# JOININGS OF THREE-INTERVAL EXCHANGE TRANSFORMATIONS

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ABSTRACT. We show that among three-interval exchange transformations there exists a dichotomy: $T$ has minimal self-joinings whenever the associated subshift is linearly recurrent, and is rigid otherwise. We build also a family of simple rigid three-interval exchange transformations, which is a step towards an old question of Veech, and a family of rigid three-interval exchange transformations which includes Katok's rank one map.

## 1. INTRODUCTION

One recurrent preoccupation of ergodicians in the last twenty years has been with joinings: the notion of *self-joinings* of a system has been introduced by Rudolph in [Ru], to generalize some useful invariants of measure-theoretic isomorphism such as the factor algebra and the centralizer. A transformation which has *minimal self-joinings* (see Definition 4.2 below), has trivial centralizer and no nontrivial proper factor, and can be used to build a so-called *counter-example machine* with surprising properties. The first example of a transformation with minimal self-joinings was given in [Ru], and a little later the famous *Chacon map* was shown in [dJ-R-S] also to have minimal self-joinings. However, both these examples may seem built on purpose, and they have no "natural", i.e. geometric, realization. Geometric examples of transformations with minimal self-joinings were sought in the category of *interval exchange transformations* (Definition 2.1 below): these were introduced by Oseledec [O], following an idea attributed to Arnold [Ad], see also Katok and Stepin [K-S]. And indeed in 1983 del Junco [dJ] built a one-parameter family of three-interval exchange transformations, depending on an irrational $\gamma$, and proved that whenever this $\gamma$ has bounded partial quotients in its continued fraction expansion the system has minimal self-joinings (the interested reader is warned that he will *not* find the terms "three-interval exchange transformation" or "minimal self-joinings" in del Junco's paper; the systems which he describes as two-point extensions of rotations are indeed three-interval exchange transformations, and the notion of "simplicity" he proves is only slightly weaker than the original notion of minimal self-joinings, and has been standing as the current definition of "minimal self-joinings " since [dJ-R1]).

But in the meantime, Veech had shown that almost all interval exchange transformation (in the sense: for a fixed permutation, for Lebesgue-almost all values of the lengths of the intervals) are *rigid* (see Definition 5.1 below), and hence have uncountable centralizers and cannot have minimal self-joinings (this is written in [V2]); thus he devised in [V1] a weakened notion of minimal self-joinings to allow for a nontrivial centralizer; the new notion, which Veech called "property S" but which is now known as "simplicity (in the sense of Veech)" (see Definition 6.1 below) is strong enough to keep many of the properties of systems with minimal self-joinings, though proving this required a lot of work [dJ-R2] [V1]. And Veech asked the following question (4.9 of [V1])

*Date*: October 28, 2003.
1991 *Mathematics Subject Classification.* Primary 37A05; Secondary 37B10.
Partially supported by NSF grant INT 9726708.

**Question 1.1.** *Are almost all interval exchange transformations simple?*

Note that if the answer is positive then almost all interval exchange transformations are simple and rigid; simplicity implies either discrete spectrum or weak mixing; though there are nontrivial examples of interval exchange transformations with discrete spectrum [Ax] [F-H-Z3], no one seriously suspects that these examples form a set a positive measure; hence to give a positive answer to Veech's question, we need to know that almost all interval exchange transformations are weakly mixing, which has been known since [K-S] for three intervals but remains unproved for a higher number of intervals, the best partial result in this direction being [N-R].

But, while Veech's question stood unanswered, examples of simple transformations remained scarce: there were of course the systems with minimal self-joinings, and some systems without minimal self-joinings but naturally related to these systems (such as the time-one map of a flow which, as a flow, has minimal self-joinings); at last in [dJ-R1], a natural generalization of Chacon's map was (very cunningly!) shown to be simple and rigid. It remained to this day the only explicit simple rigid map; no "natural" system was known to be simple and rigid, and not a single interval exchange transformation was there to give substance to Veech's conjecture, the only known simple interval exchange transformations being del Junco's ones [dJ], which have minimal self-joinings.

Among interval exchange transformations, the first nontrivial ones are the three-interval exchange transformations; such a transformation involves two interval lengths $\alpha$ and $\beta$ as parameters, but can also be viewed as an induced map of a rotation of angle $\alpha'$ on an interval of length $\beta'$. The properties of minimal self-joinings and rigidity are linked to the diophantine properties of $\alpha'$ and $\beta'$; for example, in [dJ] $\beta' = \frac{3\alpha'}{2}$, and it is shown that whenever $\alpha'$ has bounded partial quotients we have minimal self-joinings, and if $\alpha'$ has unbounded partial quotients with infinitely many even convergents we have rigidity, the remaining cases raising open questions.

Now, the structure of three-interval exchange transformations has been studied extensively in [F-H-Z1] [F-H-Z2], where an expansion $(n_k, m_k, \epsilon_{k+1})_{k\in\mathbb{N}}$ reflects the diophantine properties of $\alpha'$ and $\beta'$ and governs a representation of the system by Rokhlin stacks. This representation has been used already in [F-H-Z3] to derive new spectral properties of three-interval exchange transformations. In the present paper, which relies heavily on [F-H-Z1] [F-H-Z2] but is essentially independent of [F-H-Z3], we use this representation to solve completely the problem of minimal self-joinings for three-interval exchange transformations, by a clear dichotomy: for an explicit set of expansions, corresponding to a subset of the $\alpha'$ with bounded partial quotients, and characterized by the word-combinatorial property of *linear recurrence* (this notion was defined independently in [B] and [D], see Definition 3.1 below), the three-interval exchange transformations have minimal self-joinings, and the proof is a nontrivial generalization of the proof of minimal self-joinings for Chacon's map. Outside this set, every three-interval exchange transformation is rigid, and these provide the first known examples of rigid systems built through a family of Rokhlin stacks, but where the rigidity sequence is not the sequence of heights of these stacks; this also puts the open cases in [dJ] on the side of rigidity. And among these rigid three-interval exchange transformations, we do have an uncountable family of examples which are weakly mixing and simple, as they are measure-theoretically isomorphic to variants of del Junco-Rudolph's map; this gives us a modest but first step towards a positive answer to Veech's question. Another interesting family of three-interval exchange transformations has a structure closely related to a well-known abstract example, the so-called Katok's map (in fact, a family of maps): we show the surprising result that the classic Katok's map is indeed a three-interval exchange transformation (up to measure-theoretic isomorphism), and that the unusual properties of its Cartesian square equipped with the product self-joining (Loosely Bernoulli property, finite spectrum) are shared by the Cartesian square of several other three-interval exchange transformations; thus we see that many people have been using three-interval exchange transformations without knowing it. The simplicity of Katok's map remains a tantalizing open problem.

*Throughout the paper, every constant will be denoted by $K$.*

## 2. GENERALITIES

### 2.1. Three-interval exchange transformations.

**Definition 2.1.** *A* $k$-interval exchange transformation *is given by a probability vector* $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ *together with a permutation* $\pi$ *of* $\{1, 2, \ldots, k\}$*. The unit interval* $[0, 1)$ *is partitioned into* $k$ *subintervals of lengths* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *which are then rearranged according to the permutation* $\pi$*.*

In this paper we consider a *symmetric three-interval exchange transformation* i.e., a three-interval exchange transformation $T$ with probability vector $(\alpha, \beta, 1-(\alpha+\beta))$, $\alpha, \beta > 0$, and permutation $(3, 2, 1)$ [1] defined by

$$Tx = \begin{cases} x + 1 - \alpha & \text{if } x \in [0, \alpha) \\ x + 1 - 2\alpha - \beta & \text{if } x \in [\alpha, \alpha + \beta) \\ x - \alpha - \beta & \text{if } x \in [\alpha + \beta, 1). \end{cases} \tag{1}$$

Throughout the paper, $T$ will denote the symmetric three-interval exchange transformation on $I = [0, 1)$ defined in equation (1). It depends only on the two parameters $0 < \alpha < 1$ and $0 < \beta < 1 - \alpha$. We note that $T$ is continuous except at the points $\alpha$ and $\alpha + \beta$.

Set

$$\alpha' = \frac{1-\alpha}{1+\beta} \quad \text{and} \quad \beta' = \frac{1}{1+\beta}. \tag{2}$$

Then $T$ is induced by a rotation on the circle by angle $\alpha'$. More precisely, $T$ is obtained from the 2-interval exchange $R$ on $[0, 1)$ given by

$$Rx = \begin{cases} x + \alpha' & \text{if } x \in [0, 1 - \alpha') \\ x + \alpha' - 1 & \text{if } x \in [1 - \alpha', 1). \end{cases}$$

by inducing (according to the first return map) on the subinterval $[0, \beta')$, and then renormalizing by scaling by $1 + \beta$.

