Substitution dynamical systems on infinite alphabets

Sébastien Ferenczi
Institut de Mathématiques de Luminy
CNRS, UPR 9016
163 av. de Luminy
F13288 Marseille Cedex 9 (France)
and Fédération de Recherche des Unités de Mathématiques de Marseille
CNRS - FR 2291
ferenczi@iml.univ-mrs.fr

July 15, 2005

Abstract

We give a few examples of substitutions on infinite alphabets, and the beginning of a general theory of the associated dynamical systems. In particular, the *drunken man substitution* can be associated to an ergodic infinite measure preserving system, while substitutions of constant length with a positive recurrent infinite matrix correspond to ergodic finite measure preserving systems.

1 Substitutions

Substitutions are well-known and much used mathematical objects, and throughout the paper we refer the reader to the Old and New Testament, [QUE] and [PYT-1].

Definition 1.1 *Let A be a finite or countable set, called the* alphabet*, and its elements will always be called* letters*.*

A word *of* length $|w| = k$ *is a finite string* $w_1...w_k$ *of elements of A ; the concatenation of two words w and w' is denoted multiplicatively, by ww'. A word $w_1...w_k$ is said to* occur *at place i in the infinite sequence or finite word u if $u_i = w_1, ..., u_{i+k-1} = w_k$; we say also that w is a* factor *of u; when u is finite, we denote by $N(w,u)$ the number of these occurrences.*

Let A be a finite alphabet and $u = (u_n, n \in \mathbb{N}) = u_0u_1u_2...$ be a one-sided sequence on A.

The language *$L_n(u)$ is the set of all words of length n occurring in u, while $L(u)$ is the union of all the $L_n(u)$.*

A substitution *is an application from an alphabet A into the set $A^\star$ of finite words on A; it extends to a morphism for the concatenation by $\sigma(ww') = \sigma w \sigma w'$.*

It is primitive *if there exists k such that a occurs in $\sigma^k b$ for any $a \in A$, $b \in A$.*

It is of constant length L *if the length of* σa *is* L *for any* $a \in A$.
A fixed point *of* σ *is an infinite sequence* u *with* $\sigma u = u$.

For any sequence $u = (u_n, n \in \mathbb{N})$ on a finite alphabet A, we can define the (topological) *symbolic dynamical system* associated to u: we first take $\Omega = A^{\mathbb{N}}$, equipped with the product topology (each copy of A being equipped with the discrete topology) and T the one-sided shift

$$T(x_0x_1x_2\ldots) = x_1x_2x_3\ldots$$

then X_u is the closure of the orbit of u under T. The dynamical system associated to a primitive substitution is the symbolic system (X_u, T) associated to any of its fixed points.

In the usual case when A is finite, the theory is well-established, see for example [QUE], [PYT-1]: under the (relatively mild) assumption of primitivity, the symbolic system X_u, T is *minimal*: the closed orbit of any point under T is the whole X_u, or, equivalently, for all m there exists n such that every word of length m occurring in u occurs in every word of length n occurring in u. Under the same assumption, the system is *uniquely ergodic*: it admits a unique invariant probability measure μ. The measure-theoretic dynamical systems built from primitive substitutions give many (one could say: most) interesting examples in ergodic theory, such as the Morse, Fibonacci, Chacon and Rudin-Shapiro substitutions (or systems).

In the present paper we try to build the bases of a theory of substitutions on infinite alphabets, and a large part of it is devoted to the study of a simple (in its definition, at least) example, which we call the *drunken man substitution* (the reason of this name will be found in section 3 below, but we take this opportunity to recall the paramount influence of Gérard Rauzy in this whole field of mathematics). Here most results and techniques of the finite case fail; still, we have been able to adapt a few main tools, such as the Rokhlin stacks, the step-by-step determination of the measure of cylinders, and (to a lesser extant as it seems more difficult to generalize to other examples) the theory of complexity and special factors; thus we build from this substitution a conservative ergodic infinite measure preserving system of (Krengel) entropy zero, and introduce on the way another interesting substitution, corresponding to the induced system on a cylinder. Then we attempt to generalize this theory, using a natural classification of substitutions through the recurrence properties of their incidence matrix; we have a whole class of examples, the substitutions with constant length, a property of synchronization, and a positive recurrent irreducible aperiodic matrix, which give birth to ergodic finite measure-preserving systems. In the null recurrent case (to which the drunken man substitution belongs), only a part of the results can be generalized, while next to nothing is known in the transient case. And we illustrate this theory by examples: one of them, the infini-Bonacci substitution in Example 3.8 below, is shown to give a system which is isomorphic to a well-known one, while the others give new dynamical systems, which clamor to be studied further.

I acknowledge a great debt towards Christian Mauduit for proposing me this whole subject of research, and more precisely the drunken man substitution (together with its name), and for useful discussions during this work.

2 A fundamental example

The following broad question was asked by Christian Mauduit: what can be said of the following substitution on $A = \mathbb{Z}$?

Example 2.1 (The drunken man substitution)

$$n \to (n-1)(n+1)$$

for all $n \in \mathbb{Z}$

The first obstacle is that, if we look at the k-th image of 0, it is made only of even (resp. odd) numbers if k is even (resp. odd); this reflects the fact that the matrix has period two (see section 3 below). Hence the right substitution to consider is

Example 2.2 (The squared drunken man substitution)

$$n \to (n-2)nn(n+2)$$

for all $n \in A = 2\mathbb{Z}$

This substitution, which we denote by σ, has no fixed point; we create one artificially by adding a letter ι (for initial) and adding the rule

$$\iota \to \iota 0;$$

this gives a new substitution σ', which has a fixed point beginning by ι, where ι occurs only as the first letter; in all this section, we denote by ιu the fixed point of σ', and we say that u is the *fixed point of* σ. We define a subset X of $A^{\mathbb{N}}$ to be the set of all sequences $x = x_0x_1\dots$ such that every word occurring in x is in the language $L(u)$; or, equivalently, we can define X to be the set of all sequences $x = x_0x_1\dots$ such that every word occurring in x occurs in $\sigma^n a$ for at least one $a \in A$ and $n \geq 0$.

X is then a closed subset of the (noncompact) set $\mathbb{Z}^{\mathbb{N}}$ equipped with the product topology (each copy of $\mathbb{Z}$ being equipped with the discrete topology), and, if T is the one-sided shift defined in section 1, $TX \subset X$. We say that (X, T) is the (non-compact) symbolic system associated to the substitution σ.

If $w = w_0 \dots w_s$, the *cylinder* $[w]$ is the set $\{x \in X; x_0 = w_0, \dots, x_s = w_s\}$.

The substitution σ acts also on infinite sequences of X, and $\sigma X \subset X$

2.1 Combinatorial and topological properties

It is trivially false that, in any given sequence x of X, for all m there exists n such that every word of length m occurring in x occurs in every word of length n occurring in x; but on an infinite alphabet the minimality of the system (X, T) would be equivalent to a weaker property, namely that any word occurring in one element of X occurs in every element of X. But in fact this property is not satisfied here:

Proposition 2.3 *(X,T) is not minimal.*

Proof
Let v be the sequence beginning by $\sigma^n(2n)$ for all n; v is well defined as $\sigma^n(2n) = \sigma^{n-1}\sigma(2n) = \sigma^{n-1}(2n-2)\sigma^{n-1}(2n)\sigma^{n-1}(2n)\sigma^{n-1}(2n+2)$ and hence v is in X. But the smallest number in $\sigma(2n)$ is $2n-2$, which appears only once, and at the beginning of this word; inductively, for all $p \leq n$, the smallest number in $\sigma^p(2n)$ is $2n-2p$, which appears only once, and at the beginning of this word. Hence Tv is an infinite sequence in X without any occurrence of the letter 0, and the positive orbit of Tv does not visit the cylinder $[0]$. ♣

The sequence v is surprising, but we shall see in Proposition 2.23 below that it is in some sense an isolated counter-example: every sequence in X without any 0 is built in a similar way, and it cannot be extended to the left, hence v and similar examples have *no pre-image* by T and T is *not surjective*.

Though individual sequences may have strange properties, we are looking for good statistical properties for "typical" sequences of X. This involves looking for invariant measures; but here the situation is also different from the finite case, as

Lemma 2.4 *The number of occurrences of the letter $2p+2k$ in the word $\sigma^n(2p)$ is the binomial coefficient C_{2n}^{n+k} for $-n \leq k \leq n$, 0 for $k > |n|$.*

Proof
By induction on n: we write $\sigma^n(2p) = \sigma^{n-1}(2p-2)\sigma^{n-1}(2p)\sigma^{n-1}(2p)\sigma^{n-1}(2p+2)$ and count the occurrences in each part. ♣

Proposition 2.5 *There is no finite measure on X invariant under T.*

Proof
Because of the above lemma, the maximal number of occurrences of any letter k in the word $\sigma^n p$ is C_{2n}^n, which is smaller than $\frac{K}{\sqrt{n}}2^{2n}$ if n is large. Let v be any sequence in X, n large, q larger than n, r maximal for the relation $q \geq r2^{2n}$; $v_1 \dots v_p$ can be decomposed as $w_1 \dots w_s$ where $s \leq r+1$ and each w_i is some $\sigma^n(a_i)$ except w_1 which is a suffix of $\sigma^n(a_1)$ and w_s which is a prefix of $\sigma^n(a_s)$. Then the number of occurrences of any letter k in $v_1 \dots v_q$ is at most $(r+1)C_{2n}^n \leq \frac{K}{\sqrt{n}}\frac{r+1}{r}q$. Hence

$$\frac{1}{q}N(k, v_1 \dots v_q) \to 0$$

when $q \to +\infty$.

