EIGENVALUES AND SIMPLICITY OF INTERVAL EXCHANGE TRANSFORMATIONS

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ABSTRACT. For a class of d-interval exchange transformations, which we call the symmetric class, we define a new self-dual induction process in which the system is successively induced on a union of sub-intervals. This algorithm gives rise to an underlying graph structure which reflects the dynamical behavior of the system, through the Rokhlin towers of the induced maps. We apply it to build a wide assortment of explicit examples on four intervals having different dynamical properties: these include the first nontrivial examples with eigenvalues (rational or irrational), the first ever example of an exchange on more than three intervals satisfying Veech's simplicity (though this weakening of the notion of minimal self-joinings was designed in 1982 to be satisfied by interval exchange transformations), and an unexpected example which is non uniquely ergodic, weakly mixing for one invariant ergodic measure but has rational eigenvalues for the other invariant ergodic measure.

1. PRELIMINARIES

Interval exchange transformations have been introduced by Oseledec [32], following an idea of Arnold [1]; an exchange of d intervals is defined by a probability vector of d lengths and a permutation on d letters; the unit interval is then partitioned according to the vector of lengths, and T exchanges the intervals according to the permutation, see Sections 1.1 and 1.2 below for all definitions. Katok and Stepin [24] used these transformations to exhibit a class of systems with simple continuous spectrum. Then Keane [25] defined a condition called i.d.o.c. ensuring minimality, and was the first to use the idea of induction, which was later formalized by Rauzy [34], as a generalization of the continued fraction algorithm. These tools formed the basis for future studies of various ergodic and spectral properties for these dynamical systems. For general properties of interval exchange transformations, the reader can consult the courses of Viana [41] and Yoccoz [42] [43].

In this paper we study d-interval exchange transformations T, defined by a vector $(\alpha_1, \dots \alpha_d)$ of lengths and the *symmetric* permutation $\pi i = d+1-i$, $1 \leq i \leq d$; we call $\mathcal{I}$ the set of $(\lambda_1, \dots, \lambda_d)$ in $\mathbb{R}^{+d}$ for which T, defined by the vector $(\frac{\lambda_1}{\lambda_1+\dots\lambda_d}, \dots, \frac{\lambda_d}{\lambda_1+\dots\lambda_d})$, satisfies the i.d.o.c. condition; henceforth we shall consider only transformations satisfying this condition: let $\mathcal{U}$, resp. $\mathcal{M}'$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{S}$ be the subset of $\mathcal{I}$ for which T is uniquely ergodic, resp. topologically weakly mixing, resp. weakly mixing for at least one invariant measure, resp. not weakly mixing for at least one invariant measure, resp. simple for at least one invariant measure. A great part of the history of this area is made by the difficult results about these sets. After Keane proved $m(\mathbb{R}^{d+} \setminus \mathcal{I}) = 0$ for the Lebesgue measure m on $\mathbb{R}^d$ and the surprising result that (for $d = 4$) $\mathcal{U}^c$ (for $X \in \{\mathcal{U}, \mathcal{M}', \mathcal{M}, \mathcal{N}, \mathcal{S}\}$ we call X^c its complement in $\mathcal{I}$) is not empty [26], he conjectured that $m(\mathcal{U}^c) = 0$. This conjecture was proved by Masur [29] and Veech [39], see also Boshernitzan [6] for a combinatorial proof closer to the spirit of the present paper. Then Veech [40] proved that $m(\mathcal{M}^c) = 0$ for some

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permutations, not including the symmetric one for $d = 4$; it took quite a long time to have, for all permutations outside the rotation class, first $m(\mathcal{M}'^c) = 0$ (Nogueira-Rudolph [30]), then at last $m(\mathcal{M}^c) = m(\mathcal{N}) = 0$ (Avila-Forni [4]); whether $m(\mathcal{S}^c) = 0$ is still an open question asked by Veech [38]; note that the result on weak mixing in [4] is valid both for one invariant measure and all invariant measures because $m(\mathcal{U}^c) = 0$.

While all these extremely powerful articles establish generic results for general interval exchange transformations, here we aim to provide a detailed analysis of the dynamical behaviour of specific families of interval exchanges; more precisely, we want to address problems concerning relations between the sets defined above, nothing of which was known until recently for $d > 3$, except obvious relations as $\mathcal{M}' \subset \mathcal{M}$, $\mathcal{U} \cap \mathcal{N} \cap \mathcal{M} = \emptyset$ and $(\mathcal{U} \cap \mathcal{N}) \cup (\mathcal{U} \cap \mathcal{M}) = \mathcal{U}$. It was not known whether $\mathcal{N}$ is nonempty or even that $\mathcal{S}$, which is likely to have full measure (indeed, the whole notion of simplicity has been devised for that, and Veech's question has been much investigated), is nonempty; we can also ask about the non-emptiness of some intersections such as $\mathcal{U}^c \cap \mathcal{M}$ or (more difficult as these are two small sets) $\mathcal{U}^c \cap \mathcal{N}$. Another problem is to find explicit examples (in the sense that maybe the vector of lengths is not given, but it can be computed by an explicit algorithm), and not only existence theorems; very few of them were known: for $d = 4$, explicit elements of $\mathcal{U}^c$ are given by Keane [26] while explicit elements of $\mathcal{U}$ can be deduced from the same paper, or built from substitutions, or pseudo-Anosov maps, by a classical construction; but none were known in other sets, even in the bigger ones, until, during the preparation of the present paper, Sinai and Ulcigrai [35] found explicit elements of $\mathcal{M}$, while Yoccoz [42] built explicit elements of $\mathcal{U}^c$ for every k; other related results [22][8] were derived after preliminary versions of the present paper were circulated, see the discussion in Section 6 below.

Similar questions have been addressed for the (by unanimous consent much easier) case $d = 3$, by Veech [36], del Junco [12], and the present authors plus Holton [15][16][17][18]; the methods of these papers have had to be considerably upgraded to tackle the next case, $d = 4$. Thus we have introduced a new notion of induction, beside the classical ones due to Rauzy [34], Zorich [44], and more recently Yoccoz ([28] where a good survey of all these notions can also be found). This *self-dual* induction, studied in more details in [21], is a variant of the less well-known induction of da Rocha [27] [11], and for $d = 3$ its measure-theoretic properties and self-duality are studied in [20]. We present it in Section 2 below, and use it in Sections 3 and 4 to build families of explicit examples of four-interval exchanges; each example is described by four families of Rokhlin towers, depending on partial quotients of our induction algorithm. After a good choice of these partial quotients, our transformation will have the required properties through a measure-theoretic isomorphism with a rank one system. Whether and why this new induction was necessary to answer the questions we addressed will be discussed at the end of Section 6 below.

What we obtain in the end is some groups of examples for $d = 4$: two in $\mathcal{U} \cap \mathcal{M}' \cap \mathcal{M}^c$, one having rational eigenvalues and the other being measure-theoretically isomorphic to an irrational rotation, one in $\mathcal{U} \cap \mathcal{M}' \cap \mathcal{M} \cap \mathcal{S}$, and one in $\mathcal{U}^c \cap \mathcal{M}' \cap \mathcal{M} \cap \mathcal{N}$. We find also elements of $\mathcal{U} \cap \mathcal{M}$ which are measure-theoretically isomorphic to some of the so-called Arnoux-Rauzy systems. All the examples we produce come from expansions having (very) unbounded partial quotients in our induction algorithm. That makes our elements of $\mathcal{M}$ a priori different from Sinai-Ulcigrai's ones, these being obtained from periodic examples relative to a different induction algorithm; in particular, our examples are all rigid, and completely new; their existence was not unexpected, but the existence of an example with irrational eigenvalues for the simpler case $d = 3$ was the object of a question of Veech (1984) which was solved only in [17] (2004); our examples prove also that

Avila-Forni's result is strictly stronger than Nogueira-Rudolph's. The first example of an exchange on more than three intervals which is simple is not surprising, but this resisted the efforts of specialists during 25 years, and constitutes a first step towards Veech's open question. As for our last example, which is weakly mixing for one of the two invariant ergodic measure but has rational eigenvalues for the other, it came as a surprise even for the authors.

For generalizations (to other permutations and values of d), see Section 6 below.

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1.1. **The main definitions.**

Definition 1.1. *A* symmetric d-interval exchange transformation *is a d-interval exchange transformation T with probability vector $(\alpha_1, ..., \alpha_d)$, and permutation $\pi i = d+1-i$, $1 \leq i \leq d$ defined by*

$$Tx = x + \sum_{\pi^{-1}j<\pi^{-1}i} \alpha_j - \sum_{j<i} \alpha_j.$$

when x is in the interval

$$\Delta_i = \left[\sum_{j<i} \alpha_j, \sum_{j\leq i} \alpha_j\right[.$$

We denote by β_i, $1 \leq i \leq d-1$, the i-th discontinuity of T^{-1}, namely $\beta_i = \sum_{j=d+1-i}^{d} \alpha_j$, while γ_i is the i-th discontinuity of T, namely $\gamma_i = \sum_{j=1}^{i} \alpha_j = 1-\beta_{d-j}$. Then $\Delta_1 = [0, \gamma_1[$, $\Delta_i = [\gamma_{i-1}, \gamma_i[$, $2 \leq i \leq d-1$ and $\Delta_d = [\gamma_{d-1}, 1[$.

Definition 1.2. *T satisfies the* infinite distinct orbit condition *(or i.d.o.c. for short) of Keane* [25] *if the $d-1$ negative trajectories $\{T^{-n}(\gamma_i)\}_{n\geq 0}$, $1 \leq i \leq d-1$ of the discontinuities of T are infinite disjoint sets.*

The i.d.o.c. condition for T is (strictly) weaker than the *total irrationality* condition on the lengths, where the only rational relation between α_i, $1 \leq i \leq d$, is $\sum_{i=1}^{d} \alpha_i = 1$. As here π is primitive, the i.d.o.c. condition implies that T is *minimal* (every orbit is dense) [25].

1.2. **A few notions from ergodic theory.** A general reference for this section is [10].

Definition 1.3. *A system (X, T) is* uniquely ergodic *if it admits only one invariant probability measure.*

Definition 1.4. *Let (X, T, μ) be a finite measure-preserving dynamical system.*

A real number $0 \leq \gamma < 1$ is an eigenvalue *of T (denoted additively) if there exists a non-constant f in $\mathcal{L}^2(X, \mathbb{R}/\mathbb{Z})$ such that $f \circ T = f + \gamma$ in $\mathcal{L}^2(X, \mathbb{R}/\mathbb{Z})$; f is then an* eigenfunction *for the eigenvalue γ. As, following* [10]*, we consider only non-constant eigenfunctions, $\gamma = 0$ is not an eigenvalue if T is ergodic. T is* weakly mixing *if it has no eigenvalue.*

Definition 1.5. *(X, T) is* topologically weakly mixing *if it has no continuous (non-constant) eigenfunction.*

In the particular case of interval-exchange transformations, the topology we use here is the standard one (induced by the Lebesgue measure) on the interval $[0,1[$ (though T itself is not continuous), but the proofs in the present paper work in the same way if we view T as the shift on the symbolic trajectories, equipped with the product topology on $\{1,...,d\}^{\mathbb{N}}$; the two topologies are not equivalent, and it does not seem to be known whether a continuous eigenfunction for one has to be continuous for the other.

Definition 1.6. *(X,T,μ) is* rigid *if there exists a sequence $s_n \to \infty$ such that for any measurable set A $\mu(T^{s_n}A\Delta A) \to 0$.*

Definition 1.7. *In (X,T), a (Rokhlin)* tower *of* base *F is a collection of disjoint measurable sets called* levels *$F, TF, \dots, T^{h-1}F$. If X is equipped with a partition P such that each level T^rF is contained in one atom $P_{w(r)}$, the* name *of the tower is the word $w(0)\dots w(h-1)$.*

We shall use also the notion of *rank one*, for various definitions see [9] [14] [31]. Here we need only the definition of a particular class of rank one systems; they come equipped with a partition and an invariant measure; we use the same notation for a tower and its name, and s (for "spacers") is the name of one atom of the partition, corresponding to levels added after the initial stage:

Definition 1.8. *Let x_k and y_k be two sequences positive integers, and let the concatenation of two strings of letters v and w be denoted multiplicatively by vw, while v^k is a concatenation of k times the string v.*

The rank one system defined by the word H_0 and the towers $H_{k+1} = s^{y_{k+1}}H_k^{x_{k+1}}s^{z_{k+1}}$*, where, if h_0 is the length of H_0 and $h_{k+1} = x_{k+1}h_k + y_{k+1} + z_{k+1}$ the length of H_{k+1}, we have $\sum_{k=1}^{+\infty} \frac{y_{k+1}+z_{k+1}}{x_{k+1}h_k} < +\infty$, is the system (X,T,μ) built by* cutting and stacking *in the following way: we start from a set E of measure ξ, which is cut into H_0 equal parts to make the first tower. To get the $j+1$-tower, we cut the j-tower into x_{j+1} columns, stack these columns by putting the x_{j+1}-th above the $x_{j+1}-1$-th ... above the first, and add z_{j+1}* spacer *levels (that is, pieces of E^c with equal measure) one above the other above the top, and y_{j+1} spacer levels one above the other under the bottom. T is the transformation that sends each point in a tower, except those in the top level, to the point just above.*

The number ξ and the common measure ρ_j of the spacer levels in the j-tower are defined uniquely so that μ is a probability preserved by T, and X is partitioned so that H_0 is the name of the 0-tower, while E^c is the atom named s.