We say $T$ satisfies the *infinite distinct orbit condition* (or *i.d.o.c.* for short) of Keane [K] if the two negative trajectories $\{T^{-n}(\alpha)\}_{n\geq 0}$ and $\{T^{-n}(\alpha+\beta)\}_{n\geq 0}$ of the discontinuities are infinite disjoint sets. Under this hypothesis, $T$ is both minimal and uniquely ergodic; the unique invariant probability measure is given by the Lebesgue measure $\mu$ on $[0, 1)$ (and hence $(I, T, \mu)$ is an ergodic system).

Because of the connection with the inducing rotation $R$, the exchange $T$ does *not* satisfy the i.d.o.c. condition if and only if one of the following holds:

- $\alpha'$ is rational, or equivalently $p\alpha + q\beta = p - q$,
- $\beta' = p\alpha' - q$, or equivalently $p\alpha + q\beta = p - q - 1$,
- $\beta' = -p\alpha' + q$, or equivalently $p\alpha + q\beta = p - q + 1$

[1] All other permutations on three letters reduce the transformation to an exchange of two intervals.

for some nonnegative integer $p$, and positive integer $q$. Therefore, the i.d.o.c. condition holds for almost every $(\alpha,\beta)$.

2.2. **Results of the arithmetic algorithm.** This subsection summarizes some results from [F-H-Z1] and [F-H-Z2].

Let $I$ denote the open interval $(0,1)$, $D_0 \subset \mathbb{R}^2$ the simplex bounded by the lines $y=0$, $x=0$, and $x+y=1$, and $D$ the triangular region bounded by the lines $x=\frac{1}{2}$, $x+y=1$, and $2x+y=1$. We define two mappings on $I\times I$,

$$F(x,y)=\left(\frac{2x-1}{x},\frac{y}{x}\right) \quad \text{and} \quad G(x,y)=(1-x-y,y)\,.$$

We check that if $(\alpha,\beta)\in D_0$ is not in $D$ and is not on any of the rational lines $p\alpha+q\beta=p-q$, $p\alpha+q\beta=p-q+1$, $p\alpha+q\beta=p-q-1$ then there exists a unique finite sequence of integers $l_0$, $l_1$, …, $l_k$ such that $(\alpha,\beta)$ is in $H^{-1}D$ where $H$ is a composition of the form $G^t\circ F^{l_0}\circ G\circ F^{l_1}\circ G\ldots\circ G\circ F^{l_k}\circ G^s$, $s,t\in\{0,1\}$.

The function $H(\alpha,\beta)$ is computed recursively as follows: we start with $\alpha^{(0)}=\alpha$, $\beta^{(0)}=\beta$. Then, given $(\alpha^{(k)},\beta^{(k)})$, we have three mutually exclusive possibilities: if $(\alpha^{(k)},\beta^{(k)})$ is in $D$, the algorithm stops; if $\alpha^{(k)}>\frac{1}{2}$, we apply $G$; if $2\alpha^{(k)}+\beta^{(k)}<1$, we apply $F$.

Associated to each point $(\alpha,\beta)\in D_0$ is a sequence $(n_k,m_k,\epsilon_{k+1})_{k\geq 1}$, where $n_k$ and $m_k$ are positive integers, and $\epsilon_{k+1}$ is $\pm 1$. This sequence we call the *three-interval expansion* of $(\alpha,\beta)$, is a variant of the *negative slope expansion* defined in [F-H-Z1]; it is constructed as follows:

- For $(\alpha,\beta)$ in $D$ we put

$$x_0=\frac{1-\alpha-\beta}{1-\alpha} \quad \text{and} \quad y_0=\frac{1-2\alpha}{1-\alpha}$$

and define for $k\geq 0$

$$(x_{k+1},y_{k+1})=\begin{cases}\left(\left\{\frac{y_k}{(x_k+y_k)-1}\right\},\left\{\frac{x_k}{(x_k+y_k)-1}\right\}\right) & \text{if } x_k+y_k>1\\ \left(\left\{\frac{1-y_k}{1-(x_k+y_k)}\right\},\left\{\frac{1-x_k}{1-(x_k+y_k)}\right\}\right) & \text{if } x_k+y_k<1\end{cases}$$

$$(n_{k+1},m_{k+1})=\begin{cases}\left(\lfloor\frac{y_k}{(x_k+y_k)-1}\rfloor,\lfloor\frac{x_k}{(x_k+y_k)-1}\rfloor\right) & \text{if } x_k+y_k>1\\ \left(\lfloor\frac{1-y_k}{1-(x_k+y_k)}\rfloor,\lfloor\frac{1-x_k}{1-(x_k+y_k)}\rfloor\right) & \text{if } x_k+y_k<1\end{cases}$$

where $\{a\}$ and $\lfloor a\rfloor$ denote the fractional and integer part of $a$ respectively. For $k\geq 0$ set

$$\epsilon_{k+1}=\operatorname{sgn}(x_k+y_k-1).$$

We note that $\epsilon_1$ is always $-1$, hence we ignore it in the expansion.

- For $(\alpha,\beta)\notin D$ we let $H$ be the function above for which $(\alpha,\beta)\in H^{-1}D$ and put

$$\left(\bar\alpha,\bar\beta\right)=H(\alpha,\beta),$$

and define $(n_k,m_k,\epsilon_{k+1})$ as in the previous case, starting from $\left(\bar\alpha,\bar\beta\right)\in D$.

The following proposition sums up what we need of [F-H-Z1]; when $(\alpha,\beta)$ is in $D$, it is a translation (taking into account the fact that the initial conditions are slightly different) of results in [F-H-Z1]; in the general case, it comes from these results and the definition of $(\bar\alpha,\bar\beta)$.

**Proposition 2.1.** 1) *If $T$ satisfies the i.d.o.c. condition, then the three-interval expansion $(n_k,m_k,\epsilon_{k+1})$ of $(\alpha,\beta)$ is infinite.*

2) *An infinite sequence* $(n_k, m_k, \epsilon_{k+1})$ *is the expansion of at least one pair* $(\alpha, \beta)$ *defining a symmetric three-interval exchange transformation satisfying the i.d.o.c. condition, if and only if* $n_k$ *and* $m_k$ *are positive integers,* $\epsilon_{k+1} = \pm 1$*,* $(n_k, \epsilon_{k+1}) \neq (1, +1)$ *for infinitely many* $k$ *and* $(m_k, \epsilon_{k+1}) \neq (1, +1)$ *for infinitely many* $k$*.*
3) *Each infinite sequence* $(n_k, m_k, \epsilon_{k+1})$ *satisfying the conditions in 2) is the three-interval expansion of a countable family of couples* $(\alpha, \beta)$*, with exactly one couple in each of the disjoint triangles* $H^{-1}D$*, where* $H$ *has any of the possible forms defined earlier in this section, including the identity.*
4) $(\bar{\alpha})' = \frac{1-\bar{\alpha}}{1+\bar{\beta}} = \cfrac{1}{2 + \cfrac{1}{m_1 + n_1 - \cfrac{\epsilon_2}{m_2 + n_2 - \cfrac{\epsilon_3}{m_3 + n_3 - \dots}}}}$
5) $\alpha'$ *has bounded partial quotients (in the usual continued fraction expansion) if and only if in the three-interval expansion of* $(\alpha, \beta)$ *the* $n_k + m_k$ *are bounded, as well as the lengths of strings of consecutive* $(1, 1, +1)$*.*

2.3. **Results of the combinatorial description.** This subsection summarizes some results from [F-H-Z2].

Let $\alpha$, $\beta$, $T$ be as in equation (1). We define the natural partition

$$P_1 = [0, \alpha),$$
$$P_2 = [\alpha, \alpha + \beta),$$
$$P_3 = [\alpha + \beta, 1).$$

For every point $x$ in $[0, 1)$, we define an infinite sequence $(x_n)_{n\in\mathbb{N}}$ by putting $x_n = i$ if $T^n x \in P_i$, $i = 1, 2, 3$. This sequence, also denoted by $x$, is called the *trajectory* of $x$. If $T$ satisfies the i.d.o.c. condition, the minimality of the system implies that all trajectories contain the same finite words as factors. The shift on the set of trajectories defines a symbolic dynamical system which is called the *natural coding* of $T$.

Let $I'$ be a set of the form $\cap_{i=0}^{n-1} T^{-i} P_{k_i}$; we say $I'$ has a *name* of length $n$ given by $k_0 \dots, k_{n-1}$; note that $I'$ is necessarily an interval, and $k_0, \dots, k_{n-1}$ is the common beginning of trajectories of all points in $I'$.