But if there exists a finite invariant measure, we can find an ergodic invariant probability μ; the ergodic theorem then implies $\mu([k]) = 0$ for every k; hence $\mu(X) = 0$, which is a contradiction. ♣

In spite of all these differences, in one way σ behaves like a substitution on a finite alphabet, and this will be our main tool in the next section: we can describe it with *Rokhlin stacks*, as in Lemma 6.10 of [QUE] though we have to use a countable family of them.

To build them, we need first a notion of *synchronization*, allowing us to know how a word can be decomposed into words of the form σn, $n \in 2\mathbf{Z}$; there are different possible notions, see [CAS], [MOS] or section 3 below for further discussion, but fortunately our σ satisfies one of the strongest, which is both a property of synchronization and *recognizability*.

Definition 2.6 *Let σ be a substitution on an alphabet A, with a fixed point u: it is* left determined *if there exists N such that, if w is a word of length at least N in the language $L(u)$, it has a unique decomposition $w = w_1 \dots w_s$ where each w_i is a σa_i for some $a_i \in A$, except that w_1 may be only a suffix of σa_1 and w_s may be only a prefix of σa_s, and the a_i, $1 \le i \le s-1$, are unique.*

Lemma 2.7 *σ is left determined.*

Proof
Any word of length at least 3 contains either some $nn(n+2)$, or $n(n+2)(n+2)$, or $n(n+2)m$ for $m \ne n+2$, or $mn(n+2)$ for $m \ne n$, and this is enough to place in a unique way the bars between the w_i, hence the w_i are unique; and the last letter of σa determines the letter a.♣

Lemma 2.8 *The system (X, T, μ) is generated by a countable family of Rokhlin stacks: namely, for every $n \in \mathbb{N}$, X is the disjoint union of the $T^k\sigma^n[j]$, $j \in 2\mathbf{Z}$, $0 \le k \le 4^n - 1$.*

Proof
Let $x = x_1 \dots \in X$, M from Lemma 2.7; $x_1 \dots x_M$ has a unique decomposition $w = w_1 \dots w_s$ as in Definition 2.7; for any $n \ge M$ the unique such decomposition of $x_1 \dots x_n$ begins with $w_1 \dots w_s$, and this gives a unique infinite sequence a_i, $i \ge 1$, such that x begins by $w_1\sigma a_2 \dots \sigma a_i$ for every i. And $\sigma a_1 = w_1' w_1$, with $k_1 = |w_1'|$, $0 \le k_1 \le 3$. Then $w_1' x$ is the unique preimage of x by T^{k_1}, $w_1' x \in \sigma[a_1]$, $x \in T^{k_1}\sigma[a_1]$, and x is in no other $T^k a$ for $a \in A$ and $0 \le k \le 3$.

This proves the lemma for $n = 1$. For $n = 2$ we apply the same reasoning to the word $A = a_1 \dots a_i \dots$ which is in X again; and we proceed inductively for every n.♣

Henceforth we denote by $G_{n,2q}$ the stack $\cup_{j=0}^{4^n-1} T^j\sigma^n[2q]$. The stack $G(n, 2q)$ is made of *levels* $T^j\sigma^n[2q]$, and such a level is included in the cylinder $[a]$ (thus *labelled* by the letter a) if a is the $j+1$-th letter of $\sigma^n(2q)$. The levels of $G(n, 2q)$ are viewed as stacked above one another as j increases from 0 to $4^n - 1$.

Proposition 2.9 *The shift T is a bijection on a closed T-invariant subset of X with a countable complement.*

Proof
From the proof of Lemma 2.8 we deduce that a point $x \in X$ can be extended uniquely to the left when it is in any $T^k\sigma^n[j]$, $j \in 2\mathbf{Z}$, $1 \le k \le 4^n - 1$. Hence, if x is a point without a unique left extension, for any $n \ge 0$ there exists $q_n \in 2\mathbf{Z}$ such that x begin by $\sigma^n q_n$; because of Lemma 2.7 this implies $q_{n-1} = q_n - 2$ for all n, and there are at most countably many such points. And by deleting countably many orbits we get our result. ♣

2.2 Natural measure and ergodicity

The last tool we need is a "natural" notion of measure; this is inspired by the finite alphabet case, where the only invariant measure can be explicitly built by equation 1 below. Though there is no unicity in our case, it is most useful to be able to build an explicit invariant measure.

Definition 2.10 *For any words v and w, we say that v is an* **ancestor** *(under σ) of w with multiplicity m if w occurs in σv at m different places.*

We say that μ is a **natural measure** *on (X,T) if it is T-invariant and for each cylinder $[w]$,*

$$\mu[w] = \frac{1}{4}\sum \mu[v]m(v), \tag{1}$$

the sum being taken on all its ancestors v and $m(v)$ denoting their multiplicities.

Proposition 2.11 *There is a unique natural measure μ, if we normalize by giving to the cylinder $[0]$ the measure 1; it is an infinite measure on X invariant under T.*

Proof
First we define μ; the naturality equation 1 on one-letter words gives

$$4\mu[2n] = 2\mu[2n] + \mu[2n-2] + \mu[2n+2]$$

hence all the letters have the same measure, 1 by our normalization condition. The measures of cylinders of length 2 are defined recursively by the equation 1: words of the form $(2n)(2n)$ and $(2n)(2n+2)$ only have ancestors of length 1, then words of the form $(2n)(2n-2p)$, $p>0$ only have ancestors of the form $(2n)(2n-2p+4)$, and this exhausts all words of length 2 of nonzero measure. Then if $k>2$ each word of length k has only ancestors of length at most $k-1$, and this allows to define them by equation 1.

Then we have to check that for a word w, $\sum_{a\in A}\mu[aw] = \sum_{a\in A}\mu[wa] = \mu[w]$; because of equation 1 we need to check these relations only for 1-letter words, such as $\mu[2n] = \sum_{p=-1}^{+\infty}\mu[(2n)(2n-2p)]$, and we check this is true as $\mu[(2n)(2n-2p)] = 2^{-p-2}$ for $p\geq -1$.

Hence μ is T-invariant, and clearly $\mu(X)$ is infinite. ♣

Note that the measure μ is nonatomic, and hence T is a bijection on a set whose complement has measure 0. μ can be said to *preserve the Rokhlin stacks*: for every q, when n varies each $G(n,2q)$ has measure $\mu(G_{n,2q}) = \mu[2q]$.

Proposition 2.12 *The system (X,T,μ) is* recurrent *(or* conservative*): namely, for every set B with $0<\mu(B)$, $\mu\{x\in B; T^n x\notin B \text{ for every } n>0\} = 0$.*

Proof
We first check this result for $B=[0]$. For every n and q $\mu(G_{n,2q}) = 1$ and each level $T^j\sigma^n[2q]$ has measure 4^{-n}; hence $\mu(G_{n,2q}\cap[0]) = 4^{-n}N(0,\sigma^n(2q))$; in particular, we have $\mu(G_{n,2q}\cap[0]) \geq \mu(G_{n,2q'}\cap[0])$ if $0\leq q\leq q'$ or $q'\leq q\leq 0$. And $\mu(G_{n,2q}\cap[0]) = 0$ if $|q|>n$.

Fix $n > k$; if $-n \leq q \leq n-2k$, the upper 4^{n-k} levels of $G_{n,2q}$ correspond to the word $\sigma^{n-k}(2q+2k)$ which contains a zero as $|q+k| \leq n-k$; hence the set S_n of levels in the stacks $G_{n,2q}$, $-n \leq q \leq n$ which have no zero above them is included in the union of the upper 4^{n-k} levels of all these stacks and the whole stacks $G_{n,2q}$, $n-2k+1 \leq q \leq n$. Thus $\mu(S_n \cap [0]) \leq 4^{-k} + \frac{k}{n}$. This measure is as small as we want if we take k then n large enough, hence the claimed property is true for $B = [0]$.

The same reasoning works for $B = [2p]$ for any $p \in \mathbf{Z}$ by looking at the stacks $G_{n,2q}$, $p-n \leq q \leq p+n$, and for $B = \sigma^r[2p]$ by replacing n by $n+r$. Hence the result is true for every cylinder, hence for any (finite or countable) union of cylinders, hence for any measurable set. ♣

Lemma 2.13 *For every set B with $0 < \mu(B)$, $\mu\{x \in B; T^n x \notin [0]$ for every $n > 0\} = 0$.*

Proof

It is the same proof as for the previous proposition; we start from $B = [2p]$, look at stacks $G_{n,2q}$, $p-n \leq q \leq p+n$, which will have a zero in their upper 4^{n-k} levels as soon as $p-n \leq q \leq n-2k$, and p being fixed, take k then n large. Then the result is true for $B = \sigma^r[2p]$, every cylinder, or any (finite or countable) union of cylinders, or any measurable set. ♣

Because of the recurrence and Lemma 2.13, it makes sense to study the *induced*, or *first return*, map of (X, T, μ) on the cylinder $E = [0]$. Let (E, T_E, μ_E) be this system, defined by $T_E y = T^{r(y)} y$ where $r(y)$ is the first $r > 0$ such that $T^r y \in E$, and $\mu_E(F) = \frac{\mu(F)}{\mu(E)}$ for $F \subset E$. This system will be studied in the next section, but we can already use it to prove the ergodicity of (X, T, μ).