A standard argument proves that

Proposition 1.1. *The rank one systems defined above are rigid.*

The following necessary condition for any θ to be an eigenvalue of a rank one transformation was originally deduced (in [17]) from a condition of Choksi and Nadkarni [9]; we give it here with a new direct proof adapted from [7]:

Proposition 1.2. *If θ is an eigenvalue for the rank one system defined above by the word H_0 and the towers $H_{k+1} = s^{y_{k+1}}H_k^{x_{k+1}}s^{z_{k+1}}$, then $x_{k+1}||h_k\theta|| \to 0$ when $k \to +\infty$, where $||\quad||$ denotes the distance to the nearest integer.*

Proof

Let f be an eigenfunction for the eigenvalue θ; the σ-algebra generated by the levels of the k-tower converges to the full σ-algebra when k tends to infinity, thus for each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $k > N(\varepsilon)$ there exists f_k, which satisfies $\int ||f - f_k||d\mu < \varepsilon$ and is constant on each level of the k-tower.

Let j be any integer with $0 \leq j \leq \left[\frac{x_k+1}{2}\right]$. Let τ_k be the union of the levels of the k-tower between the $y_k + 1$-th and $y_k + [\frac{x_k}{2}]h_{k-1}$-th levels; by construction, for any point ω in τ_k, $T^{jh_{k-1}}\omega$ is in the same level of the $k-1$-tower as ω. Thus for μ-almost every ω in τ_k, $f_k(T^{jh_{k-1}}\omega) = f_k(\omega)$ while $f(T^{jh_{k-1}}\omega) = \theta j h_{k-1} + f(\omega)$; we have

$$\int_{\tau_k} ||f_k \circ T^{jh_{k-1}} - j\theta h_{k-1} - f_k||d\mu = \int_{\tau_k} ||j\theta h_{k-1}||d\mu = ||j\theta h_{k-1}||\mu(\tau_k)$$

and

$$\int_{\tau_k} ||f_k \circ T^{jh_{k-1}} - j\theta h_{k-1} - f_k||d\mu \leq \int_{\tau_k} ||f_k \circ T^{jh_{k-1}} - f \circ T^{jh_{k-1}}||d\mu + \int_{\tau_k} ||f_k - f||d\mu < 2\varepsilon.$$

As $\mu(\tau_k) \geq \frac{1}{3}$ for k large enough, the above estimates imply $||j\theta h_{k-1}|| < 6\varepsilon$, for any integer $0 \leq j \leq \left[\frac{x_k+1}{2}\right]$. Thus $||j\theta h_{k-1}|| < 12\varepsilon$ for any integer $0 \leq j \leq x_k$.

Let $\varepsilon < \frac{1}{40}$, and suppose $||x_k\theta h_{k-1}|| \neq x_k||\theta h_{k-1}||$: let i be the smallest $0 \leq j \leq x_k$ such that $||j\theta h_{k-1}|| \neq j||\theta h_{k-1}||$, then $i \geq 2$ and $||(i-1)\theta h_{k-1}|| = (i-1)||\theta h_{k-1}||$, thus $i||\theta h_{k-1}|| = (i-1)||\theta h_{k-1}|| + ||\theta h_{k-1}|| = ||(i-1)\theta h_{k-1}|| + ||\theta h_{k-1}|| < 18\varepsilon < \frac{1}{2}$ thus $||i\theta h_{k-1}|| = ||(i||\theta h_{k-1}||)|| = i||\theta h_{k-1}||$, contradiction. Thus we get $x_k||\theta h_{k-1}|| < 12\varepsilon$. □

Definition 1.9. *A* self-joining *(of order two) of a system (X, T, μ) is any measure ν on $X \times X$, invariant under $T \times T$, for which both marginals are μ.*

An ergodic system (X, T, μ) is simple *(of order two) if any ergodic self-joining of order two ν is either the product measure $\mu \times \mu$ or a measure defined by $\nu(A \times B) = \mu(A \cap U^{-1}B)$ for some measurable transformation U commuting with T.*

2. The self-dual induction

In the remainder of this paper (except for one example in Section 2.2), we call transformation T *a symmetric d-interval exchange transformation satisfying the i.d.o.c. condition and the condition of* alternate discontinuities:

$$\beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \ldots \beta_{d-1} < \gamma_{d-1}.$$

The condition of alternate discontinuities avoids introducing a lot of particular cases in the first steps of our induction; the way it can be dispensed with is discussed in Section 6 below.

2.1. Castles and induction: definitions.

Our transformation T is now fixed, on the interval $[0, 1[$. We consider its *induced maps*: an induced map of T on a set Y is the map $y \to T^{r(y)}y$ where, for $y \in Y$, $r(y)$ is the smallest $r \geq 1$ such that $T^r y$ is in Y (when such an r exists, which will be true in all cases considered in this paper).

In classical inductions, Y is generally an interval; here we consider disjoint unions of $d-1$ intervals; and as for any induction, there is a canonical way to build towers; following [11], we say that a union of towers is a *castle* (the Ornstein school used the words *stacks* and *gadgets* instead of towers and castles).

Definition 2.1. *Given $d-1$ disjoint intervals E_i, $1 \leq i \leq d-1$, let S be the induced map of T on $E_1 \cup \ldots E_{d-1}$. The* induction castle *of the E_i is the unique partition of X into* levels $T^r I_{i,t}$, *$1 \leq i \leq d-1$, $1 \leq t \leq k_i$, $0 \leq r \leq h_{i,t} - 1$, where*

- *each interval E_i is partitioned into k_i subintervals $I_{i,t}$, $1 \leq t \leq k_i$,*

- *$SI_{i,t}$ is a subinterval of $E_{j_{i,t}}$, and on $I_{i,t}$ $S = T^{h_{i,t}}$.*

A castle is indeed a union of Rokhlin towers, each tower being made with the levels $T^r I_{i,t}$, $0 \leq r \leq h_{i,t} - 1$. Note that the k_i are finite by compactness, but that in general each of the $d-1$ intervals could be partitioned in many subintervals; only for interval exchange transformations and the type of induction chosen shall we be able to bound these numbers.

We define now a new induction operation, as a way to associate $d-1$ new intervals E'_i to $d-1$ intervals E_i, $1 \leq i \leq d-1$. It was primarily motivated by considerations from word combinatorics, the $d-1$ families of subintervals corresponding to the *bispecial factors of the associated language*, which implies that their endpoints are the points where the orbit of any discontinuity of T comes close to any discontinuity of T^{-1}; this in turn implies an interesting geometric property of the natural extension of our induction, studied in [20] for $d = 3$, which prompted us to call our induction *self-dual*.

The process is discussed and described in full generality in [21]; we give here a self-contained and slightly different description, adapted to our present (mainly ergodic) aims: indeed, the result we use in the present paper is the explicit description of the induction castles, which appears only as a by-product in [21]. Our intervals will be built so that the induction castles have always a nice structure: namely, the intervals at the initial stage are the Δ_i, $1 \leq i \leq d-1$, and, as we shall see in Lemma 2.2 below, their induction castle is *binary*:

Definition 2.2. *A castle is* binary *if for each $1 \leq i \leq d-1$ $k_i = 2$ and there are exactly two $j_{l,t}$, $1 \leq l \leq d-1$, $t = 1, 2$ which are equal to i.*

When a castle is binary, we denote by $E_{i,m}$ and $E_{i,p}$ the left and right subintervals among the two $I_{i,t}$, by $E_{i,-}$ and $E_{i,+}$ the left and right subintervals among the two $SI_{l,t}$ which are in E_i. Also, we denote by $p(i)$, resp. $m(i)$, the j such that E_j contains $SE_{i,p}$, resp. $SE_{i,m}$. Finally, we denote by l_i, resp. r_i, the length of $E_{i,-}$, resp. $E_{i,+}$ for $1 \leq i \leq d-1$.

It seems likely that for all binary castles we have $SE_{i,m} = E_{m(i),+}$ and $SE_{i,p} = E_{p(i),-}$, but we have not been able to find a direct proof using the i.d.o.c. condition. Indeed, we do not know any example of a binary castle other than those built by our induction, or small variants of it (see also Section 2.2 below), and for them the above properties are true by construction, implying that p and m are bijections.

One of our aims is to keep all induction castles binary throughout the process; to achieve that, we use an auxiliary property, which at the initial stage is satisfied with s being the identity:

Definition 2.3. *A binary castle is* symmetric *if it is endowed with a bijection s on $\{1, ..., d-1\}$ such that*

$$s^{-1} = psp = msm = s$$

and that for all i, we have the relations

- $l_{s(i)} + r_{s(i)} = l_i + r_i$
- $l_i = l_{ps(i)}$,
- $r_i = r_{ms(i)}$.

The relations above are studied in depth in [21] where (in contrast with the present paper) they are used as the basic tool to define the induction.

Definition 2.4. *A relation is called* trivial *if it is* $l_{s(i)} + r_{s(i)} = l_i + r_i$ *with* $s(i) = i$*, or* $l_i = l_{ps(i)}$ *with* $ps(i) = i$*, or* $r_i = r_{ms(i)}$ *with* $ms(i) = i$*,* non-trivial *otherwise.*

We may have $s = Id$, the identity; in that case all the relations $l_{s(i)} + r_{s(i)} = l_i + r_i$ are trivial, and the only non-trivial relations are $l_i = l_{p(i)}$ for $p(i) \neq i$ and $r_i = r_{m(i)}$ for $m(i) \neq i$; this is what happens (for $d = 4$ intervals) in the first stage of the example in Section 2.2 just below, where it turns out that there are only two different non-trivial relations. It can also happen that s has a cycle of length two, as in the second stage of the example in Section 2.2; then when $s(i) \neq i$, the non-trivial relation $l_{s(i)} + r_{s(i)} = l_i + r_i$ expresses that the intervals E_i and $E_{s(i)}$ have the same length, but there are also non-trivial relations $l_i = l_{ps(i)}$ or $r_i = r_{ms(i)}$. Indeed in [21] it is proved that in all binary symmetric castles used in the induction, there are exactly $d-2$ different non-trivial relations, and we shall check this for $d = 4$ in Lemma 3.1 below. Note that in a symmetric binary castle p and m are bijections.

Binary symmetric castles are conveniently described by the following object:

Definition 2.5. *The* castle graph *of a binary symmetric castle is the oriented graph* G *whose vertices are the two-letters words* $is(i)$*,* $1 \leq i \leq d-1$*, and for each* i *there is a* positive *edge from* $is(i)$ *to* $p(i)sp(i)$ *and a* negative *edge from* $is(i)$ *to* $m(i)sm(i)$*.*

The induction associates to $d-1$ intervals E_i containing β_i, $1 \leq i \leq d-1$, a new family of intervals E'_i. For a given $1 \leq i \leq d-1$, either $E'_i = E_i$, or $E'_i = E_{i,m}$, or $E'_i = E_{i,p}$, with the notations of Definition 2.2. When E_i is cut, it is cut by the point separating $E_{i,m}$ and $E_{i,p}$, which is indeed the first point $T^{-s}\gamma_j$, $s > 0$, $1 \leq j \leq d-1$, to fall in the interior of E_i, see [21] for details; the choice of $E_{i,m}$ or $E_{i,p}$ is then made to ensure that β_i is in E'_i. The choices of cutting or not cutting E_i are made so that the induction castle of the E'_i remains binary symmetric, this will be the difficult part and this last property is the crucial one for the sequel.

Definition 2.6. *We call* self-dual induction *the following process: suppose* $E_i = [\beta_i - l_i, \beta_i + r_i[$*,* $1 \leq i \leq d-1$ *are* $d-1$ *disjoint subintervals such that their induction castle is binary symmetric and has a castle graph* G *with bijections* p, m, s*, and that for every* i $l_i - r_{s(i)} = l_{s(i)} - r_i \neq 0$*; we define the* instruction ι *by the sign (+ or −) of this last quantity,*

$$\iota i = \iota s(i) = \operatorname{sgn}(l_i - r_{s(i)}) = \operatorname{sgn}(l_{s(i)} - r_i);$$

let C *be the maximal union of same-sign circuits of* G *using only the edges starting from* $is(i)$ *and of sign* ιi*,* $1 \leq i \leq d-1$*; then we define* $d-1$ *new disjoint intervals by*

- *if* $is(i) \in C$ *and* $\iota i = +$*,* $E'_i = E_{i,p}$*,*
- *if* $is(i) \in C$ *and* $\iota i = -$*,* $E'_i = E_{i,m}$*,*
- *if* $is(i) \notin C$*,* $E'_i = E_i$*.*

2.2. **Castles and induction: examples.** It is time now to look at castles and castle graphs in concrete situations. We look first at what happens for $d = 4$ intervals, at the first stage of the induction, see Lemma 2.2 below. To draw the pictures, we assume, together with the condition of alternate discontinuities, that β_1, resp. β_2, β_3, is to the left of $T^{-1}\gamma_3$, resp. $T^{-1}\gamma_2$, $T^{-1}\gamma_1$. Figure 1 shows the induction castle of the intervals $E_1 = \Delta_1 = [0, \gamma_1[$, $E_2 = \Delta_2 = [\gamma_1, \gamma_2[$, $E_3 = \Delta_3 = [\gamma_2, \gamma_3[$; it is made of three towers, which we draw separately because we choose to forget that E_1, E_2 and E_3 are adjacent, as it happens at this stage only; to save space, we denote by $\gamma_i^{(j)}$ the point $T^{-j}\gamma_i$.