For each interval $J$, it is known (see for example [C-F-S], p. 128, Lemma 2) that the induced map of $T$ on $J$ is an exchange of three or four intervals. More precisely, there exists a partition $J_i$, $1 \leq i \leq t$ of $J$ into subintervals (with $t = 3$ or $t = 4$), and $t$ integers $h_i$, such that $T^{h_i} J_i \subset J$, and $\{T^j J_i\}$, $1 \leq i \leq t$, $0 \leq j \leq h_i - 1$, is a partition of $[0, 1[$ into intervals: this is the partition into *Rokhlin stacks* associated to $T$ with respect to $J$. The intervals $J_i$ have names of length $h_i$, called *return words* to $J$.

**Theorem 2.2.** *(Structure Theorem) Let* $T$ *be a symmetric three-interval exchange transformation as defined in equation (1), satisfying the i.d.o.c. condition, and let* $(n_k, m_k, \epsilon_{k+1})_{k\geq 1}$*, be the three-interval expansion of* $(\alpha, \beta)$*. Then there exists an infinite sequence of nested intervals* $J_k$*,* $k \geq 1$*, which have name* $w_k$ *and exactly three return words,* $A_k$*,* $B_k$ *and* $C_k$*, given recursively for* $k \geq 1$ *by the following formulas*

$$A_k = A_{k-1}^{n_k - 1} C_{k-1} B_{k-1}^{m_k - 1} A_{k-1},$$
$$B_k = A_{k-1}^{n_k - 1} C_{k-1} B_{k-1}^{m_k},$$

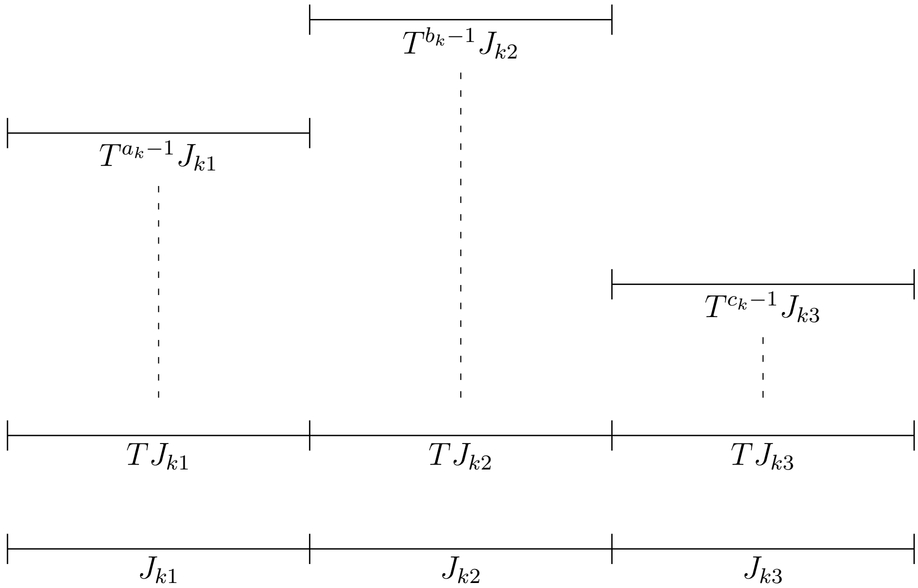


FIGURE 1. The three Rokhlin Stacks

$$C_k = A_{k-1}^{n_k-1}C_{k-1}B_{k-1}^{m_k-1}$$

*if* $\epsilon_{k+1} = +1$, *and*

$$A_k = A_{k-1}^{n_k-1}C_{k-1}B_{k-1}^{m_k},$$
$$B_k = A_{k-1}^{n_k-1}C_{k-1}B_{k-1}^{m_k-1}A_{k-1},$$
$$C_k = A_{k-1}^{n_k-1}C_{k-1}B_{k-1}^{m_k}A_{k-1}$$

*if* $\epsilon_{k+1} = -1$, *for initial words* $A_0$, $B_0$ *and* $C_0$ *such that the lengths of the first two differ exactly by one. The words* $A_k$ *and* $B_k$ *both contain* $w_k$ *as a prefix.*

The Structure Theorem gives an explicit construction of Rokhlin stacks.

Let $w_k$ be the name of $J_k$. Every trajectory under $T$ is a concatenation of words $A_k$, $B_k$, $C_k$, which we call the $k$-*words*. We say that a $k$-word occurs at its *legal* $k$-*place* in a trajectory if it is immediately followed by the word $w_k$. A concatenation of $k$-words occurs at its legal $k$-place if each of its $k$-words occur at their legal $k$-place.

We define $FA_k$ to be set of $x \in X$ such that (in the trajectory of $x$) $x_0$ is the first letter of the $k$-word $A_k$ in its legal $k$-place, and similarly $FB_k$ and $FC_k$. Note that $FA_k \cup FB_k \cup FC_k = J_k$, and these are exactly the three intervals of continuity of the induced map of $T$ on $J_k$.

Let $a_k$, $b_k$, $c_k$ be the lengths of $A_k$, $B_k$ and $C_k$; it follows from the Structure Theorem that $|a_k - b_k| = |a_0 - b_0| = 1$ and $c_k$ is either $a_k - a_{k-1} = b_k - b_{k-1}$ or $a_k + a_{k-1} = b_k + b_{k-1}$. In particular, we have $c_k \le 2a_k$. Then $X = [0, 1)$ is the disjoint union of $T^jFA_k$, $0 \le j \le a_k - 1$, $T^jFB_k$, $0 \le j \le b_k - 1$, $T^jFC_k$, $0 \le j \le c_k - 1$; we denote by $\tau A_k$ the disjoint union of $T^jFA_k$, $0 \le j \le a_k - 1$, and define similarly $\tau B_k$ and $\tau C_k$; $\tau A_k$, $\tau B_k$ and $\tau C_k$ are called the $k$-*stacks*, and the $T^jFA_k$, $0 \le j \le a_k - 1$ are the *levels* of the stack $\tau A_k$, and similarly for $B$ and $C$. The levels are intervals of small diameter, as they have names of arbitrarily large length; hence any integrable function $f$ can be approximated (in $L_1$ for example) by functions $f_k$ which are constant on each level of each $k$-stack.

Hence, in the language of *finite rank systems*, see [F], the Structure Theorem implies that

**Corollary 2.3.** *$T$ is of rank at most three, generated by the stacks $\tau A_k$, $\tau B_k$, $\tau C_k$; the recursion formulas in the Structure Theorem give an explicit construction by cutting and stacking of these stacks.*

The finite rank structure is particularly relevant in the measure-theoretic study of the system $(I, T, \mu)$, and in this framework, the following Lemma will be useful:

**Lemma 2.4.** *For any trajectory $x$, the Lebesgue measure $\mu(\tau A_k)$ is the limit when $n$ goes to infinity of $\frac{1}{n}$ times the total number of indices $0 \leq i \leq n-1$ such that $x_i$ belongs to a word $A_k$ in its legal $k$-place. Similarly for $B_k$ and $C_k$.*

*Proof.* This follows from the unique ergodicity of $T$. □

## 3. Linear recurrence

**Definition 3.1.** *A symbolic system $X$ is called* linearly recurrent *if there exists a constant $K$ such that each word of length $n$ occurring in a sequence of $X$ occurs in every word of length at least $Kn$ occurring in a sequence of $X$.*
*We say that a three-interval exchange transformation is linearly recurrent whenever its natural coding is linearly recurrent.*

**Proposition 3.1.** *A symmetric three-interval exchange transformation satisfying the i.d.o.c. condition is linearly recurrent if and only if the $m_k$, $n_k$, the lengths of the strings of $(n_k = 1, \epsilon_{k+1} = +1)$ and the lengths of the strings of $(m_k = 1, \epsilon_{k+1} = +1)$ are bounded.*

**Proof**
If the condition is satisfied, then the ratios of the length of each $k$-word and the length of each $(k-1)$-word are bounded, while each $k$-word occurs in each $(k+K)$-word for a bounded $K$. As each word occurring in a sequence of $X$ is a subword of some $k$-word, linear recurrence follows.

Conversely, if the condition is not satisfied: if $m_k$ is not bounded, the word $A_{k-1}$ in its legal place does not occur in $B_{k-1}^{m_k}$, whose length is at least $m_k(a_{k-1}-1)$, and this means that $w_{k-1}A_{k-1}$, whose length is at most $2a_{k-1}$, does not occur in $w_{k-1}B_{k-1}^{m_k}$, and this prevents linear recurrence; if there are strings of length $M$ of $(m_k = 1, \epsilon_{k+1} = +1)$, some word $A_k$ in its legal place will not occur in $B_{k+M}$ nor in $C_{k+M}$, the lengths of which are at least $M'a_k$ for some $M'$ going to infinity with $M$, and we conclude similarly. The reasoning is similar if we replace $m_k$ by $n_k$. QED

**Proposition 3.2.** *Let $\alpha'$ and $\beta'$ be defined by equation (2). If $\alpha'$ has unbounded partial quotients (in the usual continued fraction expansion), $T$ is not linearly recurrent. If $\alpha'$ has bounded partial quotients, the set of $\beta'$ such that $T$ is linearly recurrent is uncountable and of measure $0$.*

**Proof**
If $\alpha'$ has unbounded partial quotients, assertion 5) of Proposition 2.1 implies the absence of linear recurrence.