Proposition 2.14 *The system (X, T, μ) is* ergodic*: namely, for every set B with $0 < \mu(B)$ and $\mu(B \Delta TB) = 0$, either $\mu(B) = 0$ or $\mu(X/B) = 0$.*

Proof

We shall first show that (E, T_E, μ_E) is ergodic. μ_E is a finite measure on E, and we build a generating partition P in the following way: a *return word* of 0 in X is a word of the form $0w$, where w does not contain a 0 but the word $0w0$ occurs in some element of X. There is a finite or countable family of return words $0w_n$, and we define the set P_n as the set of x in E such that w_n occurs in x between $x_0 = 0$ and the next 0 in x.

We first check that the ergodic theorem is satisfied for atoms of P, and begin by the simplest one, which we call P_0, corresponding to the return word 0 ($0w0$ where w is the empty word): we have $\mu_E(P_0) = \frac{\mu[00]}{\mu[0]} = \frac{1}{4}$ and need to show that for μ_E-almost all $y \in E$,

$$\lim_{n \to =\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{P_0}(T_E^k x) = \frac{1}{4}.$$

But a visit of $T_E^n y$ to P_0 corresponds to an occurrence of the word 00 in $y \in E \subset X$, and what we want is equivalent to prove that for μ -almost all $x \in X$,

$$\frac{N(00, x_0 \dots x_{n-1})}{N(0, x_0 \dots x_{n-1})} \to \frac{\mu([00])}{\mu([0])}$$

when n goes to infinity.

We look at the Rokhlin stacks: the number of 0 in $\sigma^n(2q)$ is C_{2n}^{n-q}; 00 occurs in $\sigma^n(2q)$ iff $(-2)002$ occurs in $\sigma^n(2q)$, that is iff 0 occurs in $\sigma^{n-1}(2q)$, hence the number of 00 in $\sigma^n(2q)$ is equal to the number of 0 in $\sigma^{n-1}(2q)$, that is C_{2n-2}^{n-q-1}. Now, for given ϵ,

$$4 - \epsilon < \frac{C_{2n}^{n-q}}{C_{2n-2}^{n-q-1}} = \frac{4n^2 - 2n}{n^2 - q^2} < 4 + \epsilon$$

as soon as $|q| < \epsilon n$.

Now, for fixed n, all levels labelled 0 are included in the stacks $G(n, 2q)$, $-n \le q \le n$; among these stacks, we call *good* those in which the number of 00 divided by the number of 0 is ϵ-close to $\frac{1}{4}$: a stack is not good if $n\epsilon \le q \le n$ or $-n \le q \le -n\epsilon$. The ratio of the measure of the union of levels 0 in the stacks with $n\epsilon \le q \le n$ and the union of levels 0 in the stacks with $0 \le q \le n$ is

$$a_n = \frac{\sum_{n\epsilon \le q \le n} C_{2n}^{n-q}}{\sum_{q=0}^{n} C_{2n}^{n-q}}$$

and $a_n \to 0$ when $n \to +\infty$: to prove it, we compare the term C_{2n}^{n-i}, $0 \le i \le n\epsilon$, to the partial sum $\sum_{n\epsilon+(i-1)\frac{1-\epsilon}{\epsilon} \le j \le n\epsilon + i\frac{1-\epsilon}{\epsilon}} C_{2n}^{n-j}$, or equivalently to $C_{2n}^{n-[n\epsilon+i\frac{1-\epsilon}{\epsilon}]}$ and show that this ratio tends to infinity uniformly.

Hence, as in the proof of the recurrence, with probability at least $(1 - \epsilon)$ a point x has above it a good stack: for k fixed, the stack $G(n, 2q)$ is made of 4^{2k} copies of stacks $G(n - 2k, 2q')$, and these are all good as soon as $|q| \le \epsilon n - 2k - 1$. We take x in such n-stacks deprived of their last upper $n - k$-stack; then above x there is at least a full $n - k$-stack, hence at least 4^k full $n - 2k$-stacks, and the relative frequency of 00 is good in any of these full $n-2k$-stacks; even though it may be bad in the $n-2k$-stack containing x (because of truncation), it becomes good after averaging with the $4^k - 1$ following $n-2k$-stacks. Then by taking $\epsilon = 2^{-n}$ and Borel-Cantelli we get that for almost all y the frequency of P_0 is good along a subsequence, which is enough as it allows us to identify the limit of the Birkhoff sums (which exists by the ergodic theorem) as $\mu_E(P_0)$.

For other atoms of P, and of any $\vee_{i=0}^n S^i P$, a similar reasoning applies: for example, the return word 02 corresponds in $L(u)$ to the word 020; 020 occurs in $\sigma^n(2q)$ iff $(-2)0020224$ occurs in $\sigma^n(2q)$, that is iff 02 occurs in $\sigma^{n-1}(2q)$, that is iff 0 or 2 occurs in $\sigma^{n-2}(2q)$, while, because μ is the natural measure, $\mu[020] = 4^{-2}(\mu[0] + \mu[2])$; and we show in the same way that the Birkhoff sums for the corresponding atom of P tend to $\frac{\mu([020])}{\mu([0])}$. More generally, any atom of $\vee_{i=0}^n S^i P$ corresponds to a word $0w0$ in $L(u)$, and, because of Lemma 2.7, $0w0$ occurs in $\sigma^n(2q)$ iff a shorter word w_1 occurs in $\sigma^{n-1}(2q)$, and we can iterate this operation if the length of w_1 is at least 3; hence $0w0$ occurs in $\sigma^n(2q)$ iff a word w_k of length 1 or 2 occurs in $\sigma^{n-k}(2q)$, and this in turn is equivalent to the occurrence of either one letter a_k, or any one of two letters a_k or b_k in $\sigma^{n-k-1}(2q)$, and we have similar estimates as $\mu[0w]$ is then respectively $4^{-k-1}\mu[a_k]$ or $4^{-k-1}(\mu[a_k]+\mu[b_k])$. So the ergodic theorem is satisfied for all these atoms.

Hence for every atom F of any $\vee_{i=0}^{n} T_E^i P$,

$$\mu_E(F) = \lim_{n\to=\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_F(T_E^k x)$$

almost everywhere and in $\mathcal{L}^2$. Hence for any measurable set B in E, by making the inner product with 1_B we get

$$\mu_E(F)\mu_E(B) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu_E(T_E^k F\cap B)$$

and this is still true for any measurable set F in E by approximation. Thus any invariant set F under T_E satisfies $\mu_E(F) = (\mu_E(F))^2$ and T_E is ergodic.

Now, if B is T-invariant, then $B\cap E$ is T_E-invariant and $\mu_E(B\cap E)=0$ or $\mu_E(B^c\cap E)=0$. In the first case, this implies $\mu(T^{-n}(B\cap E))=0$ for all n; but because of Lemma 2.13

$$\mu(B) \leq \sum_{n\in\mathbb{N}} \mu(B\cap T^n E) = \sum_{n\in\mathbb{N}} \mu(T^n(B\cap E)) = 0.$$

In the second case we do the same for B^c.♣

In fact, to prove the ergodicity of T, we have proved directly that it satisfies the so-called *Hopf ergodic theorem* [HOP]: though we cannot define frequencies for words, we may define ratios of frequencies: namely, for almost all $x\in X$, and words w and w', $\frac{1}{n}N(w,x_0\dots x_{n-1})$ has limit zero when $n\to+\infty$, but $\frac{N(w,x_0\dots x_{n-1})}{N(w',x_0\dots x_{n-1})}$ does converge to $\frac{\mu[w]}{\mu[w']}$.

2.3 The induced substitution, complexity and entropy

Proposition 2.15 *The system (E, T_E, μ_E) is measure-theoretically isomorphic to the symbolic system (Y, S) associated to the substitution τ on the alphabet $A = \mathbb{N}\times\mathbb{Z}$, equipped with its natural measure ν, which is an invariant probability measure.*

where τ is the

Example 2.16 (The induced drunken man substitution)

$$(m,n) \to \prod_{j=0}^{n-1+m^+} (j,1) \quad (m,n+1) \prod_{i=-n+1+m^-}^{-1} (i,1)$$

for all $m\in\mathbb{Z}$ and $n\geq 1$.