The picture shows that the castle is binary, with $E_{1,-} = [0, \beta_1[$, $E_{1,+} = [\beta_1, \gamma_1[$, $E_{1,m} = [0, T^{-1}\gamma_3[$, $E_{1,p} = [T^{-1}\gamma_3, \gamma_1[$, $[\gamma_3, 1[= TE_{1,p}$, $E_{2,-} = [\gamma_1, \beta_2[$, and so on. The labels give the

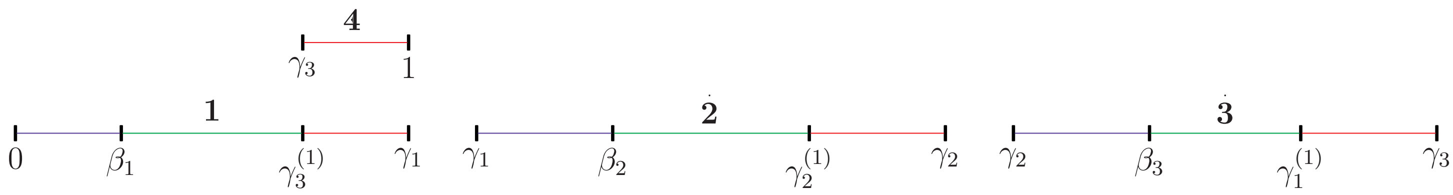

FIGURE 1. First stage of towers $1, 2$ & 3.

names of the towers: on the first tower they indicate that E_1 is a subinterval of Δ_1 and $TE_{1,p}$ is a subinterval of Δ_4, thus when we read them from bottom to top, we get the M_1 and P_1 of Lemma 2.4 below. We see also that $S = T^2$ on $E_{1,p}$, $S = T$ everywhere else.

Some more information we have not yet written is that $SE_{1,m} \subset E_3$ thus $m(1) = 3$; indeed we have $SE_{1,m} = E_{3,+}$, and similarly $SE_{1,p} = E_{1,-}$, $SE_{2,m} = E_{2,+}$, $SE_{2,p} = E_{3,-}$, $SE_{3,m} = E_{1,+}$, $SE_{3,p} = E_{2,-}$, thus $p(1) = 1$, $m(2) = 2$, $p(2) = 3$, $m(3) = 1$, $p(3) = 2$. Moreover we check that the castle is indeed symmetric for $s = Id$: this means checking $p^2 = m^2 = Id$ and the nine relations on lengths in Definition 2.3: five of them are trivial ($l_i + r_i = l_i + r_i$ for $i = 1, 2, 3$, $l_1 = l_1$, $r_2 = r_2$), the non-trivial ones are $r_1 = r_3$ and $l_2 = l_3$, each of them appearing for two values of i.

Thus the information which was not in the picture of the castle is conveniently summarized by the castle graph on the left of Figure 2, which is vertex I of the graph of graphs Γ_4, see Lemma 3.1 below.

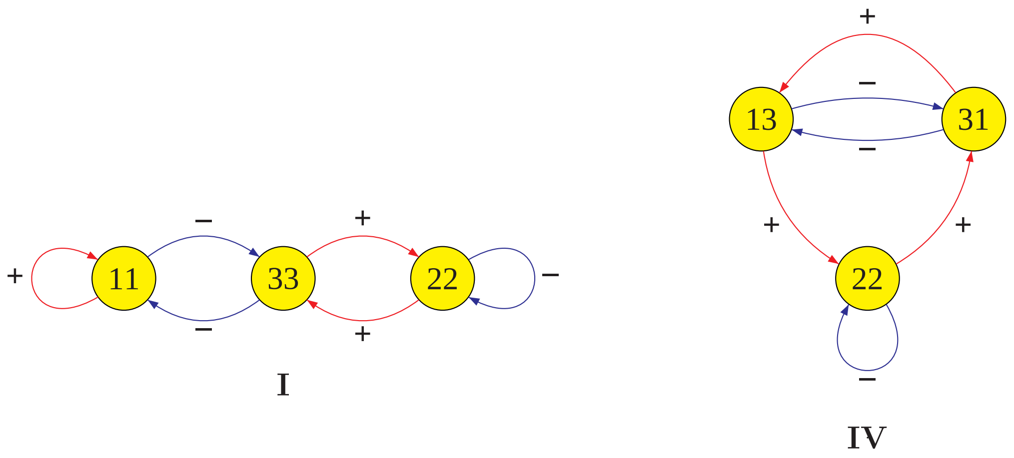

FIGURE 2. The castle graphs at first and second stage.

We look now at what happens at the next stage, assuming the condition of alternate discontinuities, and the respective positions of β_i and $T^{-1}\gamma_j$ from the first stage.

Applying Definition 2.6, we see that at the first stage the instruction is $\iota_1 = \iota_2 = \iota_3 = -$. In the castle graph there is a $-$ circuit with the vertices 11 and 33, and a $-$ loop around the vertex 22, thus $C = \{11, 22, 33\}$, and for $i = 1, 2, 3$ the E_i at second stage is the $E_{i,m}$ of first stage.

Thus we can draw the induction castle of the new E_1, E_2, E_3; to position the points, we make the extra assumption that β_1, resp. β_2, β_3, is to the left of $T^{-2}\gamma_1$, resp. $T^{-2}\gamma_2$, $T^{-2}\gamma_3$.

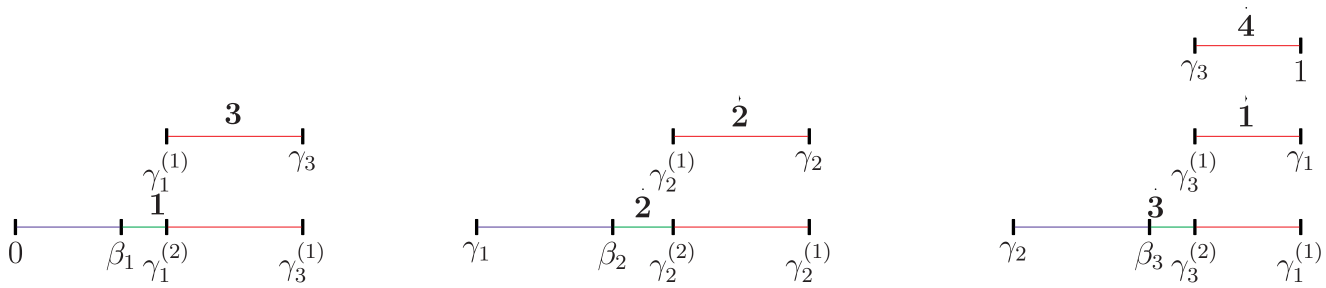

FIGURE 3. Second stage of towers $1, 2$ & 3.

The reader can now decipher this picture as in the previous stage. The extra information is that now $m(1) = 3$, $p(1) = 2$, $m(2) = 2$, $p(2) = 3$, $m(3) = 1$, $p(3) = 1$, and the new castle is symmetric with the involution $s(1) = 3$, $s(2) = 2$, $s(3) = 1$; this involves checking $psp = msm = s$ and the non-trivial relations $r_1 + l_1 = r_3 + l_3$ (the intervals E_1 and E_3 have the same length), $l_2 = l_3$. Thus the new castle graph is shown on the right of Figure 2; it is vertex IV of the graph of graphs Γ_4, see Lemma 3.1 below.

A *non-symmetric* binary castle can be found in Section 5.1 of [21], for a 4-interval exchange with permutation $\pi 1 = 4$, $\pi 2 = 3$, $\pi 3 = 1$, $\pi 4 = 2$. In the initial stage, the castle of E_1, E_2, E_3 is binary with $p(1) = 3$, $m(1) = 2$, $p(2) = 1$, $m(2) = 3$ $p(3) = 2$, $m(2) = 1$, and the relations between the parameters are $l_{i,n} + r_{i,n} = l_{p(i),n} + r_{m(i),n}$, $i = 1, 2, 3$ (these hold also for the symmetric castles considered in the present paper, see the proof of Proposition 2.1 below) but they do not yield the relations of Definition 2.3, and indeed in [21] we choose the parameters so that no relation $l_i = l_j$, $r_i = r_j$ or $l_i + r_i = l_j + r_j$ holds for $i \neq j$. Thus for no choice of s can the relations in Definition 2.3 be satisfied.

With the symmetric permutation $\pi 1 = 4$, $\pi 2 = 3$, $\pi 3 = 2$, $\pi 4 = 1$, we get non-symmetric castles when we induce against the rules of Definition 2.6, for example if at the first stage above we choose the new E_1 to be the full old E_1 instead of the old $E_{1,m}$, but such castles are not binary either.

As one of the referees pointed out, for a binary castle where m and p are bijections, the reciprocal map is also a binary castle, whose combinatorics is given by m^{-1} and p^{-1}, and, for a symmetric binary castle, the reciprocal castle has the same combinatorics up to a permutation of names by an involution. This last condition, however, is in general weaker than the symmetry we define, as it is satisfied by the non-symmetric castle defined above for a non-symmetric permutation π, with the involution $s(1) = 1$, $s(2) = 3$, $s(3) = 2$; what are missing there are the non-trivial relations between the lengths, for example $l_2 + r_2 = l_3 + r_3$. Still, it is quite possible that when the permutation π is the symmetric one every binary castle is symmetric, see the remark after Definition 2.2. We do not know whether a castle can have the same combinatorics as its reciprocal castle up to a permutation of names which is not an involution.

2.3. **Castles and induction: results.** The following proposition describes how the induction works, and gives conditions ensuring that it can be iterated.

Proposition 2.1. *If E_i, $1 \leq i \leq d-1$, is a set of disjoint intervals such that*

(1) $E_i = [\beta_i - l_i, \beta_i + r_i[$, $l_i > 0$, $r_i > 0$,
(2) *all their endpoints are of the form $T^a\gamma_b$ for $a \leq 1$ and $b = 1, 2, 3$,*
(3) *their induction castle is binary symmetric, with bijections p, m, s,*
(4) $SE_{i,p} = E_{p(i),-} = [\beta_{p(i)} - l_{p(i)}, \beta_{p(i)}[$,
(5) $SE_{i,m} = E_{m(i),+} = [\beta_{m(i)}, \beta_{m(i)} + r_{m(i)}[$.

Then we can apply the self-dual induction to the E_i, and the new E'_i satisfy (1) to (5), with new parameters l'_i, r'_i, and bijections p', m', s' given by the following rules

- *if $is(i) \in C$ and $\iota i = +$, $l'_i = l_i - r_{s(i)} = l_{s(i)} - r_i$, $r'_i = r_i$, $s'(i) = sp(i)$, $p'(i) = p(i)$, $m'(i) = mp(i)$,*
- *if $is(i) \in C$ and $\iota i = -$, $l'_i = l_i$, $r'_i = r_i - l_{s(i)} = r_{s(i)} - l_i$, $s'(i) = sm(i)$, $p'(i) = pm(i)$, $m'(i) = m(i)$,*
- *if $is(i) \notin C$, $l'_i = l_i$, $r'_i = r_i$, $s'(i) = s(i)$, $p'(i) = p(i)$, $m'(i) = m(i)$.*

Proof

We know that $E_{i,-} = [\beta_i - l_i, \beta_i[$ and $E_{i,+} = [\beta_i, \beta_i + r_i[$; the symmetry of the castle implies the relations of Definition 2.3. We know also that $E_{i,p}$ is the right subinterval of E_i with the same length as $E_{p(i),-}$, namely

$$E_{i,p} = [\beta_i - l_{p(i)} + r_i, \beta_i + r_i[.$$

Similarly

$$E_{i,m} = [\beta_i - l_i, \beta_i - l_i + r_{m(i)}[.$$

This implies the *train-track equalities* (see [33] for example) $l_i + r_i = l_{p(i)} + r_{m(i)}$, which is another way of stating the above relations (the equivalence of the set of the train-track equalities and the set of relations in Definition 2.3 is shown in [21], it is not used in the present paper).