Suppose now $\alpha'$ has bounded partial quotients; we make the proof for $(\alpha, \beta)$ in the preferred triangle $D$, the situation in the other triangles $H^{-1}D$ being deduced by the tranformation $H$ which does not change the above properties. Then $\frac{1}{3} \le \alpha' \le \frac{1}{2}$; we expand it in the form

$$\alpha' = \cfrac{1}{2 + \cfrac{1}{r_1 - \cfrac{\epsilon_2}{r_2 - \cfrac{\epsilon_3}{\dots}}}}$$

with $r_i \ge 2$ and $\epsilon_{i+1} = \pm 1$; there are many ways to do this, as at each stage we have two choices, one with $r_i = r$ and $\epsilon_i = -1$ and one with $r_i = r+1$ and $\epsilon_i = +1$ (except if $r = 1$, where the only possible choice is $r_i = 2$, $\epsilon_{i+1} = +1$). Then, because of assertions 3) and 4) of Proposition 2.1, every possible sequence $(n_k, m_k, \epsilon_{k+1})$ with $n_k + m_k = r_k$ for every $k$ is the three-intreval expansion of some $(\alpha, \beta)$ such that $\alpha' = \frac{1-\alpha}{1+\beta}$. Hence, for any expansion $(r_i, \epsilon_{i+1})$ and for any $M$ such that there are at most $M$ consecutive $(r_i, \epsilon_{i+1}) = (2, +1)$, which exists because $\alpha'$ has bounded partial quotients, we can find an uncountable number of expansions $(n_k, m_k, \epsilon_{k+1})$ with at most $M$ consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ or $(n_i, \epsilon_{i+1}) = (1, +1)$, and hence an uncountable number of $\beta'$ such that $T$ is linearly recurrent.

To prove the last assertion, we need to prove that for fixed $\alpha' = c$, where $\frac{1}{3} \le c \le \frac{1}{2}$ and the (usual) partial quotients of $c$ are bounded by $B$, and for any $M$, the set of $\beta'$ giving expansions with at most $M$ consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ or $(n_i, \epsilon_{i+1}) = (1, +1)$ is of measure zero. We use the notations of subsection 2.2 and [F-H-Z1]; let $S$ be the transformation sending $(x_k, y_k)$ to $(x_{k+}, y_{k+1})$ in subsection 2.2, and the $Q_l$ be the Farey cells coorespondings to the first $l$ values of $(n_k, m_k, \epsilon_{k+1})$, see Section 3 of [F-H-Z1]. Let $Z_M$ be the set of $(\alpha, \beta)$ in the preferred triangle $D$ having three-interval expansions with at least $M$ consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ or $(n_i, \epsilon_{i+1}) = (1, +1)$; we want to prove that each line $\frac{1-\alpha}{1+\beta} = c$ intersects $Z_M$ on a set of full linear measure.

Let $Z'_M$ be the set of $(\alpha, \beta)$ in the preferred triangle $D$ having three-interval expansions beginning with $M$ consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$; then, as the line $L_c = \{\frac{1-\alpha}{1+\beta} = c\}$, corresponds to $x_0 + y_0 = 3 - \frac{1}{c}$, the set $Z'_M$ intersects each $L_c$, where the partial quotients of $c$ are bounded by $B$, on a set of linear Lebesgue measure at least $s(B, M) > 0$. On the other hand, the intersection (assuming it is nonempty) of $L_c$ with a $k$-th level Farey quadrilateral is taken linearly by $S^k$ to the intersection of a line $L_d$ with the preferred triangle $D = Q_0$, where the partial quotients of $d$ are bounded by $B+1$, because of the formulas giving $x_k$ and $y_k$. Hence, at least $s(B+1, M)$ of the linear measure of the intersection of the line $L_c$ with every $k$-th level Farey quadrilateral $Q_k$ is taken linearly by $S^k$ to $Z'_M$; such points have a string of length $M$ of consecutive $(m_i, \epsilon_{i+1}) = (1, +1)$ beginning at place $k$ in their expansions. By applying the argument to higher level quadrilaterals, neglecting boudary points, we can thus show that almost all the line $L_c$ is in $Z_M$. QED

**Examples**

The notion of linear recurrence is, for an irrational rotation, equivalent to the boundedness of the partial quotients of the angle. Thus, a linearly recurrent three-interval exchange transformation can be viewed as a three-interval exchange transformation with bounded partial quotients; as was noticed in [B], the three-interval exchange transformations for which $\alpha'$ has bounded partial quotients are not always linearly recurrent, but they always satisfy a weaker property called *predictable recurrence* (there exists a constant bounding the ratio of the highest and lowest return times of any word in their natural coding).

In [dJ], a one-parameter family of three-interval exchange transformations was built as follows: starting with an irrational $0 < \gamma < 1$, we put $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3} - \frac{2\gamma}{3}$ if $\gamma < \frac{1}{2}$, $\beta = 1 - \frac{2\gamma}{3}$ if $\gamma > \frac{1}{2}$. Then the three-interval expansion of $(\alpha, \beta)$ is such that $m_k = n_k$ for every $k \geq 2$: if $\gamma > \frac{1}{2}$, $(\alpha, \beta)$ is in the preferred triangle $D$ with $\alpha' = \gamma$ and $\beta' = \frac{3\gamma}{2}$, and we check that $x_1 - y_1 = 1$, hence $m_1 = n_1 + 1$, $x_2 = y_2$, and $m_k = n_k$ for every $k \geq 2$; the case $\gamma < \frac{1}{2}$ splits into a countable number of subintervals according to the value of $(\bar{\alpha}, \bar{\beta})$, and we check that in each case $x_1 - y_1$ is an integer. Thus, among del Junco's examples, $T$ is linearly recurrent whenever $\gamma$ has bounded partial quotients, not linearly recurrent whenever $\gamma$ has unbounded partial quotients.

It will be proved in [F-H-Z3] that a weakly mixing three-interval exchange transformation can be linearly recurrent or not; for self-induced ones, we have always linear recurrence, as was already noticed in [B]. Note that linearly recurrent three-interval exchange transformations, when they satisfy the i.d.o.c. condition, are always weakly mixing: this comes from Proposition 3.1 and [F-H-Z3], or from Proposition 3.2 and [B-N].

## 4. Minimal self-joinings

**Definition 4.1.** *A* self-joining *(of order two) of a system $(X, T, \mu)$ is any measure $\nu$ on $X \times X$, invariant under $T \times T$, for which both marginals are $\mu$.*

**Definition 4.2.** *An ergodic system $(X, T, \mu)$ has* minimal self-joinings *(of order two) if any ergodic self-joining (of order two) $\nu$ is either the product measure $\mu \times \mu$ or a* diagonal *measure defined by $\nu(A \times B) = \mu(A \cap T^i B)$ for an integer $i$.*

The minimal self-joinings of order three or more would involve measures on higher order Cartesian powers of $X$; we conjecture our results would generalize to higher orders. Throughout this paper, we use “minimal self-joinings” for “minimal self-joinings of order two”.

**Theorem 4.1.** *A linearly recurrent three-interval exchange transformation satisfying the i.d.o.c. condition has minimal self-joinings.*

In all the sequel of this section, we shall prove this theorem, the proof being a natural generalization of the proof that Chacon's map has minimal self-joinings [dJ-R-S]; a more involved generalization of that proof was made in [Ra] for the horocycle flow, and indeed our proof uses (implicitely) Ratner's famous *R-property*.

Let $T$ be a linearly recurrent three-interval exchange transformation satisfying the i.d.o.c. condition. If we denote by $h_k$ the smallest of the heights $a_k$, $b_k$, $c_k$, we have $h_k \leq a_k \leq Kh_k$, $h_k \leq b_k \leq Kh_k$, $h_k \leq c_k \leq Kh_k$, and $h_k \leq h_{k+1} \leq Kh_k$.