Proof
We have first to determine the return words; this is done empirically, and we choose to classify them by families:

- we start by 00, which gives the return word 0, and we denote it by $(0,1)$; if we apply σ to $0*0$ (with the convention $\sigma(*)=*$), we get $(-2)002*(-2)002$ and by looking at the last zero before the star and the first zero after the star, we get $02*(-2)0$, which, deprived of its star and its last zero, gives the return word denoted by $(0,2)$; iterating this process from $02*(-2)0$ we get $(0,3)=0224(-4)(-2)(-2)$, and so on, the word $(0,n)$ having μ-measure 2^{-2n},

- applying the same process to $0*2$, we get $02*0$ giving $(1,1)=02$, then $0224*(-2)0$ giving $(1,2)$, and so on, the word $(1,n)$ having μ-measure 2^{-2n-1},

- similarly from $(-2)*0$, we get a family of return words denoted by $(-1,n)$, starting from $0*(-2)0$ giving $(-1,1)=0(-2)$, and having μ-measure 2^{-2n-1},

- we apply the same process to $(2p)*(2p)$: we iterate σ (with the convention $\sigma(*)=*$), wait until some 0 appear, then take the word from the last zero before the star to the first zero after the star and continue; by removing the stars and the last zero, we get a family of return words denoted by $(2p,n)$, of measure 2^{-2n-2p},

- similarly we get the $(-2p,n)$, of measure 2^{-2n-2p}, from $(-2p)*(-2p)$, the $(2p+1,n)$, of measure $2^{-2n-2p-1}$, from $(2p)*(2p+2)$, the $(-2p-1,n)$, of measure $2^{-2n-2p-1}$, from $(-2p-2)*(-2p)$.

The short proof that we have exhausted all the return words of 0 lies in the fact that they are all different and that the sum of their measures is $1=\mu[0]$.

And now, almost every infinite sequence in X can be written as a succession of return words of 0, $\ldots 0w_{-n}\ldots 0w_0 0w_1\ldots$; if we apply σ to this sequence, we get

$$\ldots(-2)002\sigma(w_{-n})\ldots(-2)002\sigma(w_0)(-2)002\sigma(w_1)\ldots$$

Let τ be the substitution which associates to the return word of 0 equal to $0w$ the word $002\sigma(w)(-2)$, seen as a concatenation of return words of 0. We check that its action on our words denoted by (m,n) is the one defined above. The new substitution τ has a fixed point beginning by $(0,1)$, and we can associate to it a symbolic dynamical system (Y,S); we equip it with its natural measure defined by $\nu[m,n]=\nu(S^k[m,n])=2^{-|m|-2n}$ and a cylinder $[w]$, or $S^k[w]$, has measure $\frac{1}{4}\sum\nu[v]m(v)$, the sum being taken on all its ancestors (under τ) v and $m(v)$ denoting their multiplicities. There is then a natural measure-theoretic isomorphism between (E,T,μ_E), coded by the generating partition P, and (Y,S,ν). ♣

We can now revert to the symbolic system (Y,S) and study it for its own sake

Proposition 2.17 *The system (Y,S,ν) is not minimal.*

Proof
The infinite sequence $(0,1)\dots(n,1)\dots$ is in Y, and does not contain the symbol $(0,2)$ for example. ♣

Then, of course (Y,S) cannot be uniquely ergodic; it is worth noting that we can also build sequences in Y in which every symbol occurs, but they are not well distributed; this can be done for example by taking V_1 to be $\tau(1,1)$ deprived of its initial $(0,1)$; V_2 to be $\tau^2(2,1)$ deprived of everything before its first V_1 included in a $\tau(1,1)$; generally V_q is $\tau^{q^2}(q,1)$ deprived of everything before its first V_{q-1} included in a $\tau^{(q-1)^2}(q-1,1)$. The infinite sequence beginning by V_q for each q has a frequency of $(0,1)$ smaller than $\frac{1}{4}$.

Note that τ is not of constant length, and even it can be said to be of *unbounded length* as the lengths of $\tau(m,n)$ are unbounded when (m,n) exhausts the alphabet; the lengths of further iterates are not given by straightforward formulas; we check that

$$|\tau^p(m,1)| = C^p_{2p+|m|}$$

while for $n \geq 2$, the lengths $|\tau^p(m,n)|$ involve linear combinations of binomial coefficients.

τ is not left determined, nor does it satisfy any of the usual notions of synchronization or recognizability: for example, for any $p \geq 0$, the word $(1,1)\dots(p,1)$ has no decomposition as in Definition 2.6, and the word $(0,1)\dots(p,1)$ is a prefix of infinitely many $\tau(m,n)$.

But if we drop the (countable) orbit of the sequence $(0,1)\dots(n,1)\dots$, we do get a bijective shift where the equivalent of Lemma 2.8 does hold: for any $p \geq 0$, Y is the disjoint union of the $S^k\tau^p[(m,n)]$, $m \in \mathbb{Z}$, $n \geq 1$; $0 \leq k \leq |\tau^p(m,n)| - 1$. However, the Rokhlin stacks are not useful as they have not always the same measure, the measure of the Rokhlin stack corresponding to $\tau^p(m,1)$ being $4^{-p}C^p_{2p+|m|}$, which tends to zero when p tends to infinity.

Substitution dynamical systems on a finite alphabet are of *entropy* zero, whether in the topologic or measure-theoretic sense. The classic proof [QUE] uses a finer notion, the *complexity*, see [CAS] or [FER] for a detailed theory.

Definition 2.18 *The* complexity *of an infinite sequence x is the function $p_x(n)$ which associates to each integer $n \geq 1$ the cardinality of $L_n(x)$.*

A right special *word in $L(x)$ is a word w such that wa and wb are again in $L(x)$ for two different letters a and b.* Prefixes *and* suffixes *are defined as in the usual language.*

For fixed points of primitive substitutions on a finite alphabet, the complexity is always bounded by Cn, and this implies the system has topological (and hence measure-theoretic) entropy zero. But, going back to σ, the complexity of u would be $p_u(n) = +\infty$ for all n. By *truncating* the alphabet, replacing u_k by a fixed symbol when it is larger than a prescribed N, we get a sequence on a finite alphabet, whose complexity can be computed [LeG]; but it does not yield any dynamical property, as this truncation means losing information on a part of X which is of infinite measure.

However, we can compute the measure-theoretic *Krengel entropy* [KRE] for the conservative infinite measure preserving system (X,T,μ); because of Lemma 2.13, it is just the measure-theoretic entropy $h(T_E,\mu_E)$ of the induced system on E, hence also of (Y,S,ν). It is a finite measure-preserving system, and the underlying non-compact symbolic system may be associated to the sequence v, the fixed point of τ beginning by $(0,1)$. Hence, instead of looking at u, we can now look at v, its *derived sequence* (with respect to the word 0, see [DUR]). With v, we have the same problem as with u, the complexity is infinite, but a truncation argument is now possible, losing information only on a part of Y of arbitrarily small measure. And this will allow us to prove that *the Krengel entropy of T is* 0.

Lemma 2.19 *The right special words of v of length 1 are the letters $(k,1)$, $k \geq 0$.*

For every $k \geq 0$ and any $t \geq 1$, there is one right special word $Z(k,t)$ of length t ending by $(k,1)$, and there exists $n \geq 1$ such that $Z(k,t)$ is the suffix of length t of the word

$$Y(k,n) = \prod_{j=-n}^{0} \tau^k(j,1) \prod_{p=0}^{k-1} \prod_{j=0}^{p+1} \tau^{k-1-p}(j,1).$$

The only possible letters following $Y(k,n)$ are the $(q,k+2)$ for $q \leq -n+1$.

Proof

The letter $(k,1)$ can be followed:

- (1) by $(k+1,1)$, and this occurs in any $\tau(m,n)$ with $(m > 0, m+n \geq k+2)$ or $(m \geq 0, n \geq k+2)$;
- (2) by any $(k-p, 2+p)$, $0 \leq p \leq k$, and this occurs in $\tau(k-p, 1+p)$;
- (3) by any $(q, k+2)$, $q < 0$, and this occurs in $\tau(q, k+1)$.

We look now at the possible ways to extend $(k,1)$ to the left in the language of v; we get first $(k-1,1)(k,1)$ (if $k \geq 1$), then $(l,1)\dots(k,1)$ for any $0 \leq l \leq k$. All these words are suffixes of $W_1 = (0,1)\dots(k,1)$, which is the common prefix of all $\tau(m,n)$ for the m,n) which can follow $(k-1,1)$; they have the same possible right extensions as $(k,1)$.

To go further left, we have to look at which $\tau(m',n')$ may occur before a $\tau(m,n)$ containing the word $(0,1)\dots(k,1)$: if (m,n) corresponds to the subcase (2) or (3) above, the only possible (m',n') is $(k-1,1)$; if (m,n) corresponds to the subcase (1) above, there are infinitely many possible (m',n'), but they give extensions of the form $w(0,1)\dots(k,1)$ where w has no common suffix with $\tau(k-1,1)$ and hence $w(0,1)\dots(k,1)$ can only be followed by $(k+1,1)$.

Hence the only way to get a right special word extending $(0,1)\dots(k,1)$ to the left is to extend it by $\tau(k-1,1)$ or any suffix of it; then, further left, we get the suffixes of $W_2 = \tau(0,1)\dots\tau(k-1,1)(0,1)\dots(k,1)$ All these words, from $(k-1,2)W_1$ to W_2, can be followed by any $(k-p,2+p)$, $0 \leq p \leq k$, or any $(q,k+2)$, $q < 0$. They are suffixes of the common prefix of all $\tau^2(m,n)$ for the (m,n) which can follow $(k-2,1)$.