This implies also that $l_i - r_{s(i)} = l_{s(i)} - r_i \neq 0$, as otherwise β_i would be the left endpoint of $E_{i,p}$, hence its image by S would be the left endpoint of $E_{p(i)}$, which is impossible because of (2) and the i.d.o.c. condition.

Thus we can apply the self-dual induction, with C as in the definition. Let S' be the induced map of T on $E'_1 \cup ...E'_{d-1}$.

If $is(i) \in C$, with $\iota i = +$: we say that E_i has been *cut on the left*; because of the relation $l_{p(i)} = l_{s(i)}$, we have $l_{p(i)} - r_i > 0$, and thus $\beta_i \in E_{i,p} = E'_i$. If $is(i) \in C$ with $\iota i = -$, where E_i is *cut on the right*, we use the relation $r_{m(i)} = r_{s(i)}$ to prove that $\beta_i \in E_{i,m} = E'_i$. If $is(i) \notin C$, E_i is not cut and $\beta_i \in E_i = E'_i$. Thus in each case we can define l'_i and r'_i, they are given by the claimed expression. Moreover the endpoints of the E'_i have the required form.

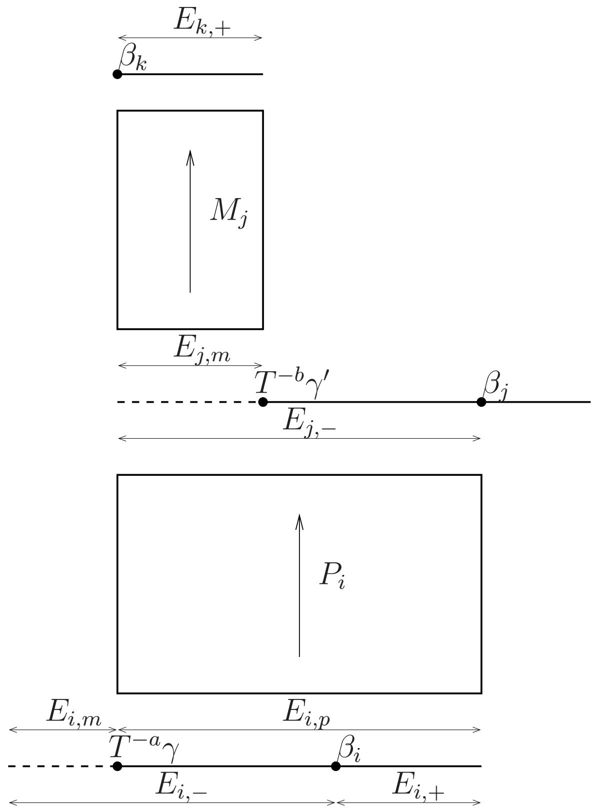

FIGURE 4. Evolution of Rokhlin towers.

We look at the action of S' on E'_i. Suppose $is(i) \in C$, with $\iota i = +$. Then the situation is completely described in Figure 4, where $j = p(i)$ and $k = m(j)$, while the P_i and M_i are the names of the towers, to be discussed in Section 2.5 below.

As $p(i)sp(i)$ is also in C, $E_{p(i),-}$ intersects both $E_{p(i),p}$ and $E_{p(i),m}$. The new $E'_{p(i)}$ is $E_{p(i),p} = [\beta_{p(i)} - l_{pp(i)} + r_{p(i)}, \beta_{p(i)} + r_{p(i)}[$. The image of E'_i by S is the interval $E_{p(i),-}$, which is not included in the new $E'_{p(i)}$: it is made of $X_i = E_{p(i),m}$, the left subinterval of $E_{p(i)}$ which is not in $E'_{p(i)}$, and $Y_i = E_{p(i),-} \cap E_{p(i),p}$.

This creates a partition of E'_i: the right subinterval of E'_i with the same length as Y_i, which we denote by $E'_{i,p}$, is sent by S on $Y_i \subset E'_{p(i)}$, and on this interval $S' = S$. The left subinterval of E'_i with the same length as X_i, which we denote by $E'_{i,m}$, is sent by S on $X_i \subset E_{p(i)} - E'_{p(i)}$; then $X_i = E_{p(i),m}$ is sent by S on $E_{mp(i),+} \subset E_{mp(i)}$. As $p(i)sp(i)$ is on the same positive circuit in C as $is(i)$, it cannot be on a negative circuit in C, hence neither can $mp(i)smp(i)$; hence $mp(i)smp(i)$ is either on a positive circuit in C, or not in C, hence either $E_{mp(i)}$ has been cut on the left or not cut; thus $E_{mp(i),+} \subset E'_{mp(i)}$. Hence on $E'_{i,m}$ we have $S' = S^2$, and S' sends $E'_{i,m}$ onto a subinterval of $E_{mp(i)}$.

Thus, in the new castle, E'_i is indeed partitioned into two subintervals, and we can define $p'(i) = p(i)$ and $m'(i) = mp(i)$. $S'E'_{i,p}$ is the interval Y_i, which is the left subinterval of $E'_{p'(i)}$ of length $l'_{p'(i)}$, hence we can call it $E'_{p'(i),-} = [\beta_{p'(i)} - l'_{p'(i)}, \beta_{p'(i)}[$. And, whether $E_{mp(i)}$ has been cut on the left or not cut, $S'E'_{i,m}$ is the interval $E'_{m'(i),+} = [\beta_{m'(i)}, \beta_{m'(i)} + r'_{m'(i)}[$.

A similar reasoning takes care of the case $is(i) \in C$ with $\iota i = -$, where we can define $p'(i)$, $m'(i)$, $E'_{p'(i),-}$ and $E'_{m'(i),+}$ by the claimed formulas.

If $is(i) \notin C$, E_i is not cut and S sends $E'_{i,p} = E_{i,p}$ on $E_{p(i),-}$, which is still in $E'_{p(i)}$ as $p(i)sp(i)$ cannot be on a positive circuit of C and hence $E_{p(i)}$ has not been cut on the left; similarly S sends $E'_{i,m}$ on $E_{m(i),+} \subset E'_{m(i)}$, thus we define $p'(i)$, $m'(i)$, $E'_{p'(i),-}$ and $E'_{m'(i),+}$ by the claimed formulas.

As p' and m' are bijections, the new induction castle is indeed binary, and our p' and m' are its defining bijections. We define now s' by $s' = sp$ on $C \cap \{\iota = +\}$, $s' = sm$ on $C \cap \{\iota = -\}$, $s' = s$ on C^c. It is then straightforward to check that the castle is symmetric: for example if $is(i) \in C$ and $\iota i = +$, $s(i)i$ is also in C (as $\iota s(i) = \iota i$), hence $ms(i)sms(i)$ cannot be on a negative circuit in C and hence $r'_{ms(i)} = r_{ms(i)}$, while $r'_i = r_i$ and $m's'(i) = mpsp(i) = ms(i)$; hence the relation $r'_i = r'_{m's'(i)}$ is satisfied, and similarly for the other cases and relations.

Thus we have proved (1) to (5) for the E'_i. □

Lemma 2.2. *If we put $E_i = \Delta_i$, $1 \leq i \leq d-1$, they satisfy (1) to (5), and their castle graph G_0 is defined by the bijections $s(i) = i$, $1 \leq i \leq d-1$, $m(i) = d-i$, $1 \leq i \leq d-1$, $p(i) = d+1-i$, $2 \leq i \leq d-1$, $p(1) = 1$.*

Proof

The proof consists of a simple verification, using the relative positions of the β_i and γ_i which are assumed in the condition of alternate discontinuities. □

2.4. **The graph of graphs.** As for the classical inductions, the self-dual induction is represented by paths in a graph; each vertex of this graph is not a permutation as in the case of the Rauzy induction, but a castle graph:

Definition 2.7. *Given a castle graph G with bijections p, m, s, an* instruction *on G is a map from the set of vertices of G to $\{-,+\}^{d-1}$ such that $\iota \circ s = \iota$; the castle graph $J_\iota G$ is the castle graph defined by the bijections p', m', s' described in Proposition 2.1.*

Let G_0 be as in Lemma 2.2, let $\mathcal{G}(G_0)$ be the smallest set of castle graphs which contains G_0 and is stable by the map J_ι for all possible instructions ι. The graph of graphs Γ_d *is the oriented graph whose vertices are the elements of $\mathcal{G}(G_0)$, with an edge labeled by ι from G to $J_\iota(G)$.*

If E_i are intervals satisfying (1) to (5), and their castle graph is a vertex a of the graph of graphs; if we apply the self-dual induction, the castle graph of the intervals E'_i is the vertex b such that from a to b there is an edge labeled by the instruction ι of Proposition 2.1.

Definition 2.8. *Let Γ be an infinite path in the graph of graphs; let $G_n, n \in \mathbb{N}$ be its vertices; for each n, let ι_n be the instruction labeling the edge from G_n to G_{n+1}, let s_n, p_n, m_n be the bijections defining the castle graph G_n, let C_n be the maximal union of same-sign circuits of G_n using only the edges starting from $is_n(i)$ and of sign $\iota_n i$, $1 \leq i \leq 3$.*

Γ *is* admissible *if*

- *G_0 is as in Lemma 2.2,*
- *if $i \notin C_n$, $\iota_{n+1} i = \iota_n i$,*
- *for each i, $\iota_n i = +$ for infinitely many n,*
- *for each i, $\iota_n i = -$ for infinitely many n.*

The following theorem is proved in [21]; the proof uses elaborate combinatorial tools; in the next section we give a simpler proof for $d = 4$, to make the present paper self-contained.

Theorem 2.3. *Every transformation T defines an admissible infinite path in the graph of graphs. Every admissible infinite path in the graph of graphs is the path of at least one transformation T.*

2.5. **Names.** The self-dual induction gives a way to generate any transformation T by $2d-2$ families of Rokhlin towers; when we know the path of T in the graph of graphs, we know how to build these towers recursively, or, equivalently, how to build their names for the partition of $[0,1[$ into Δ_i, $1 \leq i \leq d$.

In the initial castle, and hence in all the castles we consider, each level $T^r E_{i,p}$ is contained in one interval $\Delta_{w(r,i,p)}$, $w(r,i,p) \in \{1,...d-1\}$, and the same holds if we replace p by m. Thus we can define the names of our towers as in Definition 1.7; there are $2d-2$ names, we denote by P_i and M_i the names of the towers of bases $E_{i,p}$ and $E_{i,m}$.

Lemma 2.4. *In the initial castle, $P_1 = 1d$, $M_i = i$, $1 \leq i \leq d-1$, $P_i = i$, $2 \leq i \leq d-1$.*

If we apply the self-dual induction to a castle with names P_i and M_i, the new names P'_i and M'_i are given by

- *if $is(i) \in C$ and $\iota i = +$, $P'_i = P_i$, $M'_i = P_i M_{p(i)}$;*
- *if $is(i) \in C$ and $\iota i = -$, $P'_i = M_i P_{m(i)}$, $M'_i = M_i$;*
- *if $is(i) \notin C$, $P'_i = P_i$, $M'_i = M_i$.*

Proof

The proof can be obtained by following the steps of the proof of Proposition 2.1, adding the names M_i and P_i of the towers as in Figure 4. □

In classical inductions, we generate T by only d families of Rokhlin towers; this is possible also for the self-dual induction, by inducing T further on one of our $d-1$ subintervals; but, as will be seen in Lemma 4.2, this requires the knowledge of the path in the graph of graphs some way beyond the stage we are considering, thus we shall do it only for some particular families of examples; more generally, the reasoning of Lemma 4.2 and the result in its corollary can be repeated for any given infinite path in the graph of graphs.

3. Structure of symmetric 4-interval exchange transformations

Throughout the remainder of this paper, we restrict ourselves to $d = 4$.