For a point $x$, we say equivalently that $x$ is in the $i$-th level of the stack $\tau W_k$ ($W_k$ being $A_k$, $B_k$ or $C_k$), or that $x_0$ is in position $i$ in the word $W_k$. This word $W_k$ is a subword of a $k+1$-word occurring in its legal $k$-place according to the recursion formulas of the Structure Theorem; we say it has *order* $j$ if it is the $j$-th $k$-word in this $k+1$-word ; $j$ is an integer between $1$ and $m_{k+1} + n_{k+1}$, or $m_{k+1} + n_{k+1} + 1$, or $m_{k+1} + n_{k+1} - 1$. If $x$ and $y$ are in the same stack $\tau W_k$, we say $x_0$ and $y_0$ are in $k$-words of the same type ($A$, $B$ or $C$). If $x$ and $y$ are in the same stack and in the same column, then $x_0$ and $y_0$ are in $k$-words of the same type and of the same order inside $k+1$-words of the same type. If $x$ and $y$ are in the same stack but in different columns, then $x_0$ and $y_0$ are in $k$-words of the same type and with different orders, or else with the same order but inside $k+1$-words of different types.

If $x_0$ is at position $i$ in its $k$-word which we denote $X_k(0)$, $y_0$ at position $j$ in the word $Y_k(0)$, the *overlap* between $X_k(0)$ and $y_k(0)$ is $\omega = i \wedge j + (|X_k(0)| - i) \wedge (|Y_k(0)| - j)$. We write then, for a given $r$, the $r+1$ first $k$-words in $x$ and $y$ above each other:

$$X_k(0)X_k(1)\dots X_k(r),$$

$$Y_k(0)Y_k(1)\dots Y_k(r).$$

We call $(X_k(i), Y_k(i))$ the pairs of *corresponding* $k$-words, the initial one being for $i = 0$. Note that there is not necessarily a positive overlap between $X_k(s)$ and $Y_k(s)$ for $1 \leq s \leq r$, in the sequel we shall use only the fact that the initial overlap is large enough.

**Definition 4.3.** *For integers $k > 0$ and $t > 0$, $x$ and $y$ have a $k$-*forcing *of* shift $t$ *if there exist integers $l \geq \frac{h_k}{K}$ and $-Kh_k \leq i_1 < i_2 \leq Kh_k - l$, and subwords $u$ and $v$ of $x$ and $y$ such that on $(x,y)$ from $i_1$ to $i_1 + l$ we see $(u_0 \dots u_l, v_0 \dots v_l)$ while on $(x,y)$ from $i_2$ to $i_2 + l - t$ we see $(u_0 \dots u_{l-t}, v_t \dots v_l)$ or $(u_t \dots u_l, v_0 \dots v_{l-t})$.*

**Lemma 4.2.** *There exists a set $E_0$ of measure 1 such that if $x$ and $y$ are in $E_0$ and belong to different orbits, then, for infinitely many values of $k$, $x_0$ and $y_0$ are in $k$-words of different types or of different orders, with an overlap between these $k$-words larger than $h_k/K$.*

**Proof**

We check that for $x$ in a set $E_0$ of full measure, there exist infinitely many $l$ such that $x_0$ is not in the first nor in the last $\frac{h_l}{K}$ positions in its $l$-word (this come from the independence of these events when $k$ varies). We start from $x$ in $X_0$ and take such an $l$. If, for every $k \geq l$, $x_0$ and $y_0$ are in the same $k$-word, they are always in the same column of the $k$-stack, and their difference of level $i(x,y)$ is constant: then $x$ and $T^i y$ are not separated by the partition into levels of the stacks and $x$ and $y$ are on the same orbit. So we take the first $k \geq l$ such that $x_0$ and $y_0$ are in different $k$-words. Then: if $k = l$, the position of $x_0$ in its $l$-word guarantees an overlap of at least $\frac{h_l}{K}$ to the left or to the right of $(x_0, y_0)$; if $k > l$, $x_0$ and $y_0$ are in a $k-1$-word of same order, which guarantees an overlap of at least $h_{k-1}$, except in the case where a $k$-word is made with only one $k-1$-word, which happens only when $C_k = C_{k-1}$ because $(n_k, m_k, \epsilon_{k+1}) = (1, 1, +1)$, and in that case we wait until the first $l$ such that $(n_l, m_l, \epsilon_{l+1}) \neq (1, 1, +1)$, which is at most $l + K$, and the overlap will be at least $\frac{h_l}{K}$. QED

**Lemma 4.3.** *If $x$ and $y$ are in $E_0$ and belong to different orbits, there exist an integer $t > 0$ and infinitely many integers $k > 0$ such that $x$ and $y$ have a $k$-forcing of shift $t$.*

**Proof**

We take one of the $k$ in the conclusion of Lemma 4.2. Take for example a $k$ such that $A_k$ is the longest of the two words $A_k$ and $B_k$; we know that $A_k$, $B_k$ and $C_k$ are return words of the word $w_k$; $w_k$ is a prefix of $A_k$ and $B_k$, and either $w_k$ is a prefix of $C_k$ or $C_k$ is a prefix of $w_k$ and $w_k$, $A_k$, $B_k$, $C_k$ have respectively lengths $h+2$, $h+1$, $h$, $c$ for some $h$ and $c$ ; each time $A_k$, $B_k$ or $C_k$ occur in their legal position in an orbit, they are followed by $w_k$. Occasionally, we shall write $A$ for $A_k$ and so on.

*Case 1*: $x_0$ and $y_0$ are in $k$-words of different type, $x_0$ is in $A_k$, $y_0$ is in $B_k$. Suppose for example $x_i \dots x_{i+h+1} = A_k$, and $y_{i+d} \dots y_{i+d+h} = B_k$ with $0 < d < h$. Then on $(x,y)$ from $i+d$ to $i+h-1$ we see $(w_d \dots w_{h-1}, w_0 \dots w_{h-d-1})$ while on $(x,y)$ from $i+d+h+1$ to $i+2h$ we see $(w_{d-1} \dots w_{h-2}, w_0 \dots w_{h-d-1})$. $h-d$ is the overlap and is at least $\frac{h_k}{K}$ while $i+d+h+1$ is between $0$ and $Kh_k$, so we have a $k$-forcing with shift $1$. Of course, the same is true for negative values of $d$ and cases when $A_k$ is shorter than $B_k$.

*Case 2*: $x_0$ and $y_0$ are in $k$-words of different type, $x$ is in $A_k$, $y$ is in $C_k$. Suppose first that $m_{k+1} \geq 2$; then after a $C$ there is always a $B$, followed possibly by other $B$s, then by a string of $n_{k+1}$ or $n_{k+1} - 1$ $A$s and a $C$, while, after an $A$, we see a string of length at most $n_{k+1} - 1$ of $A$s then a $C$. Thus in $x$ from place $i$ we see $AA^rC$ and in $y$ from place $i + d$ we see $CBD_1 \dots D_r$ where $0 \leq r \leq K$ and each $D_i$ can be either an $A$ or a $B$.

Suppose we see $AC$ in $x$ starting at place $i$, over $CB$ in $y$, starting at place $i + d$; note that in fact we see $ACw$ in $x$, $CBw$ in $y$. Let $h'$ be the smallest of $h$ and $c$. Then on $(x, y)$ from $i + d$ to $i+h'-1$ we see $(w_d \dots w_{h'-1}, w_0 \dots w_{h'-d-1})$ while on $(x, y)$ from $i+d+c+h+1$ to $i+c+h+h'$ we see $(w_{d-1} \dots w_{h'-2}, w_0 \dots w_{h'-d-1})$; for the same resaons as in the last case, this gives a forcing of shift one. Suppose now we see $AAC$ in $x$ starting at place $i$, over $CBB$ in $y$, starting at place $i + d$; note that in fact we see $AACw$ in $x$, $CBBw$ in $y$. Then on $(x, y)$ from $i + d$ to $i + h' - 1$ we see $(w_d \dots w_{h'-1}, w_0 \dots w_{h'-d-1})$ while on $(x, y)$ from $i + d + c + 2h + 2$ to $i + c + 2h + h' + 1$ we see $(w_{d-2} \dots w_{h'-3}, w_0 \dots w_{h'-d-1})$; here we have a forcing of shift 2. Similarly, the general case gives a forcing of shift $0 < r' < K$, the length $l$ being at least the initial overlap minus $r'$.

*Csae 3*: $x$ is in $A_k$, $y$ is in $C_k$, $m_{k+1} = 1$ and $\epsilon_{k+2} = -1$. The $k$-words occur in strings of the form $A^{n-1}CA$, $A^{n-1}CB$, and $A^{n-1}CBA$; we wait until we see a $B$ in $x$ or $y$, which will occur $r$ words after the initial $A$ or $C$, and this $r$ is bounded as at least one $A^{n-1}CBA = C_{k+1}$ will occur in each $k + 2$-word.