Continuing our extension to the left, a similar reasoning gives, as the only possible right special words, the suffixes of W_p for any $1 \leq p \leq k+1$, where W_p is the common prefix of $\tau^p(m,n)$ for the (m,n) which can follow $(k-p,1)$.

$$W_{p+1} = \tau^p(0,1)\dots\tau^p(k-p,1)W_p,$$

and if a right special word w is a suffix of W_{p+1} but not of W_p, it can be followed by any $(k-q, 2+q)$, $p-1 \le q \le k$, or any $(q, k+2)$, $q < 0$.

In the next step to the left, we use the fact that $(-1,1)(0,1)$ is the only left extension of $(0,1)$ to be right special, and it can be followed by any of the successors of $(0,1)$ except $(1,1)$; more generally, for $n \ge 1$, the only right special word of length $n+1$ ending by $(0,1)$ is $(-n,1)\dots(0,1)$, and it can be followed by every $(p,2)$, $p \le -n+1$.

Hence the left extensions of W_{k+1} which are right special are suffixes of

$$\tau^k(-n,1)\dots\tau^k(-1,1)W_{k+1}$$

for $n \ge 1$. As n grows, we lose possible right extensions: namely, if a right special word w is a suffix of $\tau^k(-n,1)\dots\tau^k(-1,1)W_{k+1}$ and not of $\tau^k(-n+1,1)\dots\tau^k(-1,1)W_{k+1}$ it can be followed by every $(q, k+2)$, for $q \le -n+1$, and also by $(-n+2, k+2)$ (replaced by $(1, k+1)$ if $n=1$) in the case where w is a common suffix of $\tau^k(-n,1)\tau^k(-n+1,1)\dots\tau^k(-1,1)W_{k+1}$ and $\tau^k(-n+1,2)\tau^k(-n+1,1)\dots\tau^k(-1,1)W_{k+1}$. In particular, if $w = \tau^k(-n,1)\dots\tau^k(-1,1)W_{k+1}$, it can only be followed by $(q, k+2)$ for $q \le -n+1$. ♣

Let $M \ge 2$ be an integer. Let A_M be the finite alphabet whose letters are (m,n), $-M+1 \le m \le M-1$, $1 \le n \le M-1$, and the letters η and ω. We define a letter-to-letter map F_M from $A = \mathbb{N} \times \mathbb{Z}$ to A_M by putting

- $F_M(m,n) = \eta$ if $|m| \ge M$ and $n \ge 2$, or if $n \ge M$,
- $F_M(m,n) = \omega$ if $|m| \ge M$ and $n = 1$,
- $F_M(m,n) = (m,n)$ in all other cases.

Let $V(M) = F_M(v)$.

Lemma 2.20 *For fixed M, if w is a right special word of $V(M)$ which contains at least one letter other than η and ω, there exists at most one letter $a \in A_M$ such that aw is a right special word of $V(M)$.*

Proof

M being fixed, we omit it in the mention of F and V. Like in the previous lemma, we shall find all the right special words of V, moving to the left from their possible last letters.

1) right special words ending with $(0,1)$*:* any left extension $w(0,1)$ which is still right special must be such that $w = F(w_1) = F(w_2)$ where $w_1(0,1)$ can be followed by (m,n) and $w_2(0,1)$ can be followed by (m',n') for some $(m,n) \neq (m',n')$; we shall show that there exists a number C_0 such that the only solution to this problem is $w(0,1) = FZ(0,t)$ if $|w|+1 = t$ is at most C_0, and that there is no solution if $t > C_0$.

Indeed, $(m,n)(0,1)$ is right special in $L(V)$ only if it is so in $L(v)$, $\omega(0,1)$ does not occur, and we have to check whether $\eta(0,1)$ is right special; but it is not the case, $\eta(0,1)$, though it has infinitely many different preimages under F, is always followed by $(1,1)$; hence, in the language of V, the only right special word of length 2 ending by $(0,1)$ is $(-1,1)(0,1) = FZ(0,2)$.

Similarly, as $\eta(-n,1)\dots(0,1)$ is not right special, there is only one way to continue to the left while staying right special, until we reach $(-M+1,1)\dots(0,1)$. At the next step, we check again that $\eta(-M+1,1)\dots(0,1)$ is not right special; the only possible right special word extending $(-M+1,1)\dots(0,1)$ to the left is $F[(-M,1)(-M+1,1)\dots(0,1)] = \omega(-M+1,1)\dots(0,1)$; this word has only one preimage under F, and, because of the previous lemma, can only be followed by $F(q,2)$ for $q \leq -M+1$, hence by η or $(-M+1,2)$. But at the next stage, no left extension of $\omega(-M+1,1)\dots(0,1)$ is right special: we check that $\eta\omega(-M+1,1)\dots(0,1)$, whose preimages by F are of the form $a(-M,1)\dots(0,1)$ with $F(a)=\eta$, is not right special; as for $\omega\omega(-M+1,1)\dots(0,1)$, whose only preimage by F is $(-M-1,1)\dots(0,1)$, it is not right special because it can be followed only by $F(q,2)$ for every $q \leq -M$, and this is just η. And no further left extension can be right special.

2) right special words ending with $(1,1)$*:* we show the same result as in the previous case, the only right special left extensions are $FZ(1,t)$ (though at some stage $FZ(1,t)$ will have infinitely many preimages under F) for t smaller than a number C_1, and after that no left extension can be right special.

Indeed, there is nothing new as we add successively to the left of $Z(1,t)$ the words $(0,1)$, $\tau(0,1)$, $\tau(-1,1)$, $\dots \tau(-M+1,1)$: $FZ(1,t)$ has no η nor ω and hence has only one preimage by F, and no other word than $FZ(1,t)$ can be right special, as the only other possibility would be replacing the first letter of $FZ(1,t)$ by an η; the resulting word $G(1,t)$ does exist when the first letter of $FZ(1,t)$ is a $(-n,2)$, or when the first letter of $FZ(1,t)$ is a $(-n,1)$ and there is no other $(-n,1)$ in $FZ(1,t)$, but $G(1,t)$ is never right special.

Now we extend $Z(1,t)$ left by $\tau(-M,1)$; this means adding to the left of $FZ(1,t)$, successively, the letters $(-1,1)$, $(-2,1)\dots(-M+1,1)$, and nothing new happens; then we add an ω, but nothing new happens as, in this position, it has only one preimage by F; then we add η, and for this length t_0 $FZ(1,t_0)$ is still the only possible right special word, but it has infinitely many preimages by F; hence we have to check that only one left extension of $FZ(1,t_0)$ is right special, but this is the case, and it is indeed $FZ(1,t_0+1)$. And $FZ(1,t_0+1)$ has again only one preimage by F, as $(0,1)\eta\omega(-M+1,1)\dots(-1,1)$ has only one preimage, $\tau(-M,1)$.

The same reasoning works when we extend $Z(1,t_0+1)$ left by $\tau(-M-1,1)$; but as soon as we have added strictly more than the common suffix of $F\tau(-M-1,1)$ and $F\tau(-M,2)$, our $FZ(1,t)$ can only be followed by $F(q,3)$ for every $q \leq -M$, and this is just η, and no further extension will be right special.

3) right special words ending with $(k,1)$*,* $2 \leq k \leq M-1$*:* we show the same result with $FZ(k,t)$ and C_k; the reasoning is similar to the last case, with only two differences. First, words $G(k,t)$ deduced from $FZ(k,t)$ by changing its first letter into an η exist when the first letter of $FZ(k,t)$ is a $(-n,k)$, or when the first letter of $FZ(k,t)$ is a $(-n,i)$ for $1 \leq i \leq k-1$ and there is no other $(-n,i)$ in $FZ(k,t)$, but $G(k,t)$ is still never right special. Second, for $k=M-1$, some η will appear in $FZ(k,t)$ before we have added $F\tau^k(-M,1)$: these are the images by F of the $(-n,M)$ in $\tau^{M-1}(-n,1)$. When such an η is the first letter of an $FZ(k,t_1)$, it has infinitely many preimages by F. But still only one left extension of $FZ(k,t_1)$ is right special, it is indeed $FZ(k,t_1+1)$, and, the preimage of η being forced if we know the letter after and the letter before, $FZ(k, t_1 + 1)$ has again only $Z(k, t_1 + 1)$ as a preimage.

4) right special words ending with $(M-1, 1)$ *followed by a string of* ω*:* if a word ends by $(M-1, 1)\omega^p$, then its preimages by F end by $(M-1, 1)\dots(M+p-1, 1)$, and these right special words are $FZ(k = M+p-1, t)$ for $t \leq C_k$ as in the cases before. The only new phenomenon is that, when some $FZ(k, t_2)$ begins by η, its only right special left extension $FZ(k, t_2 + 1)$ may have also infinitely many preimages by F as it begins by $\omega\eta$; but then again its only right special left extension is $FZ(k, t_2 + 2)$, and if we continue to the left we still follow the $F(k, t)$; as the preimage of η is forced if we know the letter after it (in this case, even if it is ω as it has only one preimage), and either the letter before it or the number of ω between the last $(M-1, 1)$ and it, for some $q \leq k - M$ $FZ(k, t_2 + q)$ has again only one preimage.

