes: this is \emph{usine}, the logic factory. When I write the identity « axiom » \( P(t) \vdash P(t) \), my sole concern is the identity between the two \( P \) and the equality between the two \( t \). \emph{Usine} only makes sure that we are respecting this restriction. And for this, only very rudimentary dependencies are needed: in the binary case, \( P \) becomes a 4-ary multiplicative connective \( \mathcal{C} \), and there are not that many choices. If we were interested in Martin-Löf’s type theory — which requires a more important investment —, we should perhaps be coerced into making a switching between the two basic forms \((\Sigma/\Pi)\) of dependency.
The actual, hard-boiled, dependency occurs at the level of usage, i.e., logical consequence through cut-elimination. The question becomes the relevance of the sampling done at the usine stage. For instance, a first attempt led us to linear predicates: l’usage in this case must be limited to linear abstraction terms, typically \{x; t = u\} when \(x, t, u\) are themselves linear in \(x\). But our more elaborate non-linear sampling will justify l’usage of general abstraction terms.

Now, the question: how can a limited, finite, sampling guarantee incredibly complex phenomena? This is the mystery of epidicticity, this mysterious balance between the two sides of a gabarit, rights vs. duties. It is not because it cannot be fully rational — this is the basic lesson of incompetence — that it should be fully irrational, i.e., axiomatic.\

In order to understand the epidictic mystery, I think that one turn our attention to arithmetic, even very weak systems: we understand so little that any progress in that direction can be crucial.

\[6\] In modern Greek, axiomatikos means officer; the axiomatic approach to logic is a sort of martial law.
References