Let us show that *we do not see the first $B$ at the same time in $x$ and $y$*. Suppose for example this $B$ occurs in $y$. The first $B$ in each of the two types of $k + 1$-word where it may occur has the same order (considering the order of a $k$-word inside its $k + 1$-word). And $C_{k+1}$ contains $B$, hence none of the pairs of corresponding $k + 1$-words between (strictly) the initial one and the one containing the first $B$ can have a $C_{k+1}$; as for the initial one, it can have a $C_{k+1}$, but then it contains the first $B$, except in the exceptional case where the word $A$ containing $x_0$ is the last $k$-word of its $C_{k+1}$. Now, the initial $k$-words $A$ in $x$ and $C$ in $y$ have different orders; we cannot overcome this difference of order unless we see one $C_{k+1}$, as $A_{k+1}$ and $B_{k+1}$ have the same number of $k$-words; hence, if there is any pair of corresponding $k + 1$-words after (strictly) the initial one and before the one containing the first $B$, when we move to this second pair of corresponding $k + 1$-words, two cooresponding $k$-words have also different orders, either because we have not seen a $C_{k+1}$, or because we are in the exceptional case above and check that it must still be so; after that and until the pair of corresponding $k + 1$-words containing the first $B$, we do not see a $C_{k+1}$; hence two corresponding $k$-words inside the $k + 1$-words containing the first $B$ cannot have the same order, so there cannot be another $B$ above the first $B$.

As for the words $C$, as $(n_{k+1}, m_{k+1}, \epsilon_{k+2}) \neq (1, 1, +1)$ they are always isolated. Then, if the first $B$ is in $y$, we write the words of $(x, y)$ until the end of the first $C$ in $x$ after this $B$ in $y$ (note that this will not bring another $B$ in $x$ or $y$ as the $B$s are isolated and occur only after $C$s); if this $B$ is in $x$, we stop at its end. Because we have not seen a $C_{k+1}$, we have seen the same number of $k + 1$-words above and below, and the number of $C$ above and below until we stop is the same. So we have two strings of corresponding $k$-words, made with one $B$, and some words $A$ and $C$ with the same number of $C$ above and below. We compare as in the previous case $(x, y)$ from $i + d$ to $i + h' - 1$ with $(x, y)$ just after we stopped, and we get a $k$-forcing of shift 1, because the length of the string we compare is at least $\omega - K$ and the position of the furthest one is bounded by $Kh_k$.

*Case 4*: $x$ is in $A_k$, $y$ is in $C_k$, $\epsilon_{k+2} = +1$ and $m_{k+1} = 1$; this cannot be solved at the level of $k$-words. But the strings of $(m_l = 1, \epsilon_{l+1} = +1)$ have bounded length: so let $z$ be the first $l \geq k+1$ such that $(m_{l+1}, \epsilon_{l+2} \neq (1, +1)$; $z$ is smaller than $k+K$, $B$ does not occur in any $A_l$ or $C_l$ for $k+1 \leq l \leq z$, but occurs in $B_{z+1}$, $C_{z+1}$ and $A_{z+1}$ if $m_{z+1} > 1$, in $A_{z+1}$ and $C_{z+1}$ otherwise. Hence we see a $B$ in $x$ or $y$ after a bounded number of $k$-words. From the formulas giving the $z+1$-words as concatenations of $k$-words, we check that the first $B$ in each of the two or three types of $z+1$-words where it may occur has the same order, considering orders of $k$-words inside $z+1$-words, and that the initial words $A$ in $x$ and $C$ in $y$ have different orders (if we see different type of words with the same order, it can only be an $A$ and a $B$); again, we cannot overcome this difference of order unless we see one $C_{z+1}$, and $C_{z+1}$ contains $B$, hence none of the pairs of $k+1$-words we see in $x$ and $y$ between (strictly) the initial one and the one containing the first $B$ can have a $C_{k+1}$. We can now show, as in Case 3, that there is no $B$ under (or above) the first $B$ we see, with only one trouble: the case when the initial pair of $z+1$-words has a $C_{z+1}$ but does not contain a $B$ after the initial position; note that this can only happen if $\epsilon_{z+2} = -1$ and $x_0$ or $y_0$ is in the last $A_z$ of $C_{z+1}$; also, if this pair is made with two $C_{z+1}$, the trouble is avoided. Otherwise, for example $x_0$ is in the last $A_z$ of $C_{z+1}$, while $y_0$ is in a $B_z$ (and then we conclude by Case 1, as the overlap is a bounded proportion of the length of the $z$-words), or in a $C_z$ (and then we conclude by Case 2 or 3, as $(m_{z+1}, \epsilon_{z+2} \neq (1, +1))$, or in an $A_z$ ; in that last case, there will still be a difference of orders of the matched $k$-words in the next pair of $z+1$-words (considering order of $k$-words inside $z+1$-words), and we can continue as in Case 3, unless (maybe) $y_0$ is in the last $A_z$ of $B_{z+1}$; in that last subcase, $B_{z+1} = A_z^{t-1} C_z B_z^{u-1} A_z$ and $C_{z+1} = A_z^{t-1} C_z B_z^{u} A_z$ with $u \geq 1$, so we see a $B_z$ above a $C_z$ with $\epsilon_{z+2} = -1$, hence we conclude by the symmetric of Case 2 or 3. And then we finish Case 4 in the same way as Case 3, except that the $C$ are not isolated; but, as we see the same number of $C_{z+1}$ above and below, again we can stop in such a way that we see the same number of $C$ above and below.

*Symmetrics of Case 2, 3, 4*: if $x$ is in $B_k$, $y$ is in $C_k$, a similar reasoning holds, replacing the $m_k$ by the $n_k$: in Case 2S ($n_{k+1} \geq 2$), we go left instead of going right from $(x_0, y_0)$, using that before a $C$ there is an $A$. In Case 3S, we also go left, using that the first $A$ we see on the left is the last $k$-word in any of the $k+1$-word where it may occur. In Case 4S, $z$ is the first $l \geq k+1$ such that $(n_{l+1}, \epsilon_{l+2} \neq (1, +1)$, and we go right if $n_{z+1} \geq 2$, left if $n_{z+1} = 1$ and $\epsilon_{z+2} = +1$.

Of course, $x$ and $y$ play symmetric parts.

*Other cases, when $x$ and $y$ are in the same $k$-stacks, in different columns*: for example $x_0$ is in $A_k$, $y_0$ is in $A_k$, but not in the same $k$-word inside their $k+1$-word. Then a translation sends us to one of the cases 2 to 4, or 2S to 4S.

Now, as we have infinitely many $k$-forcings with shift $0 < t_k < K$, there exists one $t$ such that we have infinitely many $k$-forcings with shift $t$. QED

**Lemma 4.4.** *(*[dJ-R-S]*, Proposition 2, p. 278) If $(Y, S, \rho)$ is an ergodic transformation, $\nu$ an ergodic measure on $Y \times Y$, with marginals $\rho$ on each copy of $Y$, invariant under $S \times I$, where $I$ is the identity, then $\nu = \rho \times \rho$.*

**Proof of Theorem 4.2**

Let $\nu$ be an ergodic joining; we choose a point $(x, y)$ generic for $\nu$, such that $x$ and $y$ are in $E_0$ (it is possible because the marginals are $\mu$ ). If $x$ and $y$ are on the same orbit under $T$, we check that

$\nu$ is diagonal. Henceforth we suppose $x$ and $y$ are not on the same orbit. Then let $t$ be given by Lemma 4.3.

Let $P$ and $Q$ be two arbitrary cylinders, of names $p$ and $q$, and let us show that $\nu(P \times T^t Q) = \nu(P \times Q)$. If this holds for every cylinder, we shall deduce that $\nu$ is $I \times T^t$-invariant, and so is the product measure, as $T^t$ is ergodic because $T$ is weakly mixing.

We fix an $\epsilon$, and take a $k$ such that we have a $k$-forcing with shift $t$; If $k$ is large enough, genericity ensures that on every segment $(x,y)_0, ..., (x,y)_l$ or $(x,y)_{-l}, ..., (x,y)_0$, where $l > \frac{\epsilon h_k}{K}$, the pair of words $(p,q)$ appear with a frequency $\frac{\epsilon}{2K^3}$-close to $\nu(P \times Q)$. We deduce that these pairs appear with a frequency $\frac{\epsilon}{K^3}$-close to $\nu(P \times Q)$ in every segment of $(x,y)$ of length at least $\frac{h_k}{K}$ containing the origin.