5) right special words ending with $(0, 1)$ *followed by a string of* ω *and* η*:* the word $(0, 1)\eta$ exists in the language of V, and is the image by F of $(0, 1)(-n, 2)$ for any $n \geq M$. After such $(0, 1)(-n, 2)$ we see $(-n, 1)(-n+1, 1)\dots(-1, 1)$. After taking the image by F, we see that $(0, 1)\eta$ is always followed by ω, but $(0, 1)\eta\omega$ is right special: it is followed by $(-M+1, 1)$ when it is the image of $(0, 1)(-M, 2)(-M, 1)$, or by $\omega = F(-n+1, 1)$ when it is the image of $(0, 1)(-n, 2)(-n, 1)$ for $n > M$.

Any left extension $w(0, 1)\eta\omega$ which is still right special must be such that $w = F(w_1) = F(w_2)$ where $w_1(0, 1)$ can be followed by $(-M, 2)$ and $w_2(0, 1)$ can be followed by $(-n, 2)$ for some $n > M$; this is a similar problem to the one in case 1), and we show that this implies $w = FZ(0, t)$ for $t \leq C_{0,1}$, while there is no solution for $|w| \geq C_{0,1}$. The only difference is that $C_{0,1} = C_0 + 1$ as we can extend as far as $t = M + 1$, the last t for which $Z(0, t)$ can be followed by $(-M, 2)$.

Similarly, $(0, 1)\eta\omega^p$ is right special for every $p \geq 1$, as it is followed by $(-M+1, 1)$ when $\eta\omega^p = F((-M-p+1, 2)(-M-p+1, 1)\dots(-M, 1))$ and by ω when $\eta\omega^p = F((-n, 2)(-n, 1)\dots(-n+p-1, 1))$ for some $n > M+p-1$. By the same reasoning, its only right special left extension is $FZ(0, t)\eta\omega^p$ for $t \leq C_{0,p}$, and then no further left extension can be right special. This exhausts the case 5).

6) right special words ending with $(k, 1)$ *followed by a string of* ω *and* η*:* for $k = 1$ we have right special words $(1, 1)\eta\omega^p$ for every $p \geq 2$, followed by $(-M+1, 1)$ when $\eta\omega^p = F((-M-p+2, 3)(-M-p+1, 1)\dots(-M, 1))$ and by ω when $\eta\omega^p = F((-n, 3)(-n+1, 1)\dots(-n+p, 1))$ for some $n > M+p-2$. By the same reasoning, its only right special left extension is $FZ(1, t)\eta\omega^p$ for $t \leq C_{1,p}$, and then no further left extension can be right special.

The same situation happens for $1 \geq k \geq M-3$, with right specials $(k, 1)\eta\omega^p$ for every $p \geq k+1$, which extend to the left into $FZ(k, t)\eta\omega^p$ for $t \leq C_{k,p}$, and then no further left extension can be right special.

For $k = M-2$, in $(k, 1)\eta$ η can be $F(-n, M)$ for any $n \leq 0$ (and not only for $n \leq M$). This gives right specials $(k, 1)\eta\omega^p$ for every $p \geq 0$, and again they extend to the left into $FZ(k, t)\eta\omega^p$ for $t \leq C_{k,p}$, and then no further left extension can be right special.

For $k = M - 1$, in $(k,1)\eta$ η can be $F(-n, M+1)$ for any $n \leq 0$ and also $F(1, M)$. This gives the same situation as in the previous case.

The case $k = M$ corresponds to right specials $(M-1,1)\omega\eta\omega^p$ for every $p \geq 0$, with $\omega = F(M,1)$, $\eta = F(-n, M+2)$ for any $n \leq 0$, $\eta = F(1, M+1)$ or $\eta = F(2, M)$; they extend to the left into $FZ(k,t)\eta\omega^p$ for $t \leq C_{k,p}$, and then no further left extension can be right special.

This last situation generalizes to $k = M+q$, $q \geq 1$, with right specials $(M-1,1)\omega^{q+1}\eta\omega^p$ for every $p \geq 0$, with $\omega^{q+1} = F((M,1)\ldots(M+q-1,1))$, $\eta = F(-n, M+q+1)$ for any $n \leq 0$, or $\eta = F(r, M+q+1-r)$ for any $r \geq 0$; they extend to the left into $FZ(k,t)\eta\omega^p$ for $t \leq C_{k,p}$, and then no further left extension can be right special. ♣

Proposition 2.21 *The complexity of $V(M)$ is bounded by $K(M)n^3$.*

Proof

Let M be fixed, $p(n)$ the complexity function of $V(M)$, $q(n)$ the number of its right special words of length n. A right special word of length $n+1$ is a left extension of a right special word of length n; the only ones for which more than one left extension is right special are those which are made only with η and ω; as between two η there is always at least a $(0,1)$, the only possible ones are strings of ω and zero or one η; this makes $n+1$ different words and each of these has at most $M' = (M-1)(2M-1)+2$ left extensions, hence

$$q(n+1) \leq q(n) + (M'-1)(n+1)$$

while

$$p(n+1) \leq p(n) + (M'-1)q(n),$$

and this gives the desired estimate. ♣

Corollary 2.22 *The measure-theoretic entropy $h(S,\nu)$, and hence the Krengel entropy of (X,T,μ) are equal to* 0.

Proof

The shift on the language of $V(M)$ is of polynomial complexity, hence has a topological entropy equal to 0. Now we equip the system (Y,S) with the partition P_M whose atoms are

- $P_M(\eta) = \{y_0 = (m,n)$ for some (m,n) with $|m| \geq M$ and $n \geq 2$, or with $n \geq M\}$,
- $P_M(\omega) = \{y_0 = (m,n)$ for some (m,n) with $|m| \geq M$ and $n = 1\}$,
- $P_M(m,n) = \{y_0 = (m,n)\}$ if $|m| < M$ and $n < M$.

The measure-theoretic entropy $h(P_M, S)$ is the measure-theoretic entropy of the shift on the language of $V(M)$ for the measure induced by ν, hence it is zero. When M goes to infinity, P_M converges, in the sense of the usual distance between partitions, to the partition P of Y into cylinders $\{x_0 = (m,n)\}$, $m \in GZ$, $n \geq 1$, hence $h(P,S) = 0$, hence $h(S,\nu) = 0$ as P is a generating partition. ♣

Remarks
The proof of Lemma 2.20 allows us to compute more precisely the complexity of $V(M)$, but, as the truncation is somewhat arbitrary, the exact value is not of dynamical interest.

Both the language of v (by Lemma 2.19) and the language of u ([LeG]) satisfy the property that, if w is a right special word, there exists at most one letter a such that aw is a right special word; this, of course, does not prevent their complexity function to be infinite, and this property is in general *not* preserved by taking the derived sequence; also, it is not preserved by truncation, even after restricting it to right special words made with letters which have only one preimage in the non-truncated alphabet. But still it is a possible generalization to infinite alphabets of various notions of low complexity (such as those described in [FER]), and it would be interesting to know which information it gives on a system in general.

2.4 The two-sided shift

We can also take the subset X' of $A^{\mathbb{Z}}$ to be the set of all sequences $x = (x_n, n \in \mathbb{Z})$ such that every finite word occurring in x is in $L(u)$, and consider the *two-sided shift* defined by $(T'x)_n = x_{n+1}$ for all $n \in GZ$.

Proposition 2.23 *(X', T') is minimal.*

Proof
Suppose that a sequence $x \in X'$ is such that $x_n \neq 0$ for all n large enough, and look at the one-sided sequence without 0 $y = (x_n, n \geq N)$. y is in X and, because of Lemma 2.7 and as in the proof of Lemma 2.8, we can define without ambiguity two sequences $q_n \in 2\mathbb{Z}$ and $k_n \in \{1, 2, 3, 4\}$ such that the beginning of y falls into the word $\sigma^n q_n$, and into the k_n-th word in its decomposition $\sigma^n q_n = \sigma^{n-1}(q_n - 2)\sigma^{n-1}(q_n)\sigma^{n-1}(q_n)\sigma^{n-1}(q_n + 2)$. Because these sequences are unique, we get that $q_{n-1} = q_n - 2$ if $k_n = 1$, $q_{n-1} = q_n$ if $k_n = 2$ or $k_n = 3$, $q_{n-1} = q_n + 2$ if $k_n = 4$. Because the sequence y contains the full words $\sigma^{n-1}(q)$ situated to the right of the k_n-th in $\sigma^n(q_n)$, these cannot contain any 0, and this implies the following conditions

- if $k_n = 1$ (hence $q_n = q_{n-1} + 2$), $q_n \geq 2n$ or $q_n \leq -2n - 2$,
- if $k_n = 2$ (hence $q_n = q_{n-1}$), $q_n \geq 2n$ or $q_n \leq -2n - 2$,
- if $k_n = 3$ (hence $q_n = q_{n-1}$), $q_n \geq 2n - 2$ or $q_n \leq -2n - 2$,
- if $k_n = 4$ (hence $q_n = q_{n-1} - 2$), no condition.

But these conditions can be satisfied only if $k_n = 1$ for all n large enough or $k_n = 4$ for all n large enough. In the first case we have $q_n = Q + 2n$ for $n \geq N_1$, and y begins with $\sigma^n(Q + 2n)$. Before y in x there must be a $\sigma^n r_n$, r_n can be any number larger or equal to $Q + 2n - 2$; as $\sigma^n r_n$ ends by $\sigma^{n-1}(r_n + 2)$ we must have $r_n = r_{n-1} - 2$, and it is impossible to find such a sequence, y cannot be extended to the left. By the same reasoning, in the case where $k_n = 4$ for all n large enough, $(x_n, n \leq N - 1)$ could not have been extended to the right.