Lemma 3.1. *The graph of graphs Γ_4 is the graph whose vertices are*

I $s = (123)$, $p = (132)$, $m = (321)$, *with nontrivial relations* $r_1 = r_3$, $l_2 = l_3$,
II $s = (123)$, $p = (321)$, $m = (213)$, *with* $r_1 = r_2$, $l_1 = l_3$,
III $s = (123)$, $p = (213)$, $m = (132)$, *with* $r_2 = r_3$, $l_2 = l_1$,
IV $s = (321)$, $p = (231)$, $m = (321)$, *with* $l_1 + r_1 = l_3 + r_3$, $l_2 = l_3$,
V $s = (213)$, $p = (231)$, $m = (213)$, *with* $l_1 + r_1 = l_2 + r_2$, $l_1 = l_3$,
VI $s = (132)$, $p = (231)$, $m = (132)$, *with* $l_2 + r_2 = l_3 + r_3$, $l_2 = l_1$,
VII $s = (132)$, $p = (132)$, $m = (312)$, *with* $l_2 + r_2 = l_3 + r_3$, $r_1 = r_2$,
VIII $s = (321)$, $p = (321)$, $m = (312)$, *with* $l_1 + r_1 = l_3 + r_3$, $r_1 = r_3$,
IX $s = (213)$, $p = (213)$, $m = (312)$, *with* $l_1 + r_1 = l_2 + r_2$, $r_2 = r_3$,

and whose edges, labeled by instructions $(\iota 1, \iota 2, \iota 3)$, are the following

from I $(-,+,-)$ *and* $(-,-,-)$ *to* IV, $(-,+,+)$ *and* $(+,+,+)$ *to* VII, $(-,-,+)$, $(+,-,-)$, $(+,-,+)$, $(+,+,-)$ *to* I,

from II $(-,-,+)$ *and* $(-,-,-)$ *to* V, $(+,-,+)$ *and* $(+,+,+)$ *to* $VIII$, $(-,+,-)$, $(-,+,+)$, $(+,-,-)$, $(+,+,-)$ *to* II,
from III $(+,-,-)$ *and* $(-,-,-)$ *to* VI, $(+,+,-)$ *and* $(+,+,+)$ *to* IX, $(-,-,+)$, $(-,+,-)$, $(-,+,+)$, $(+,-,+)$ *to* III,
from IV $(+,+,+)$ *to* V, $(-,+,-)$ *and* $(-,-,-)$ *to* I, $(+,-,+)$ *to* IV,
from V $(+,+,+)$ *to* VI, $(-,-,+)$ *and* $(-,-,-)$ *to* II, $(+,+,-)$ *to* V,
from VI $(+,+,+)$ *to* IV, $(+,-,-)$ *and* $(-,-,-)$ *to* III, $(-,+,+)$ *to* VI,
from VII $(-,-,-)$ *to* IX, $(-,+,+)$ *and* $(+,+,+)$ *to* I, $(+,-,-)$ *to* VII,
from $VIII$ $(-,-,-)$ *to* VII, $(+,-,+)$ *and* $(+,+,+)$ *to* II, $(-,+,-)$ *to* $VIII$,
from IX $(-,-,-)$ *to* $VIII$, $(+,+,-)$ *and* $(+,+,+)$ *to* III, $(-,-,+)$ *to* IX.

Proof
The proof follows from straightforward computations, applying Definition 2.6. In each case, the knowledge of m, p and s allows us to write the set of non-trivial relations of Definition 2.4.

A simplified graph of graphs is shown in Figure 5: we have omitted the edges going from one vertex to itself, an edge $+.+$ denotes two edges, $(+,-,+)$ and $(+,+,+)$, and similarly for other edges labeled with points.

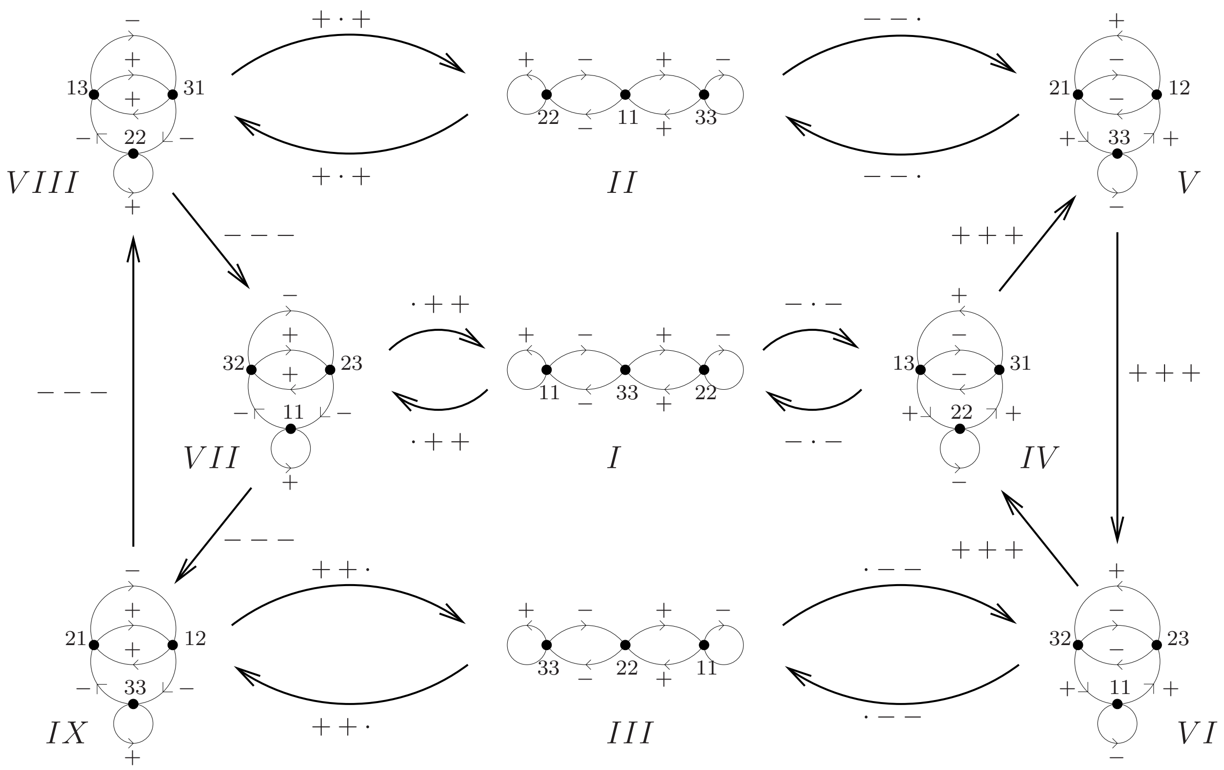

FIGURE 5. The graph of graphs.

We prove now Theorem 2.3 for $d = 4$. Only Proposition 3.4 and Lemma 3.3, restricted to some of the particular cases studied in the proof of the lemma, are necessary for the sequel, but we wish to give the reader the complete recipe to make his own examples.

Proposition 3.2. *Every transformation T defines an admissible infinite path in the graph of graphs.*

Proof
Given T, we start from $E_i = \Delta_i$ and apply the self-dual induction recursively; we get an infinite path in the graph of graphs (though only admissibility will prove that it does not become stationary). The first condition of admissibility (Definition 2.8 above) is satisfied because of Lemma 2.2 and the second one because of Definition 2.6 and Proposition 2.1.

Let $E_{i,n}$ be the interval E_i at stage n; let us prove first that *whenever $E_{i,n}$ is cut for infinitely many n, then it is cut on the right for infinitely many n and cut on the left for infinitely many n*. Indeed, by construction, the left and right endpoints of $E_{i,n}$ are respectively $T^{a(n)}\gamma_{b(n)}$ and $T^{a'(n)}\gamma_{b'(n)}$, for integers $a(n) \leq 1$ and $a'(n) \leq 1$, and there is no point $T^x\gamma_{b(n)}$ or $T^{x'}\gamma_{b'(n)}$ inside $E_{i,n}$ for $a(n) \leq x \leq 1$ and $a'(n) \leq x' \leq 1$. If $E_{i,n}$ is cut infinitely often, $a(n) \to -\infty$ or $a'(n) \to -\infty$, and thus there exists j and $c(n) \to -\infty$ such that $E_{i,n}$ does not contain any $T^x\gamma_j$ for $c(n) \leq x \leq 1$. But this contradicts minimality if $E_{i,n}$ is ultimately not cut to the right (resp. left).

We prove now that each $E_{i,n}$ is indeed cut for infinitely many n; this is done by looking precisely at the possible paths in the graph of graph. There are 27 cases to consider, we look at two of the most significant.

Suppose for some N G_N is vertex I, and let us show that $E_{1,n}$ will be cut at least once for $n \geq N$. If $\iota_N 1 = +$, $1s_N(1) = 11$ is in C_N, because there is a $+$ loop around 11 in the castle graph I, and we are done for $n = N$.

We suppose now that $\iota_N 1 = -$. If $\iota_N 3 = -$, we are done as 11 is in C_N, because of the $-$ circuit $\{11, 33\}$. For all $n \geq N$ if $E_{1,n}$ *is never cut* we have $\iota_n 1 = -$ and we can go only from I to VII, I to I, VII to I, or VII to VII; hence $\iota_n 3 = +$ for all $n \geq N$, as otherwise $1s_n(1)$ is in C_n, either because we are in I with a $-$ circuit $\{11, 33\}$, or because we are in VII with a $-$ circuit $\{11, 23, 32\}$ and $\iota_n 2 = \iota_n 3$ because $s_n(2) = 3$. Then, if there exists N' such that for all $n \geq N'$ $E_{3,n}$ *is not cut*, we have $\iota_n 2 = -$ for $n \geq N'$, otherwise $3s_n(3)$ is in C_n, as both in I and VII there is a $+$ circuit $\{2s(2), 3s(3)\}$. Then for all $n \geq N'$ G_n is vertex I (if it was vertex VII we would have $\iota_n 2 = \iota_n 3$), and $2s_n(2) = 22$ is in C_n because there is a $-$ loop around 22, thus $E_{2,n}$ is cut infinitely often but ultimately only on the right, which as we just proved is impossible. So $E_{3,n}$ has to be cut infinitely often, hence infinitely often on the right, and, for some $n > N$, $\iota_n 3 = -$, contradiction.

Suppose for some N G_N is vertex IV, and let us show that $E_{2,n}$ will be cut at least once for $n \geq N$. As there is a $-$ loop around 22 in the castle graph IV, this implies that $\iota_n 2 = +$ for all $n \geq N$. As there is a $+$ circuit $(13, 22, 31)$ in the castle graph IV, this implies in turn that $\iota_n 1 = \iota_n 3 = -$ for all $n \geq N$ such that we are in IV. As $E_{2,n}$ is never cut, we can only go from IV to I, from I to IV, and from I to I (but not from IV to IV); this implies that we are in I for infinitely many n, and that $\iota_n 3 = -$ also for all $n \geq N$ such that we are in I, because of the $+$ circuit $(22, 33)$ in I. Let N' be an n for which we are in I; if $E_{3,n}$ was never cut for $n \geq N'$, this would mean $\iota_n 1 = +$ for all $n \geq N'$, we would stay always in I, and $E_{1,n}$ would be cut infinitely often (thanks to the loop around 11 in I) but only on the left, and this is impossible. Thus $E_{3,n}$ is cut for some $n \geq N'$, thus for infinitely many n, thus $E_{3,n}$ is cut on the left for infinitely many n, and this contradicts the assumption on $\iota_n 3$.

The same reasoning applies, mutatis mutandis, for any $E_{i,N}$ when G_N is any vertex, and we have proved the last two conditions of admissibility. □

Definition 3.1. *We say that* i takes $+$, resp $-$, at stage n, *if* $\iota_n i = +$, *resp* $-$, *and* $is_n(i) \in C_n$.

We say that ij takes $+$, resp $-$, at stage n *if* i *takes* $+$, *resp* $-$, *at stage* n *and* $s_n(i) = j$.

Lemma 3.3. *Let* $G_0, \dots G_n, \dots$ *be a given admissible path in the graph of graphs. For any* $1 \leq i \leq 3$, $1 \leq j \leq 3$ *and any pair of signs* $e \in \{-,+\}$, $e' \in \{-,+\}$, *there exist a positive integer* t *and a finite sequence* $1 \leq j_s \leq 3$, $0 \leq s \leq t$, *such that*

- $j_0 = i$, $e_0 = e$, $j_t = j$, $(-1)^t e = e'$,
- *for all* $1 \leq s \leq t$, $j_{s-1}j_s$ *takes* $(-1)^s e$ *at infinitely many stages.*

Proof

The result is clearly true for paths where each ij takes $+$ and $-$ at infinitely many stages; admissibility implies each i takes $+$ and $-$ at infinitely many stages, but it is not always true for each ij, and we must prove the lemma individually for paths where each of the 18 possibilities does not occur.

Note that if an admissible path visits all vertices ultimately, to allow the transitions each ij has to take $+$ and $-$ infinitely often, and the lemma is proved. Now we look at admissible paths who do not visit all vertices. An admissible path cannot visit only one vertex ultimately, as, when we go from I to I, 3 cannot take $+$ nor $-$, and similarly for the other vertices. An admissible path cannot visit ultimately only two adjacent vertices: if they are I and IV, 2 and 3 cannot take $+$ ultimately, and all the other possibilities are similar.

We look now at a path which ultimately visits only the vertices I, IV and VII: to allow the transitions 22, 33, 23, 32 take $+$ infinitely often, 11, 33, 13, 31 take $-$ infinitely often and, to ensure admissibility, 11 takes $+$ infinitely often and 22 takes $-$ infinitely often, and this is enough to satisfy the lemma: for example, take $i = 1$, $e = -$; then by putting $j_1 = 1$, $j_2 = 3$ we get the result for $(j, e') = (1, +)$ and $(j, e') = (3, -)$; by putting $j_3 = j_4 = 2$, we get the result for $(j, e') = (2, +)$ and $(j, e') = (2, -)$, while by putting $j_3 = 3$ we get the result for $(j, e') = (3, +)$; and a similar computation works for other (i, e).