Let us take now the string from $i_1$ to $i_1 + l$ given by the $k$-forcing: let $\xi$ be the density of $z$ in $(i_1, i_1 + l)$ for which $T^z x \in P, T^z y \in Q$: we have $|\xi - \nu(P \times Q)| < \frac{\epsilon}{2}$, for otherwise there would be a proportion bigger than $\frac{\epsilon}{2K^2} - \frac{\epsilon}{K^3}$ of errors on the segment $(0 \wedge i_1, i_1 + l)$ or on the segment $(i_1, 0 \vee i_1 + l)$. Similarly we can ensure that, if $\xi'$ is the density of $z$ in $i_2, i_2 + l$ for which $T^z x \in P, T^z y \in T^t Q$, we have $|\xi' - \nu(P \times T^t Q)| < \frac{\epsilon}{2}$. But the forcing implies $\xi = \xi'$, so we deduce that $|\nu(P \times Q) - \nu(P \times T^t Q)| < \epsilon$. QED

## 5. Rigidity

**Definition 5.1.** *A system* $(X, T, \mu)$ *is* rigid *if there exists a sequence* $s_n \to \infty$ *such that for any measurable set* $A$

$$\mu(T^{s_n} A \Delta A) \to 0.$$

**Theorem 5.1.** *A non-linearly recurrent three-interval exchange transformation satisfying the i.d.o.c. condition is rigid.*

**Proof**

It is enough to prove that for a given set $E$, there exists a sequence $s_n \to \infty$ such that $\mu(T^{s_n} E \Delta E) \to 0$; such a sequence will be called a *rigidity sequence* for $T$.

*Case 1*: $\limsup \frac{n_k}{m_k} = +\infty$ or $\limsup \frac{m_k}{n_k} = +\infty$.

Suppose for example that $\frac{n_k}{m_k} \to +\infty$ for $k$ in a sequence $S$ ; then the measure of the stack $\tau A_{k-1}$ is at least $\frac{n_k - 1}{m_k + n_k + 1}$, hence tends to one on $S$, and, for a level $L$ of this stack, $\mu(T^{a_{k-1}} L \Delta L) \leq \frac{\mu(L)}{n_k}$. Hence, by approximating $A$ by levels of $\tau A_{k-1}$, we get the desired relation with the rigidity sequence $a_{k-1}, k \in S$.

*Case 2*: $\frac{n_k}{m_k}$ and $\frac{m_k}{n_k}$ are bounded, but $m_k + n_k$ are unbounded.

Suppose for example $n_k \to +\infty$ for $k \in S$; we have also $m_k \to +\infty$ for $k \in S$, as $m_k \geq K n_k$; we can also suppose that for $k \in S$, $b_{k-1} = a_{k-1} + 1$. Note that for a large enough $k \in S$, the proportion of the word $A_{k-1}$ inside each of the three $k$-words is approximately the same, and hence close to $\mu(A_{k-1})$, the same is true for $B_{k-1}$ and $C_{k-1}$; hence, for a large enough $k \in S$, each level of a $k'$-stack, $k' \leq k - 1$ is $\epsilon$-independent of each $k''$-stack, $k'' \geq k$.

Suppose we have found some $s$ and some $0 < t \leq 1$ such that $\mu(T^s A \Delta A) < t\mu(A)$, and fix $\epsilon$. For $k \in S$ large enough, $E$ is $\epsilon$-close to a set $E_1 \cup E_2$, where $E_1$ is a union of levels of $\tau A_{k-1}$

and $E_2$ is a union of levels of $\tau B_{k-1}$ (the measure of $\tau C_{k-1}$, being at most $\frac{2}{n_k+m_k-1}$, tends to $0$ on $S$). By going to a larger $k \in S$ if necessary, we can ensure that for $i = 1$, $2$, $\mu(E_i)$ is close to $K_i\mu(E)$, where $K_1 = \mu(\tau A_{k-1})$ is bounded away from $0$ and $1$ and $K_2 = 1 - K_1$. Similarly, we can ensure that $T^sE \cap E$ is cut into $E_1'$ and $E_2'$, unions of levels of the two stacks, with $\mu(E_i')$ close to $K_i\mu(T^sE \cap E)$; and, if $s$ is snall compared with $a_{k-1}$, $E_1'$ is $\epsilon$-close to $(T^sE_1 \cap E_1)$ and $E_2'$ to $(T^sE_2 \cap E_2)$. If $k$ is large enough, $s$ is also small compared with $n_k$ and $m_k$.

Let now $s' = sa_{k-1} + s = sb_{k-1}$; by $T^{s'}$, each level of $E_1$, except the ones situated at a height in $\tau A_{k-1}$ bigger than $a_{k-1} - s$, and except for a proportion of each level of at most $\frac{s}{n_k}$, is sent into a level of $\tau A_{k-1}$ situated $s$ levels above; by $T^{s'}$, each level of $E_2$, except for a proportion of each level of at most $\frac{s}{m_k}$, is sent into the same level of $\tau B_{k-1}$. Hence

$$\mu(T^{s'}E \cap E) \geq \mu(T^sE_1 \cap E_1) + \mu(E_2) - K\epsilon \geq K_1(1-t)\mu(E) + K_2\mu(E) - K\epsilon;$$

and $\mu(T^{s'}E\Delta E) \leq K_1t - K\epsilon \leq Kt$ for a constant $K < 1$. So we get the required relation with a rigidity sequence of the form $b_{k_1-1}\dots b_{k_n-1}$, for some subsequence $k_i$ of $S$.

*Case 3*: there are unbounded strings of $(n_k = 1, m_k = 1, \epsilon_{k+1} = +1)$.
Then for for $k$ in a sequence $S$ there exists $p(k) \to +\infty$ such that $A_{k+p} = C_k^pA$, $B_{k+p} = C_k^pB$, $C_{k+p} = C_k$. And as in Case 1, the rigidity holds with the rigidity sequence $c_{p(k)}$, $k \in S$.

*Case 4*: the strings of $(n_k = 1, m_k = 1, \epsilon_{k+1} = +1)$ are bounded but there are unbounded strings of $(m_k = 1, \epsilon_{k+1} = +1)$ or of $(n_k = 1, \epsilon_{k+1} = +1)$.
We take a long string of $(m_k = 1, \epsilon_{k+1} = +1)$; thus $A_k = A_{k-1}^{n_k-1}C_{k-1}A_{k-1}$, $C_k = A_{k-1}^{n_k-1}C_{k-1}$, and $B_k$ has small measure as long as we are far from the (upper) end of the string. We do not change anything (and in particular the measure of the stacks) if we replace these rules by $A_k = A_{k-1}^{n_k}C_{k-1}$, $C_k = A_{k-1}^{n_k-1}C_{k-1}$. As in Case 2, we take a set $E$ and a $k_0$ large enough so that $E$ is almost made of levels of $\tau A_{k_0-1}$ and of $\tau C_{k_0-1}$ and we suppose $k_0$ is the beginning of a string of $(m_k = 1, \epsilon_{k+1} = +1)$.

By $T^{a_{k_0-1}}E$, every level of $E$ is sent on a level of $E$, except those which are in $C_{k_0-1}$ and the part of those in $A_{k_0-1}$ corresponding to the last $A_{k_0-1}$ in a string of $A_{k_0-1}^{n_{k_0}-1}$ or $A_{k_0-1}^{n_{k_0}}$. By $T^{a_{k_0}}E$, every level of $E$ is sent on a level of $E$, except those corresponding to the last $A_{k_0-1}$ and the $C_{k_0-1}$ in a string of $A_{k_0-1}^{n_{k_0}}$ followed by a string of $A_{k_0-1}^{n_{k_0}-1}$. By $T^{a_{k_0+1}}E$, every level of $E$ is sent on a level of $E$, except those corresponding to the last $A_{k_0-1}$ and the $C_{k_0-1}$, inside a string of $A_{k_0-1}^{n_{k_0}}$ followed by a string of $A_{k_0-1}^{n_{k_0}-1}$, which are themselves inside a string of $A_{k_0}^{n_{k_0+1}}$ followed by a string of $A_{k_0}^{n_{k_0+1}-1}$. By continuing, we see that $\mu(T^{a_{k_0+l}}E\Delta E)$ is small whenever we see between $k_0$ and $k_0 + l$ a large enough number of $n_i \geq 2$; but, as we are not in Case 3, this happens for infinitely many $k_0$ and $l$ big enough. QED

**Corollary 5.2.** *A three-interval exchange transformation satisfying the i.d.o.c. condition either has minimal self-joinings or is rigid. In particular, its centralizer is either trivial or uncountable.*

**Examples**
From Section 2 and 4 we get that del Junco's examples are rigid whenever $\gamma$ has unbounded partial quotients, which was conjectured but not proved in [dJ].