Hence in any sequence in X' there are 0s arbitrarily far to the right and to the left; this is true in the same way for each letter $2q$, $q \in \mathbb{Z}$, and, by applying σ^p, this is true also for each $\sigma^p(2q)$, hence for each word $w \in L(u)$. Hence any orbit visits any cylinder, hence it is dense. ♣

As far as invariant measures go, the two-sided shift behaves as the one-sided shift T: there is no finite invariant measure, and the natural measure μ' can be defined on (X', T') exactly as μ on (X, T).

Note that, though (X', T') is minimal, there are uncountably many infinite T'-invariant measures not proportional to μ'; indeed, take any $C \geq 4$, and define, with the notations of Lemma 2.11,

$$\mu_C[w] = \frac{1}{C} \sum \mu_C[v] m(v),$$

and fix for example $\mu_C[0] = \mu_C[2] = 1$; we get another infinite measure on X' invariant under T': we check, as in Lemma 2.11, that each one-letter cylinder has a positive measure (this part would not be true for $C < 4$), and that for every n $\mu[2n] = \sum_{p=-1}^{+\infty} \mu[(2n)(2n-2p)]$; this last relation is transformed by the defining equation into $\mu[2n] = \sum_{q=0}^{+\infty} C^{-q-1}(C\mu[2n-2q] - \mu[2n-2q-2])$, and hence is satisfied. The condition on the left extensions of $[2n]$ is checked similarly, while compatibility conditions for words of length at least 3 come directly from the defining equation.

Note that μ_C can be defined similarly on the one-sided shift, giving uncountably many non-proportional T-invariant measures concentrated on the same set as μ.

μ_C is not ergodic, as it has the same support as μ' and μ' is ergodic, because Proposition 2.9 implies that (X, μ, T) and (X', μ', T') are *measure-theoretically isomorphic*.

Note that, similarly, the two-sided shift associated to the substitution τ is minimal but not uniquely ergodic; we can equip it with the natural measure ν', getting a system which is measure-theoretically isomorphic to (Y, S, ν).

3 General theory

Definition 3.1 *The* matrix *of a substitution σ is defined by $M = ((m_{ab}))$, $a \in A$, $b \in A$, where m_{ab} is the number of occurrences of the letter b in the word σa.*

Definition 3.2 *Let M be a matrix on a countable alphabet. We denote by $m_{ij}(n)$ the coefficients of M^n; M is* irreducible *if for every (i, j) there exists l such that $m_{ij}(l) > 0$. An irreducible M has* period d *if for every i $d = GCD\{l; m_{ii}(l) > 0\}$, and is* aperiodic *if $d = 1$.*

An irreducible aperiodic matrix admits a Perron-Frobenius *eigenvalue λ defined as*

$$lim_{n \to +\infty} m_{ij}(n)^{\frac{1}{n}}.$$

M is transient *if*

$$\sum_n m_{ij}(n) \lambda^{-n} < +\infty,$$

recurrent *otherwise. For a recurrent* M*, we define* $l_{ij}(1) = m_{ij}$*,* $l_{ij}(n+1) = \sum_{r\neq i} l_{ir}(n)m_{rj}$*;* M *is* null recurrent *if*

$$\sum_n n l_{ii}(n)\lambda^{-n} < +\infty,$$

and positive recurrent *otherwise.*

The reference for all the definitions and results on infinite matrices above is section 7.1 of [KIT]. The vocabulary comes from the theory of random walks: when it is stochastic, a matrix is positive recurrent if it is the matrix of a random walk which returns to each point with probability one and the expectation of the waiting time is finite, it is null recurrent if it is the matrix of a random walk which returns to each point with probability one and the expectation of the waiting time is infinite, and it is transient if it is the matrix of a random walk which does not return to each point with probability one. And the matrix of the drunken man substitution is a multiple of the matrix of the famous random walk of the same name, though the dynamical systems we can associate to these two objects are completely different.

Now, for a given substitution σ on a countable alphabet A, with an irreducible matrix, we define a language $L(u)$, either through a fixed point, possibly artificial as in section 2.1, or as the set of words which occur in $\sigma^n a$ for at least one $a \in A$ and $n \geq 0$; then we define the dynamical system associated to σ in the same way as in section 2.1 (to fix ideas, we take the one-sided shift, but the following results apply as well to the two sided-shift as in section 2.4).

Proposition 3.3 *If* σ *is of constant length and has an irreducible aperiodic positive recurrent matrix, the associated system* (X, T) *admits a natural invariant measure which is a probability.*

Proof

By Theorem 7.1.3 of [KIT], M admits a Perron-Frobenius eigenvalue λ, with positive left and right eigenvectors l and r such that the scalar product lr is finite. Moreover, as $|\sigma a| = L$ for all letters, we have $\lambda = L$ and r is a multiple of $(\ldots 1, 1 \ldots)$. We define the natural measure on letters by taking $(\mu[a], a \in A)$ to be the left eigenvector l normalized to have $\sum_{a\in A} l(a) = 1$.

For longer words, we use the substitution $\sigma_{(k)}$ defined in [QUE], section 5.4, on the alphabet $A_{(k)} = L_k(u)$ by associating to the k-letter $v_1 \ldots v_k$ the word on $A_{(k)}$

$$(y_1 \ldots y_k)(y_2 \ldots y_{k+1}) \ldots (y_{t-k+1} \ldots y_t)$$

if the $|\sigma v_1| + k - 1 = L + k - 1$ first letters of $\sigma(v_1 \ldots v_k)$ are $y_1 \ldots y_t$. We denote by $M_{(k)}$ the matrix of this substitution.

$M_{(2)}$ is also irreducible and aperiodic, and has L as its Perron-Frobenius eigenvalue, with left and right eigenvectors $l_{(2)}$ and $r_{(2)}$, and we can take $r_{(2)} = (\ldots 1, 1 \ldots)$. We can approximate M and $M_{(2)}$ by finite submatrices $M^{(n)}$ and $M_{(2)}^{(n)}$, on alphabets $A^{(n)}$ and $A_{(2)}^{(n)}$, with Perron-Frobenius eigenvalues $\lambda^{(n)}$ and $\lambda_{(2)}^{(n)}$, left eigenvectors $l^{(n)}$, $l_{(2)}^{(n)}$, normalized so that for every $a \in A^{(n)}$,

$$\sum_{b\in A^{(n)};ab\in A^{(n)}_{(2)}} l^{(n)}_{(2)}(ab) = \sum_{b\in A^{(n)};ba\in A^{(n)}_{(2)}} l^{(n)}_{(2)}(ba) = l^{(n)}(a).$$

By Theorem 7.1.4 of [KIT], $l^{(n)}$ and $l^{(n)}_{(2)}$ tend respectively to l and $l_{(2)}$, hence we can normalize $l_{(2)}$ so that for every $a \in A$,

$$\sum_{b\in A;ab\in A_{(2)}} l_{(2)}(ab) = \sum_{b\in A;ba\in A_{(2)}} l_{(2)}(ba) = l(a).$$

As a consequence, we get that the scalar product $l_{(2)}r_{(2)}$ is finite, and, by Theorem 7.1.3 of [KIT], this implies that $M_{(2)}$ is positive recurrent. And we define the natural measure on two-letter words by taking $(\mu[ab], ab \in A_2)$ to be the left eigenvctor $l_{(2)}$ normalized as above.

Similarly $M_{(k)}$ is positive recurrent and we define $(\mu[w], w \in A^k)$ to be its left eigenvctor for the Perron-Frobenius eigenvalue L, normalized inductively by the compatibility relations; and $\mu(T^k[w]) = \mu[w]$ for all cylinders. The naturality equation 1 is then a translation of the relations $l_{(k)}M_{(k)} = Ll_{(k)}$. ♣

Note that, because of the above proof, the easiest way to check that the matrix of a substitution of constant length L is positive recurrent (or not) is to check whether the equation $lM = Ll$ has (or not) a solution such that $\sum_{a\in A} l(a) = 1$; this equation is also, with the notations of section 2, for every letter $a \in A$,

$$l_a = \frac{1}{L}\sum l(b)m(b),$$

the sum being taken on all the ancestors b of a and $m(b)$ denoting their multiplicities.

Proposition 3.4 *If σ is of constant length, left determined, and has an irreducible aperiodic positive recurrent matrix, the natural invariant probability is ergodic.*

Proof

Under this hypothesis, Lemma 2.8 holds (with 4 replaced by the length L everywhere): we have stacks $G(n, a)$, $a \in A$, and $G(n, a)$ has measure $\mu([a])$.

Let 0 be a fixed element of A. Let $f(n, a)$ be the number of levels 0 in the stack $G(n, a)$, which is also the number of occurrences of 0 in $\sigma^n a$; by assertion (ii) of Lemma 7.1.19 in [KIT],

$$\frac{f(n, a)}{L^n} \to \mu([0])$$

when $n \to +\infty$ for fixed a; the limit is identified by the definition of μ and its normalization.