For a path which ultimately visits only the vertices IV, V and VI: then, to allow the transitions, 13, 22, 31, 11, 23, 32, 12, 21, 33 take $+$ infinitely often, and, to ensure admissibility, 11, 22 and 33 take $-$ infinitely often, and we check this is again enough to satisfy the lemma. Let us now take a path which ultimately visits only the vertices I, IV, V, II, $VIII$, VII, I, and always in that circular order; then, to allow the transitions, 13, 31, 22, 11, 33, 23, 32 take $+$ infinitely often, 11, 33, 12, 21, 13, 31, 22 take $-$ infinitely often, and again this is enough to satisfy the lemma.

Other cases are similar to one of these or contain more possibilities. □

Proposition 3.4. *Every admissible infinite path in the graph of graphs is the path of at least one transformation* T.

Proof

The proof (in contrast with the general proof in [21] which uses word combinatorics) follows the strategy of [26]: we find the coordinates $l_{1,0}, r_{1,0}, l_{2,0}, r_{2,0}, l_{3,0}, r_{3,0}$ defining our transformation by showing that some intersection of open cones is nonempty, though here we have to take its further intersection with a subspace of dimension 4 because of the nontrivial relations defined in Definition 2.4 and expressed in Lemma 3.1.

Let $G_0, \dots G_n, \dots$ be a given admissible path. Let ι_n and C_n be the associated instructions and unions of same-sign circuits.

We need to find a sequence of strictly positive vectors

$$v_n = (l_{1,n}, r_{1,n}, l_{2,n}, r_{2,n}, l_{3,n}, r_{3,n})$$

such that for each n,

- the coordinates of v_n satisfy the two non-trivial relations corresponding to G_n as stated in Lemma 3.1,
- $v_{n+1} = U_n v_n$,

where the linear operator U_n from $\mathbb{R}^6$ to $\mathbb{R}^6$ is defined by $U_n(l_1, r_1, l_2, r_2, l_3, r_3) = (l'_1, r'_1, l'_2, r'_2, l'_3, r'_3)$ with

- if $is_n(i) \in C_n$ and $\iota_n i = +$, $l'_i = l_i - r_{s_n(i)}$, $r'_i = r_i$,
- if $is_n(i) \in C_n$ and $\iota_n = -$, $l'_i = l_i$, $r'_i = r_i - l_{s_n(i)}$,
- if $is_n(i) \notin C_n$, $l'_i = l_i$, $r'_i = r_i$.

A direct consequence of the formulas is that U_n is invertible and the matrix of U_n^{-1} has nonnegative entries. What we shall show now is that *for any k, any n large enough, $W_{k,n} = U_k^{-1} \dots U_n^{-1}$ has a matrix whose all entries are strictly positive.*

We look at how v_n is deduced from v_{n+1}; if $is_m(i) \in C_m$ and $\iota_m i = +$, we have $r_{i,m} = r_{i,m+1}$ and

$$l_{i,m} = l_{i,m+1} + r_{s_m(i),m} = l_{i,m+1} + r_{s_m(i),m+1}$$

as $s_m(i)i$ is in the same positive circuit in C_m as $is_m(i)$. Similarly, if $is_m(i) \in C_m$ and $\iota_m i = -$, we have $l_{i,m} = l_{i,m+1}$ and $r_{i,m} = r_{i,m+1} + l_{s_m(i),m+1}$, and if $is_m(i) \notin C_m$ we have $l_{i,m} = l_{i,m+1}$ and $r_{i,m} = r_{i,m+1}$. Hence $l_{i,m+1}$ appears always in the expression of $l_{i,m}$, and hence in every $l_{i,p}$ for $p \leq m$; it appears also in the expression of $r_{j,m}$ when ij takes $-$ at stage m, and if there exists $q \leq m$ such that ij has taken $-$ at stage q, it appears in every $r_{i,p}$ for $p \leq q$.

Let $k > 0$ be fixed. We take i and j and two signs e and e', and choose j_1, ..., j_t as in Lemma 3.3. As $j_{s-1}j_s$ takes $(-1)^s e$ at infinitely many stages, we can find $k < k_1 < \dots k_t$ such that $j_{s-1}j_s$ takes $(-1)^s e$ at stage k_s for all $1 \leq s \leq t$. And if $n > k_t$, this implies that $l_{j,n}$ if $e' = -$, resp. $r_{j,n}$ if $e' = +$, appears in the coordinate $l_{i,k}$ of v_k if $e = +$, resp. $r_{i,k}$ if $e = -$. By doing the same for every choice of $1 \leq i \leq 3$, $1 \leq j \leq 3$, $e \in \{-, +\}$, $e' \in \{-, +\}$ and taking n larger than all the corresponding k_t, we get our assertion on $W_{k,n}$.

We write now the reasoning of [26], in a little more explicit way; let $\Omega = \{l_i > 0, r_i > 0, i = 1, 2, 3\}$ be the open positive cone in $\mathbb{R}^6$, $\overline{\Omega} = \{l_i \geq 0, r_i \geq 0, i = 1, 2, 3\}$ its closure, $K_n = W_{1,n}\Omega$, $\overline{K_n} = W_{1,n}\overline{\Omega}$, $K'_n = \overline{K_n} \setminus \{0\}$; we have $K_n \subset K'_n \subset \overline{K_n}$. The condition on the matrices ensures that for all k and $n > k$, if v is in $\overline{\Omega}$ with at least one strictly positive coordinate, then $W_{k,n}v$ is in Ω, thus

$$\cap_{n \geq 1} K_n = \cap_{n \geq 1} \overline{K_n} \setminus \{0\} = \cap_{n \geq 1} K'_n.$$

The last part of Keane's reasoning (which will not be used here but imitated) says that each K'_n is invariant by $v \to \lambda v$ for any scalar λ, thus the K'_n are decreasing compact sets in a projective space, thus their infinite intersection is non-empty; thus $\cap_{n \geq 1} K_n$ is non-empty.

We introduce now the relations: let Ξ_n be the subset of $\mathbb{R}^6$ made of vectors $(l_1, r_1, l_2, r_2, l_3, r_3)$ whose coordinates satisfy the two non-trivial relations corresponding to G_n in Lemma 3.1; in particular $\Xi_0 = \{r_1 = r_3, l_2 = l_3\}$ as G_0 is vertex I. It follows from Proposition 2.1, and can also be re-checked by direct computation, that

$$\Xi_{n+1} = U_n \Xi_n.$$

Now, the above considerations imply that $\Xi_0 \cap \cap_{n\geq 1} K_n = \Xi_0 \cap \cap_{n\geq 1} K'_n$. We look at the intersections of the K'_n with the space $\Xi_0 = \{r_1 = r_3, l_2 = l_3\}$: they are nonempty as, because of the expression of the relations in Lemma 3.1, each $(\Omega \cap \Xi_{n+1})$ is non-empty, thus also its image by $W_{1,n}$, and we have $W_{1,n}\Omega \cap \Xi_0 = W_{1,n}(\Omega \cap \Xi_{n+1}) = \Xi_0 \cap K_n \subset \Xi_0 \cap K'_n$. Each $\Xi_0 \cap K'_n$ is invariant by $v \to \lambda v$ for any scalar λ, thus the $\Xi_0 \cap K'_n$ are decreasing compact sets in a projective space, thus their infinite intersection is non-empty. Thus the infinite intersection $\cap_{n=1}^{+\infty}(\Xi_0 \cap W_{1,n}\Omega) = \cap_{n=1}^{+\infty} W_{1,n}(\Omega \cap \Xi_{n+1})$ is non-empty.

A vector v_0 in this latter set is such that v_n has strictly positive coordinates for all n, and satisfies the required relations for all n. After normalization by $l_{1,0}+r_{1,0}+l_{2,0}+r_{2,0}+l_{3,0}+r_{3,0}+l_{1,0} = 1$, we define a symmetric 4-interval exchange transformation by $\alpha_1 = l_{1,0} + r_{1,0}$, $\alpha_2 = l_{2,0} + r_{2,0}$, $\alpha_3 = l_{3,0} + r_{3,0}$, $\alpha_4 = l_{1,0}$, and the required inequalities on the β_i and γ_j are satisfied.

By construction the self-dual induction is iterated infinitely, defining the path $G_0, \dots, G_n, \dots$ and by admissibility each E_i is cut infinitely often on the left and on the right; thus the height of each tower tends to $+\infty$; as the negative orbits of the discontinuities of T appear as the endpoints of levels in the castles, while the negative orbits of the discontinuities of T^{-1} appear in the interiors of these levels, the i.d.o.c. condition is satisfied. □

4. Uniquely ergodic examples

In this section, we define a family of examples depending on three sequences of integers m_k, n_k, p_k, which we call the *partial quotients for the self-dual induction*: m_k (resp n_k, p_k) is the number of consecutive times when 22 (resp. 33, 11) takes $-$, the $-$ edge from 22 (resp. 33, 11) being a loop in the castle graph.

Definition 4.1. *Given* $m = \{m_k, k \in \mathbb{N}\}$, $n = \{n_k, k \in \mathbb{N}^\star\}$, $p = \{p_k, k \in \mathbb{N}^\star\}$, *let* $\Gamma(m, n, p)$ *be the admissible path defined as follows, which starts from* I, *then makes infinitely many circuits through vertices* IV, V, VI: *laps are numbered from* $k = 0$; *before lap* 0, *we go from* I *to* IV *by* $(-, -, -)$; *for all* $k \geq 0$, *at the beginning of lap* k *we are in* IV; *we apply instruction* $(+, -, +)$ m_k *times if* $k > 0$, *resp.* $m_0 - 1$ *times if* $k = 0$, *staying in* IV, *then go to* V *by* $(+, +, +)$, *then apply instruction* $(+, +, -)$ n_{k+1} *times, staying in* V, *then go to* VI *by* $(+, +, +)$, *then apply instruction* $(-, +, +)$ p_{k+1} *times, staying in* VI, *then go to* IV *by* $(+, +, +)$.

All transformations T *in this section are such that their path in the graph of graphs is a* $\Gamma(m, n, p)$.

Note that in Definition 4.1, and hence in Lemma 4.1 below, when we look at what happens between vertex IV in lap k and vertex IV in lap $k+1$, we have chosen to use m_k, n_{k+1} and p_{k+1}. This is intentional, because the fundamental Corollary 4.3 below, which depends on what happens between just after vertex IV in lap k and just after vertex IV in lap $k+1$, will thus depend on n_{k+1}, p_{k+1} and m_{k+1}, and that corollary will be used extensively in the sequel. The case $k = 0$ is special, as 22 takes $-$ when we go from the initial state to vertex IV in lap 0, thus 22 has to take $-$ only $m_0 - 1$ times in the latter situation.

Lemma 4.1. *The names of towers* $P_i(k)$ *and* $M_i(k)$ *when we are in vertex* IV *at the beginning of lap* k *are given by the following rules:*

- $P_1(k+1) = (P_1(k)M_2(k)^{m_k}P_2(k)M_3(k))^{p_{k+1}}P_1(k)$
- $P_2(k+1) = M_2(k)^{m_k}P_2(k)$
- $P_3(k+1) = (P_3(k)M_1(k))^{n_{k+1}}P_3(k)$,
- $M_1(k+1) = (P_1(k)M_2(k)^{m_k}P_2(k)M_3(k))^{p_{k+1}}P_1(k)M_2(k)^{m_k}P_2(k)P_3(k)M_1(k)$,
- $M_2(k+1) = M_2(k)^{m_k}P_2(k)(P_3(k)M_1(k))^{n_{k+1}}P_3(k)P_1(k)M_2(k)$,
- $M_3(k+1) = (P_3(k)M_1(k))^{n_{k+1}}P_3(k)P_1(k)M_2(k)^{m_k}P_2(k)M_3(k)$;

with m_k *replaced by* $m_0 - 1$ *if* $k = 0$, *and initial values* $P_1(0) = 13$, $P_2(0) = 22$, $P_3(0) = 314$, $M_1(0) = 1$, $M_2(0) = 2$, $M_3(0) = 3$.

Proof
The proof comes from applying Lemma 2.4 at each stage.

As was announced in Section 2.5, we replace the six towers by four:

Lemma 4.2. *Let* $E_1(k)$ *be the interval* E_1 *when we are in vertex* IV *at the beginning of lap* k*; its induction castle is made of four towers, whose names are*

- $A_k = M_1(k)P_3(k)$,
- $B_k = P_1(k)M_2(k)^{m_k}P_2(k)M_3(k)$,
- $C_k = P_1(k)M_2(k)^{m_k}P_2(k)P_3(k)$,
- $D_k = P_1(k)M_2(k)^{m_k+1}P_2(k)P_3(k)$,

with all m_k *replaced by* $m_0 - 1$ *if* $k = 0$.