There exist systems which do not satisfy the dichotomy in the above corollary; for example, for the shift $T$ associated to the usual *Morse sequence*, it is proved in [L] that the centralizer is generated by the powers $T^n$ and the *flip* map $\phi$, hence is countable but not trivial. Hence we get a new, completely different, proof of a result of [B-C-F]:

**Corollary 5.3.** *A three-interval exchange transformation satisfying the i.d.o.c. condition cannot be measure-thoretically isomorphic with the shift associated to the usual Morse sequence.*

## 6. SIMPLICITY

**Definition 6.1.** *An ergodic system $(X,T,\mu)$ is* simple of order two *if any ergodic self-joining of order two $\nu$ is either the product measure $\mu\times\mu$ or a measure defined by $\nu(A\times B)=\mu(A\cap S^{-1}B)$ for some measurable transformation $S$ commuting with $T$.*

Again, we shall here use "simplicity" for "simplicity of order two".

**Theorem 6.1.** *There exist uncountably many weakly mixing, simple, rigid three-interval exchange transformations.*

**Proof**
Build a three-interval exchange transformation such that, for every $k$, $n_k$ is at least $2^k m_k$, and $a_k=(m_k+n_k)a_{k-1}+1$. There are uncountably many ways to build such a system: for example, choose $\epsilon_{k+1}=+1$ for all $k$, $b_0=a_0+1$ (by selecting a suitable triangle $H^{-1}D$ for the parameters), choose for all $k$ $m_k=a_{k-2}+2$ and some $n_k>2^k m_k$; the required value of $a_k$ follows then from the facts that $b_{k-1}=a_{k-1}+1$ and $c_{k-1}=a_{k-1}-a_{k-2}$.

Thus the stacks $\tau B_k$ and $\tau C_k$ have measure at most $2^{-k}$. The system $(X,T,\mu)$ is then of *rank one* as the sequence of stacks $\tau A_k$ generate the whole space, see for example [F] for precise definitions. The formula

$$A_k=A_{k-1}^{n_k-1}C_{k-1}B_{k-1}^{m_k-1}A_{k-1},$$

together with those giving $B_k$ and $C_k$, defines the system up to measure-theoretic isomorphism.

We define the rank one system $(X',T',\mu')$ by the formula

$$A'_k=(A'_{k-1})^{n_k-1}s^{c_{k-1}+b_{k-1}(m_k-1)}A'_{k-1},$$

starting from $A'_0=A_0$. We build a measure-theoretic isomorphic between $(X,T,\mu)$ and $(X',T',\mu')$, by sending the $j$-th level of $\tau A_k$ to the $j$-th level of the $k$-th stack $\tau A'_k$ for $T'$: it is consistent by construction, and is defined almost everywhere because the total proportion of letters $s$ ("spacers") in $A'_k$, and the total porportion of words $B_{k-1}$ and $C_{k-1}$ in $A_k$ are general terms of a convergent series.

In the same way, the rank one system $(X'',T,\mu'')$ defined by the formula

$$A''_k=(A''_{k-1})^{n_k}s(A''_{k-1})^{m_k}$$

is measure-theoretically isomorphic to $(X,T,\mu)$ and $(X',T',\mu')$; and this last system is weakly-mixing, rigid and simple exactly in the same way as del Junco - Rudolph's map in [dJ-R1] (which has the same definition, but with $n_k$ and $m_k$ both replaced by $2^k$). QED

Note that there are many other possible families of parameters giving weakly mixing, simple and rigid three-interval exchange transformations: all what is needed to reproduce the above reasoning is that $n_k$ is much bigger than $m_k$, and that $c_{k-1}+b_{k-1}(m_k-1)=p_k a_{k-1}+1$ for an integer $p_k$. By modifying slightly the reasoning of [dJ-R1], we can get the same result while weakening this condition to $c_{k-1}+b_{k-1}(m_k-1)=p_ka_{k-1}+q_k$, where $q_k$ is a bounded integer, provided that a condition on the $a_k=(n_k+p_k)a_{k-1}+q_k$ ensures weak mixing, for example $q_k-q_{k-1}=\pm 1$ or $q_k-2q_{k-1}+q_{k-2}=\pm 1$ for all $k$. Similar formulas ensure the same results if we choose $m_k$ much bigger than $n_k$.

However, we have only a small class of simple rigid maps to show, and thus we are quite far from a positive answer to Veech's question; the structure of maps in our class is always similar to the one of del Junco-Rudolph's map. As for del Junco-Rudolph's map itself, we do not know whether it is actually measure-theoretically isomorphic to a three-interval exchange transformation, as the $2^k$ in the defining formula do not grow fast enough to allow us to prove it.

## 7. Ubiquity of three-interval exchange transformations

The fact that a version of the well-known del Junco-Rudolph's map is indeed a three-interval exchange transformation was completely unexpected, and even unsuspected. Another abstract construction which is apparently not connected with interval exchange transformations is *Katok's rank one transformation*, described in [G]): this is in fact a family of rank one systems, each one being described by the recursion formula

$$W_k=(W_{k-1}s)^{p_k}(W_{k-1})^{p_k}$$

for a given sequence $(p_k)$ going to infinity fast enough.

The same reasoning as in Case 2 of Theorem 5.1 above proves the following result, which was noticed some time ago by one of the authors (in answer to a question of M. Lemańczyk) but did not appear in print:

**Proposition 7.1.** *For any sequence $(p_k)$ going to infinity, the corresponding Katok's map is rigid.*

Indeed, the abstract structure of Katok's map seems to be widely spread among rigid three-interval exchange transformations; in particular

**Proposition 7.2.** *If $(p_k)$ is a sequence of integers growing fast enough, the corresponding Katok's map is measure-theoretically isomorphic to a three-interval exchange transformation.*

**Proof**

Let $W_0$ be the initial word of the rank-one map $T'$, and $h_k$ the length of $W_k$, and suppose

$$\sum_{k=1}^{+\infty}\frac{h_{k-2}}{p_k}<+\infty.$$

We build a three-interval exchange transformation $T$ defined by a word $B_0$ of length $h_0$ and word $A_0$ of length $h_0+1$ (this is possible by selecting a particular triangle $H^{-1}D$ for the parameters), and then inductively by $\epsilon_{k+1}=+1$, $m_k=p_k-1-b_{k-1}+c_{k-1}$, $n_k=p_k+1+b_{k-1}-c_{k-1}$

Thus we have $a_k=h_k+1$, $b_k=h_k$ for all $k$, and $b_{k-1}-c_{k-1}=h_{k-2}$ for $k\geq 2$.

We modify $T$ into $T_1$ by changing all the letters of each $C_k$ and the last letter of each $A_k$ into spacers $s$ (remember that $A_k$ deprived of its last letter is just $B_k$). $T_1$ is measure-theoretically isomorphic to $T$ in the same way as in the proof of Theorem 6.1 (we have just changed a small part of $A$ to a spacer), and is a rank-one map given by the formulas

$$B'_k=(B'_{k-1}s)^{n_k-1}s^{c_{k-1}}(B'_{k-1})^{m_k}.$$

And now, still with the same method of proof, both $T'$ and $T_1$ are measure-theoretically isomorphic to the transformation $T_2$ defined by

$$B''_k = (B''_{k-1}s)^{p_k} s^{c_{k-1}+h_{k-2}(1+b_{k-1})} (B''_{k-1})^{m_k}.$$

QED

The first interest of Katok's maps resides in the particular properties, shown by Katok and written in [G], of a specific self-joining, the Cartesian square $(T \times T, \mu \times \mu)$; among all transformations, this was the first known Cartesian square to be Loosely Bernoulli (other ones are built in [G]), and, up to this date, the only known Cartesian square to have a spectrum of finite multiplicity (namely, at most four), both these phenomena occurring whenever $p_k$ is large compared to $h_{k-1}$, thus, in view of the previous result, for maps which are indeed three-interval exchange transformations. These properties both come from the stronger notion of *local rank one*, which is used unknowingly in [G], and for which we refer the reader to [F]. And in fact these properties are shared by many other three-interval exchange transformations:

**Proposition 7.3.** *If a three-interval exchange transformation $T$ is such that, on an infinite set of $k$, $\frac{m_k+n_k}{a_{k-1}} \to +\infty$ and $\frac{1}{K} < \frac{m_k}{n_k} < K$, then its Cartesian square $(T \times T, \mu \times \mu)$ has local rank one, hence is Loosely Bernoulli, and has a spectrum with multiplicity at most $\lfloor 2 + K + \frac{1}{K} \rfloor$.*

**Proof**

Then we can reproduce the proof in [G]: for a $k$ large enough, we can consider the stack for $T \times T$, of basis $FA_{k-1} \times FB_{k-1}$ and of height $a_{k-1}b_{k-1}$; it can be used to approximate every partition on a proportion $\rho$ of the space close to $\frac{m_k n_k}{(m_k+n_k)^2}$. This implies local rank one, the Loose Bernoulli property, and spectral multiplicity bounded by $[\frac{1}{\rho}]$. QED

**Question 7.1.** *Is Katok's map simple?*

This question seems difficult, but an answer (positive or negative) would certainly be a substantive step towards Veech's question.

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