Fix ϵ, choose k and n to be precised later. Choose a finite subset B of A, of cardinality M, such that $\sum_{a\in B^c} \mu([a]) < \epsilon$. Then choose n such that the frequency of 0 is ϵ-good ($f(n, a)$ is ϵ-close to $\mu([0])$) in all stacks $\sigma^{n-k}a$, $a \in B$. Take the union of stacks $G(n, a)$, $a \in B$, which is of measure at least $1 - \epsilon$; each word $\sigma^n a$ is a concatenation of L^k words $\sigma^{n-k}b$, or equivalently each stack $\sigma^n a$ is a union of L^k copies of stacks $G(n-k,b)$; call a word or column $\sigma^{n-k}b$ *good* if $b \in B$; in the whole picture there are ML^k words $\sigma^{n-k}b$, of which at least $ML^k(1-\epsilon)$ are good. Call a level of a stack $G(n,a)$ *good* if there exist at least k words $\sigma^{n-k}b$ above it, not counting the one in which it is, and at least $k(1-\epsilon)$ of them are good. The total number of good levels is (by a Fubini-type argument: for each level, we count the number of bad $n-k$-words in the next k above it, and when we sum all that, we find k times the total number of bad $n-k$-words, up to boundary effects) at least $ML^k(1-\epsilon)-MkL^{-k}$. If x is in a good level, the frequency of 0 in the orbit of x up to the top of the k-th word $\sigma^{n-k}b$ above it is good up to $\epsilon+\epsilon+k^{-1}$ (to allow for the initial truncation and the bad $n-k$-words).

Thus with probability at least $(1-2\epsilon)$ a point x has above it a piece of orbit where the frequency of 0 is 3ϵ-good. Then by taking $\epsilon = 2^{-n}$ and Borel-Cantelli we get that for almost all x there exists a subsequence $k_n(x)$ such that the frequency of 0 is 2^{-n}-good along the orbit $x, \dots, T^{k_n}x$; and this allows us to identify the limit in Birkhoff's theorem as $\mu([0])$. Hence we have proved the ergodic theorem for the cylinder [0], or any one-letter cylinder; it holds for any cylinder of length k, either by computing its number of occurrences in the stacks through its ancestors and using the naturality equation, or directly by using the same machinery for the substitution $\sigma_{(k)}$. And the validity of the ergodic theorem on a dense class of subsets implies ergodicity.♣

In the hypothesis, the notion of left determination can be replaced by weaker notions: in particular, if we take the two-sided shift (or if we can restrict ourselves to a suitable set to have a bijective one-sided shift), we need only what we may call *strong bilateral recognizability* : there exists M such that, if $w = x_1 \dots x_s$ is a word in the language $L(u)$, the word $w' = x_M \dots x_{s-M}$ has one decomposition $w = w_1 \dots w_s$ where each w_i is a σa_i for some $a_i \in A$, except that w_1 may be only a suffix of σa_1 and w_s may be only a prefix of σa_s, and the w_i and a_i, $1 \le i \le s$, are uniquely determined by w; this property is convenient on finite alphabets, as it is true as soon as σ is primitive and has a nonperiodic fixed point, see [MOS]. It is not clear how (and for which notion of primitivity) this result may be extended to infinite alphabets; but, in most examples we can build in constant length, a notion of synchronization can be proved directly as in Lemma 2.7: the usual way is to find a subword which fixes the position of bars, and then use a property of injectivity on letters to get recognizability.

Here are examples of positive recurrent situations:

Example 3.5 (The one step forward, two step backwards, substitution)

$$n \to (n-1)(n-1)(n+1)$$

for all $n \ge 1$, *and*

$$0 \to 111.$$

As its matrix is of period two,

Example 3.6 (The squared one step forward, two step backwards, substitution)

$$n \to (n-2)(n-2)n(n-2)(n-2)nnn(n+2)$$

for all even $n \geq 2$*, and*

$$0 \to 002002002.$$

This substitution is left determined, has a positive recurrent irreducible aperiodic matrix, hence the system has a natural invariant ergodic probability measure, which gives measure $\frac{1}{3}$ to $[0]$ and 2^{-2n+1} to $[2n]$, $n \geq 1$. But it is still not uniquely ergodic.

Example 3.7 (The golden ratio substitution)

$$n \to (n-2)(n+1)$$

for all $n \geq 2$*,*

$$0 \to 01,$$

$$1 \to 02.$$

It is left determined, has a positive recurrent irreducible aperiodic matrix, and the natural invariant ergodic probability gives to $[n]$ the measure $\frac{2^n(3-\sqrt{5})}{2(1+\sqrt{5})^n}$.

Example 3.8 (The infini-Bonacci substitution)

$$n \to 1(n+1)$$

for all $n \geq 1$*.*

This is a very special case, as the symbolic system is minimal (even if we take the one-sided shift) and uniquely ergodic (indeed, if u is the fixed point, beginning by 1, for every word w $\frac{1}{n}N(w, u_p \dots u_{p+n-1})$ has a limit $f(w)$ uniformly in p when n goes to infinity; hence, there is one invariant probability, giving measure $f(w)$ to the cylinder $[w]$). Measure-theoretically, the system is isomorphic to the *dyadic odometer*, with an explicit coding to and from the system generated by the *period-doubling substitution* on two letters, $1 \to 12$, $2 \to 11$ (just send the even digits of u to 2 and the odd digits to 1). From the combinatorial point of view, the infini-Bonacci fixed point was defined by Cassaigne ([CAS], section 6) and used to build many interesting new sequences: it thus earned the unofficial nickname of *the universal counter-example*, and will be studied in depth in the forthcoming [PYT-2].

In the null recurrent case, a typical example is the drunken man substitution, whose matrix is proved to be null recurrent in Example 7.1.28 of [KIT], unsurprisingly in view of the above considerations on random walks, while the squared drunken man substitution has a null recurrent irreducible aperiodic matrix.

For this class, we can just generalize the beginning of section 2: if σ is of constant length, and has an irreducible aperiodic null recurrent matrix, we can define a natural measure by Perron-Frobenius left eigenvectors, and it is an infinite invariant measure; with a property of synchronization as in the last paragraph, we can generate the system by Rokhlin stacks, and prove it is recurrent; however, the proof of the ergodicity in section 2 uses precise estimates which are not clearly generalizable.

As for the transient case, it does not lead to general results: eigenvectors for the Perron-Frobenius eigenvalue measure may exist or not, see Remark 7.1.12 in [KIT]. Note also that, except in section 2.3, we have not considered substitutions of nonconstant length: natural invariant measures might be defined through Perron-Frobenius left eigenvectors, but the finiteness of the measure may not be linked to positive recurrence (as the right eigenvector is not $(\ldots, 1, 1, \ldots)$) and the all-important tool of Rokhlin stacks seems difficult to use. Still, we finish by an example which is of non-constant length with a transient matrix:

Example 3.9 (The positive drunken man substitution)

$$n \to (n-1)(n+1)$$

for all $n \geq 1$, *and*

$$0 \to 1.$$

The matrix of this substitution is shown to be transient in Example 7.1.29 of [KIT]; it is of period 2, hence we study the square.

Example 3.10 (The squared positive drunken man substitution)

$$n \to (n-2)nn(n+2)$$

for all even $n \geq 2$, *and*

$$0 \to 02.$$

The matrix is irreducible, aperiodic and transient; there exists an infinite invariant measure, defined by left eigenvectors for the Perron-Frobenius eigenvalue 4; after we normalize it by $\mu[0] = 1$, its value on letters is given by $\mu[2n] = 2n + 1$, and on cylinders by equation 1. Hand computations suggest that, despite a misleading vocabulary, the infinite measure preserving dynamical system is *recurrent*, and, when we induce it on the cylinder [0], what we get is isomorphic to the system associated to

Example 3.11 (The induced positive drunken man substitution)

$$n \to 123\ldots(n+1)$$

for all $n \geq 1$.

References

[CAS] J. CASSAIGNE, *Complexité et facteurs spéciaux*, (French) [Complexity and special factors] Journées Montoises (Mons, 1994), Bull. Belg. Math. Soc. Simon Stevin 4 (1997), no. 1, 67–88.

[DUR] F. DURAND, *A characterization of substitutive sequences using return words*, Discrete Math. 179 (1998), 89–101.

[FER] S. FERENCZI, *Complexity of sequences and dynamical systems*, Discrete Math. 206 (1999), 145–154.

[HOP] E. HOPF, *Ergodentheorie* (1937), Springer-Verlag.

[KIT] B. KITCHENS, *Symbolic dynamics. One-sided, two-sided and countable state Markov shifts*, Universitext. (1998), Springer-Verlag.

[KRE] U. KRENGEL, *Entropy of conservative transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 7 (1967), 161–181.

[LeG] M. LE GONIDEC, *Complexité des automates infinis*, Ph. D. Thesis, Institut de Mathématiques de Luminy, in preparation.

[MOS] B. MOSSÉ, *Puissances de mots et reconnaissabilité des points fixes d'une substitution*, (French) [Powers of words and recognizability of the fixed points of a substitution] Theoret. Comput. Sci. 99 (1992), no. 2, 327–334.

[PYT-1] N. PYTHEAS FOGG, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Math. vol. 1794 (2002), Springer-Verlag.

[PYT-2] N. PYTHEAS FOGG, *The universal counter-example*, in preparation.

[QUE] M. QUEFFÉLEC, *Substitution dynamical systems - Spectral analysis*, Lecture Notes in Math. vol. 1294 (1987), Springer-Verlag.