Proof
The induced map of T on $E_1(k)$ is an induced map of the induced map of T on $E_1(k) \cup E_2(k) \cup E_3(k)$, whose castle is vertex IV. To find the castle we want, we look at concatenations of towers starting from $E_1(k)$ and coming back to it, and this corresponds to paths in the castle graph IV: starting from 13, we can go to 31 by M_1 and come back to 13 by M_3 or P_3, or else go to 22 by P_1, make an unknown number of times the loop M_2 around 22, then go to 31 by P_2 and come back to 13 by M_3 or P_3; thus the possible names of our concatenations of towers are $M_1(k)P_3(k)$, $M_1(k)M_3(k)$, $P_1(k)M_2(k)^sP_2(k)M_3(k)$, and $P_1(k)M_2(k)^tP_2(k)P_3(k)$ for (a priori) any positive integers s and t. But the same formulas hold with k replaced by $k+1$, while concatenations of towers starting from $E_1(k+1)$ and coming back to it are also concatenations of the above concatenations starting from $E_1(k)$ and coming back to it. Taking into account the formulas of Lemma 4.1, we see that $M_1(k)M_3(k)$ does not occur, and that there are only two possible values for t, $t = m_k$ and $t = m_k + 1$, and one possible value for s, $s = m_k$ (with the usual modification for $k = 0$). □

Corollary 4.3. *The above names are given by the formulas*

- $A_{k+1} = B_k^{p_{k+1}}C_kA_k^{n_{k+1}+1}$,
- $B_{k+1} = B_k^{p_{k+1}}C_k(A_k^{n_{k+1}}D_k)^{m_{k+1}}A_k^{n_{k+1}}B_k$,
- $C_{k+1} = B_k^{p_{k+1}}C_k(A_k^{n_{k+1}}D_k)^{m_{k+1}}A_k^{n_{k+1}}$
- $D_{k+1} = B_k^{p_{k+1}}C_k(A_k^{n_{k+1}}D_k)^{m_{k+1}+1}A_k^{n_{k+1}}$.

with initial values $A_0 = 1314$, $B_0 = 132^{m_0-1}223$, $C_0 = 132^{m_0-1}22314$, $D_0 = 132^{m_0}22314$.

In all the sequel we denote by a_k, b_k, c_k, d_k the lengths of the names A_k, B_k, C_k, D_k; these are also the heights of the corresponding towers, which we denote by tower A_k, tower B_k, tower C_k, tower D_k, each of these being a k-tower.

By minimality, for each ϵ, if k is large enough, the lengths of the intervals are all less than ϵ; hence any integrable function f can be approximated (in $\mathcal{L}_1$ for example) by functions f_k which are constant on each level of each k-tower. Thus the above formulas give a complete description of T as a system of *rank at most four by intervals* (the original reference on finite rank is [31], but finite rank by intervals was not defined in print before [14]). From these formulas, T is determined up to measure-theoretic and topological isomorphisms.

Now, the secret for building interesting examples is to play on our partial quotients; we shall first ensure that our system is of rank one, the tower A_k being the only one which is not of very small measure (for any invariant measure μ, but this fact by itself ensures that μ is unique). Moreover, this is a rank one system as in Definition 1.8, and all its properties come from the values of a_k. In Theorem 4.6 we ensure that the a_k are the denominators of the convergents (for the Euclid algorithm) of an irrational θ, and thus T has θ as an eigenvalue, and even is measure-theoretically isomorphic to the irrational rotation of angle θ. In Theorem 4.7, each a_k will be a multiple of an integer N, and T has $\frac{1}{N}$ as an eigenvalue. In both cases, as the tower B_k is not negligible from the topological point of view, a relation between a_k and b_k will ensure topological weak mixing.

Proposition 4.4. *If for infinitely many k, there exist positive integers a'_k, b'_k such that $a'_k a_k - b'_k b_k = 1$, and we have $n_{k+1} > a'_k$, $p_{k+1} > b'_k$; then the transformation T is topologically weakly mixing.*

Proof
Recall that the union of the bases of the towers A_k, B_k, C_k, D_k is the interval $E_1(k)$, and, by minimality, for each ϵ, if k is large enough, the lengths of the intervals are all less than ϵ. Let θ be an eigenvalue with a continuous eigenfunction f; then, for given ϵ, if k is large enough, $|f(z)-f(y)| < \epsilon$ (in $\mathbb{R}/\mathbb{Z}$) if z and y are in $E_1(k)$. Because in the formulas of Corollary 4.3 $A_k^{n_{k+1}}$ occurs in (for example) A_{k+1}, there exists x in the basis of the tower A_k such that $T^{a'_k a_k}x$ is again in $E_1(k)$ hence

$$||\theta a'_k a_k|| = |f(T^{a'_k a_k}x) - f(x)| \leq \epsilon;$$

similarly there exists y in the basis of the tower B_k such that $T^{b'_k b_k}y$ is again in $E_1(k)$, and we get

$$||\theta(a'_k a_k - b'_k b_k)|| < \epsilon,$$

hence $\theta = 0$, which is not possible as T is minimal and the existence of a continuous non-constant eigenfunction for $\theta = 0$ would imply the existence of a non-trivial closed invariant subset. □

Proposition 4.5. *If*

$$\sum_{k=1}^{+\infty} \frac{(p_{k+1}+1)b_k + c_k + d_k}{n_{k+1}a_k} < +\infty,$$

then T is uniquely ergodic and (X,T,μ) is measure-theoretically isomorphic to the rank one system (X',T',μ') defined (as in Definition 1.8) by the word A_0 and the towers

$$A'_{k+1} = s^{a_{k+1}-(n_{k+1}+1)a_k}(A'_k)^{n_{k+1}+1}.$$

Proof
Note that the above condition uses d_k and not $m_{k+1}d_k$ as it is enough since both $A_k^{n_{k+1}}$ and D_k have their lengths multiplied by m_{k+1} in the formulas of Corollary 4.3.

Let μ be any invariant probability for T: each level in a given tower has the same measure, hence the above condition ensures that the towers B_k, C_k, D_k, have measure at most ϵ_k, the k-th term in the above series, in each tower A_{k+1}, B_{k+1}, C_{k+1}, D_{k+1}, hence in the whole space, where $\sum_{k=0}^{+\infty}\epsilon_k < +\infty$. The system (X,T,μ) is then of *rank one by intervals* as the sequence of towers

A_k generate the whole space, see for example [14] for precise definitions. We build a measure-theoretic isomorphism between (X,T,μ) and (X',T',μ'), by sending the j-th level of the tower A_k to the j-th level of the tower A'_k for T': it is consistent by construction, as the length of A'_k is a_k, and is defined almost everywhere because of the condition on ϵ_k. The unique ergodicity is a consequence of the rank one by intervals: as is mentioned in Definition 1.8, the definition of μ' ensures that it is the unique invariant probability measure on (X',T'), and any invariant measure $\nu \neq \mu$ on (X,T) would define an invariant measure $\nu' \neq \mu'$ on (X',T') through the above isomorphism. □

Theorem 4.6. *One can construct recursively sequences m, n, p such that the corresponding transformation T is uniquely ergodic, topologically weakly mixing, and measure-theoretically isomorphic to an irrational rotation on $\mathbb{T}_1$.*

Proof

We build the partial quotients for the self-dual induction recursively as follows: we choose m_0 such that a_0 and b_0 are coprime, and we have $b_0 > a_0$.

At stage k, we assume a_k and b_k are coprime, and $b_k > a_k$; by Bezout's identity we can find positive integers a'_k and b'_k such that $a'_k a_k - b'_k b_k = 1$. We choose first p_{k+1}, such that

$$p_{k+1} > b'_k \quad \text{and}$$

$$p_{k+1}b_k + c_k \equiv a_{k-1} \mod a_k;$$

this is possible as b_k is invertible modulo a_k; then we choose n_{k+1} large enough for

$$n_{k+1}a_k > 2^k((p_{k+1}+1)b_k + c_k + d_k),$$

$$n_{k+1} > a'_k,$$

and such that

$$(n_{k+1}+1)a_k + p_{k+1}b_k + c_k \quad \text{is coprime with} \quad b_k - a_k;$$

this is possible as a_k is invertible modulo $b_k - a_k$; finally we choose

$$m_{k+1} = t_{k+1}a_{k+1} \quad \text{for some} \quad t_{k+1} \in \mathbb{N}^\star.$$

As by Corollary 4.3 $b_{k+1} - a_{k+1} = b_k - a_k + m_{k+1}(n_{k+1}a_k + d_k)$, we have $b_{k+1} - a_{k+1} \equiv b_k - a_k$ modulo a_{k+1} by choice of m_{k+1} as in the previous equation; as $b_k - a_k$ is invertible modulo a_{k+1}, so is $b_{k+1} - a_{k+1}$, and thus a_{k+1} and b_{k+1} are again coprime, and $b_{k+1} > a_{k+1}$.

Our transformation T satisfies the hypothesis of Proposition 4.5, thus is uniquely ergodic and measure-theoretically isomorphic to the rank one system T'. T is topologically weakly mixing by Proposition 4.4.

Because of the second equation in the choice of p_{k+1} above $a_{k+1} = y_{k+1}a_k + a_{k-1}$ for positive integers y_{k+1}. We choose the irrational θ whose partial quotients (for the Euclid algorithm) are $y_0, y_1, \ldots$ so that the a_k are the denominators of its convergents. For the rotation of angle θ, the standard Sturmian trajectories (see [19] for example) are concatenations of words $A"_k$ and $C"_k$ with $C"_{k+1} = A"_k$ and $A"_{k+1} = C"_k(A"_k)^{y_{k+1}}$. As $\sum_{k=1}^{+\infty} \frac{a_{k-1}}{y_{k+1}a_k} < +\infty$ because the hypothesis of Proposition 4.5 is satisfied, this rotation is measure-theoretically isomorphic to the rank one system defined by the word $A"_0$ and the towers $A"_{k+1} = s^{a_{k-1}}(A"_k)^{y_{k+1}}$, by the same proof as in Proposition 4.5.

And T' and $T"$ are measure-theoretically isomorphic as in the proof of Proposition 4.5, as build an isomorphism between $T"$ and T' by sending some $A"_k$ to strings of spacers of length a_k, on a part of the space of measure ϵ_k with $\sum_{k=1}^{+\infty} \epsilon_k < +\infty$. □

Theorem 4.7. *For any integer $N \geq 2$, one can construct recursively sequences m, n, p such that the corresponding transformation T is uniquely ergodic, topologically weakly mixing, and has $\frac{1}{N}$ as an eigenvalue.*

Proof

We build the partial quotients for the self-dual induction as follows: at stage $k \geq 1$, we assume a_k and b_k are coprime, $b_k > a_k$ and a_k is a multiple of N; by Bezout's identity we can find positive integers a'_k and b'_k such that $a'_k a_k - b'_k b_k = 1$. We choose first p_{k+1}, such that $p_{k+1} > b'_k$ and

$$p_{k+1} b_k + c_k \equiv 0 \mod N;$$

this is possible as b_k is invertible modulo a_k hence modulo N; then we choose n_{k+1} large enough for $n_{k+1} a_k > 2^k((p_{k+1}+1)b_k + c_k + d_k)$, $n_{k+1} > a'_k$, and such that $(n_{k+1}+1)a_k + p_{k+1} b_k + c_k$ is coprime with $b_k - a_k$; this is possible as a_k is invertible modulo $b_k - a_k$; finally we choose $m_{k+1} = t_{k+1} a_{k+1}$ for some $t_{k+1} \in \mathbb{N}^\star$, hence a_{k+1} and b_{k+1} are again coprime, and $b_{k+1} > a_{k+1}$, while a_{k+1} is a multiple of N.

At the initial stage, if $N = 2$ or $N = 4$ we can choose m_0 such that b_0 is coprime with a_0, and our assumptions are satisfied at stage 0, so we begin the above process at $k = 0$. Otherwise, our assumptions will be satisfied at stage 1, in the following way: we choose m_0 such that m_0 and $m_0 + 4$ are both coprime with $4N$ (let $4N = \prod_{i=0}^{s} \pi_i^{\alpha_i}$ be the decomposition of $4N$ into prime factors, with $\pi_0 = 2 < \pi_1 < ...$; for $0 \leq i \leq s$, let Ψ_i be the set of $0 < m < 4N$ such that m and $m + 4$ are coprime with π_0, ..., π_i: we have $\#\Psi_0 = 2N$, and, by the Chinese remainder theorem, $\#\Psi_{i+1} = \#\Psi_i(1 - \frac{1}{2\pi_{i+1}})$, thus $\#\Psi_s = \prod_{i=0}^{s} \pi_i^{\alpha_i - 1} \prod_{i=1}^{s}(\pi_i - 2) > 0$, and any m_0 in Ψ_s is convenient). Thus m_0 is coprime with N and with 4 and $m_0 + 4$ is coprime with $m_0 N$, and for any n_1 and any element x of $\mathbb{Z}/m_0 N\mathbb{Z}$ we can find p_1 such that $a_1 = (m_0+4)p_1 + 4n_1 + m_0 + 10 \equiv x \mod Nm_0$. Hence we choose any n_1, and then p_1 such that a_1 is a multiple of N and coprime with $m_0 = b_0 - a_0$, then, with $m_1 = t_1 a_1$ for some $t_1 \in \mathbb{N}^\star$, we get that a_1 and b_1 are coprime.

Our transformation T satisfies the hypothesis of Proposition 4.5, thus is uniquely ergodic and measure-theoretically isomorphic to the rank one system T' in Proposition 4.5. T is topologically weakly mixing by Proposition 4.4.

On (X', T'), for $k \geq 1$ we put $\phi_k(x) = \frac{j}{N}$ if x lies in the $pN + j$-th level of the tower A'_k, for integers $0 \leq p \leq \frac{a_k}{N} - 1$, $0 \leq j \leq N - 1$. Because a_k is a multiple of N, this is consistent and the ϕ_k converge in $L^2(X, \mathbb{R}/\mathbb{Z})$ to a function ϕ, which satisfies $T'\phi = \frac{1}{N} + \phi$. Thus T' and T have the required eigenvalue. □

We can also build such a transformation T with both rational and irrational eigenvalues, by building a θ such that the a_k are the denominators of its convergents, multiplied by N.

We turn now to weakly mixing examples; the first one imitates the famous rank one system of del Junco-Rudolph [13] by ensuring a recurrence relation $a_{k+1} = y_{k+1} a_k + 1$.

Theorem 4.8. *One can construct recursively sequences m, n, p such that the corresponding transformation T is uniquely ergodic, weakly mixing, and simple (of order two).*

Proof

We build the partial quotients for the self-dual induction recursively as follows: we choose m_0 such that a_0 and b_0 are coprime, and we have $b_0 > a_0$.

At stage k, we assume a_k and b_k are coprime, and $b_k > a_k$; we choose p_{k+1} such that

$$p_{k+1} b_k + c_k \equiv 1 \mod a_k;$$

this is possible as b_k is invertible modulo a_k; then we choose n_{k+1} large enough for $n_{k+1}a_k > 2^k((p_{k+1}+1)b_k + c_k + d_k)$, and such that $(n_{k+1}+1)a_k + p_{k+1}b_k + c_k$ is coprime with $b_k - a_k$, and $m_{k+1} = t_{k+1}a_{k+1}$ for some $t_{k+1} \in \mathbb{N}^\star$. Thus a_{k+1} and b_{k+1} are again coprime, and $b_{k+1} > a_{k+1}$.

Our transformation T satisfies the hypothesis of Proposition 4.5, thus is uniquely ergodic and measure-theoretically isomorphic to the rank one system T'.

By construction $a_{k+1} = y_{k+1}a_k + 1$ for positive integers $y_{k+1} > 2^{k+1}$. Thus T' is measure-theoretically isomorphic to the rank one system T" defined by the word A_0 and the towers $A"_{k+1} = s(A"_k)^{y_{k+1}}$, as we build an isomorphism between T" and T' by sending some $A"_k$ to strings of spacers of length a_k, on a part of the space of measure ϵ_k with $\sum_{k=1}^{+\infty} \epsilon_k < +\infty$.

This last system is weakly mixing and simple exactly in the same way as del Junco - Rudolph's map [13], which is the rank one system defined by some H_0 and the towers $H_{k+1} = H_k^{2^k} s H_k^{2^k}$ (this defines a transformation by an appropriate modification of Definition 1.8); the main (and quite involved) argument in Theorem 1 of [13] uses only the fact that there are isolated spacers between long concatenations of the same tower. □

Note that we deduce from [13] that this system is also *prime* (it has no nontrivial invariant sub-σ -algebra) and rigid.

Of course, as most transformations T are weakly mixing, we may expect to find many more examples with this property. Indeed, we can build a lot of them by adapting to the family of transformations T in the present section the method described in the proof of Theorem 5.5 below.

Another unexpected way is to use the so-called *Arnoux-Rauzy systems* [3]. These are symbolic systems defined by three names X_k, Y_k, Z_k, build recursively by using a sequence of combinatorial rules; by rule 1, $X_{k+1} = X_k$, $Y_{k+1} = Y_kX_k$, $Z_{k+1} = Z_kX_k$; by rule 2, $X_{k+1} = X_kY_k$, $Y_{k+1} = Y_k$, $Z_{k+1} = Z_kY_k$; by rule 3, $X_{k+1} = X_kZ_k$, $Y_{k+1} = Y_kZ_k$, $Z_{k+1} = Z_k$. At the beginning, $X_0 = 1$, $Y_0 = 12$, $Z_0 = 13$. Here we restrict ourselves to a paréar class of Arnoux-Rauzy systems, built by applying successively rule 1 q_{3l+1} times, rule 2 q_{3l+2} times, rule 3 q_{3l+3} times, then rule 1 q_{3l+4} times and so on, starting from $l = 0$; this gives a uniquely ergodic (by Boshernitzan's result using complexity [5]) system (Y, S), and, when the q_k grow to infinity fast enough, as a straightforward consequence of the definition, this system is measure-theoretically isomorphic to a rank one system defined by the word H_0 and the towers $H_{k+1} = s^{t_k}(H_k)^{q_{k+1}}$, where, for $k = 3l+1$ (resp. $k = 3l+2$, $k = 3l+3$) H_k has name $Y_{q_1+\ldots q_k}$ (resp. Z, X), and t_k is the length of $Z_{q_1+\ldots q_k}$ (resp. X, Y). These systems are proved to be weakly mixing in [7].

Proposition 4.9. *One can construct recursively sequences m, n, p such that the corresponding transformation T is uniquely ergodic, weakly mixing, and measure-theoretically isomorphic to an Arnoux-Rauzy system.*

Proof

We build simultaneously m_k, n_k, p_k defining our transformation T and q_k defining our Arnoux-Rauzy system.

At each stage, a_k and b_k are coprime, $b_k > a_k$, and $a_k = h_k$, h_k being the length of H_k. At the beginning, we choose the first parameters so that the assumptions are satisfied at stage 1. At stage k choose first p_{k+1}, such that, if t_k is defined above from q_1, ... ,q_k and the rules defining an Arnoux-Rauzy system, as the length of $Z_{q_1+\ldots q_k}$, resp. X, Y according to the class of k modulo 3,

$$p_{k+1}b_k + c_k \equiv t_k \mod a_k;$$

then we choose n_{k+1} large enough for satisfying the hypothesis of Proposition 4.5 and such that $(n_{k+1}+1)a_k + p_{k+1}b_k + c_k$ is coprime with $b_k - a_k$, then $m_{k+1} = u_{k+1}a_{k+1}$, for a positive integer u_{k+1}, so that a_{k+1} and b_{k+1} are again coprime, and $b_{k+1} > a_{k+1}$. Then we choose q_{k+1} so that $h_{k+1} = a_{k+1}$. We conclude as in the proof of Theorem 4.6. □

Note that all the examples in this section are rigid by Proposition 1.1.

5. Non uniquely ergodic examples

Definition 5.1. *Given $m = \{m_k, k \in \mathbb{N}\}$, $n = \{n_k, k \in \mathbb{N}^\star\}$, with $n_{k+1} > m_k > n_k$, let $\Gamma(m,n)$ be the admissible path defined as follows, which starts from I and then follows infinitely many times a path $IV - I - VII - I - IV$:*

let $f_0 = n_1$, $e_k = m_k - f_{k-1}$ and $f_k = n_{k+1} - e_k$ for $k \geq 1$, thus $e_k > 0$ and $f_k > 0$. At the beginning of step k we are in IV; we go to I by $(-,-,-)$, then apply instruction $(+,-,+)$ $e_k - 1$ times, staying in I, then then go to VII by $(+,+,+)$, then go to I by $(+,+,+)$, apply instruction $(+,-,-)$ $f_k - 1$ times, staying in I, then go back to IV by $(-,-,-)$. Before step 1, starting from I we apply instruction $(+,-,-)$ $f_0 - 1$ times, staying in I, then go to IV by $(-,-,-)$.

All transformations T in this section are such that their path in the graph of graphs is a $\Gamma(m,n)$.

Indeed, in this definition m_k is the number of consecutive times when 22 takes $-$, the $-$ edge from 22 being a loop in the castle graph; 22 does take $-$ when we are in I or IV but not when we are in VII, so m_k counts also the number of times we are consecutively in I, IV, and I again, between two passages in VII. Similarly n_k is the number of consecutive times when 11 takes $+$, the $+$ edge from 11 being a loop in the castle graph, and that happens when we are in I or VII.

The e_k and f_k can be seen as auxiliary quantities with $m_k = e_k + f_{k-1}$ and $n_{k+1} = e_k + f_k$; the indexing has been chosen so that Lemma 5.1 depends on e_k, f_k, and Corollary 5.3 will thus depend on n_{k+1} and m_{k+1}, and only that corollary will be used in the sequel.

In the same way as in the previous section we prove

Lemma 5.1. *The names of towers $P_i(k)$ and $M_i(k)$ when we are in vertex IV at the beginning of step k are given by the following rules:*

- $P_1(k+1) = (M_1(k)P_3(k))^{e_k+f_k}M_1(k)M_3(k)P_1(k)$,
- $P_2(k+1) = (M_2(k)^{e_k}P_2(k)M_3(k)P_1(k)M_2(k))^{f_k}M_2(k)^{e_k}P_2(k)$,
- $P_3(k+1) = M_3(k)P_1(k)M_2(k)^{e_k}P_2(k)M_3(k)M_1(k)P_3(k)$,
- $M_1(k+1) = (M_1(k)P_3(k))^{e_k+f_k}M_1$,
- $M_2(k+1) = M_2(k)^{e_k}P_2(k)M_3(k)P_1(k)M_2(k)$,
- $M_3(k+1) = M_3(k)P_1(k)M_2(k)^{e_k}P_2(k)M_3(k)$;

with initial values $P_1(0) = (14)^{f_0-1}13$, $P_2(0) = 2^{f_0+1}$, $P_3(0) = 314$, $M_1(0) = (14)^{f_0-1}1$, $M_2(0) = 2$, $M_3(0) = 3$.

Note that $P_2(k+1)$ does indeed contain $M_2(k+1)$, and even $M_2(k+1)^{f_k}$, as a strict prefix, as the last instruction is $(---)$ from I, and the instruction for 2 has been $-$ f_k times.

Lemma 5.2. *Let $E_1(k)$ be the interval E_1 when we are in vertex IV at the beginning of step k; its induction castle is made of four towers, whose names are*

- $A_k = M_1(k)P_3(k)$,
- $B_k = M_1(k)M_3(k)$,

- $C_k = P_1(k)M_2(k)^{e_k}P_2(k)M_3(k)$,
- $D_k = P_1(k)M_2(k)^{e_k+1}P_2(k)M_3(k)$.

Corollary 5.3. *The above names are given by the formulas*

- $A_{k+1} = A_k^{n_{k+1}}B_kC_kA_k$,
- $B_{k+1} = A_k^{n_{k+1}}B_kC_k$,
- $C_{k+1} = A_k^{n_{k+1}}B_kC_kD_k^{m_{k+1}}C_k$,
- $D_{k+1} = A_k^{n_{k+1}}B_kC_kD_k^{m_{k+1}+1}C_k$.

with initial values $A_1 = (14)^{n_1}1314$, $B_1 = (14)^{n_1-1}13$, $C_1 = (14)^{n_1-1}132^{m_1+1}3$, $D_1 = (14)^{n_1-1}132^{m_1+2}3$.

Towers and lengths are denoted as in the previous section. Now we shall fix our partial quotients so that the towers A_k and the towers D_k behave like independent systems, so that the transformation T has two ergodic invariant measures, one mainly concentrated on the towers A_k and giving a rank one system with this family of towers, and the other doing the same with the towers D_k. By ensuring the a_k are even, we get an eigenvalue $\frac{1}{2}$ for the first system, while the lengths d_k will ensure the second one is weakly mixing by contradicting the criterion in Proposition 1.2.

Proposition 5.4. *If*

$$\sum_{k=1}^{+\infty} \frac{b_k + c_k}{n_{k+1}a_k} < \frac{1}{4},$$

$$\sum_{k=1}^{+\infty} \frac{n_{k+1}a_k + b_k + 2c_k}{m_{k+1}d_k} < \frac{1}{4},$$

then T has exactly two ergodic invariant probability measures μ_1 and μ_2; (X, T, μ_1) is measure-theoretically isomorphic to the rank one system defined by the word A_1 and the towers

$$A'_{k+1} = (A'_k)^{n_{k+1}} s^{a_{k+1}-n_{k+1}a_k};$$

(X, T, μ_2) is measure-theoretically isomorphic to the rank one system defined by the word D_1 and the towers

$$D'_{k+1} = s^{d_{k+1}-(m_{k+1}+1)d_k-c_k}(D')_k^{m_{k+1}+1}s^{c_k}.$$

Proof

Let ϵ_k and η_k be respectively the k-th term of the first and second series above. Let μ be any invariant probability for T: each level in a given tower has the same measure, hence the above conditions and the formulas in Corollary 5.3 ensure that the tower A_k has measure at least $1 - \epsilon_k$ in the tower A_{k+1} and the tower D_k has measure at least $1 - \eta_k$ in the tower D_{k+1}, while the towers B_k, C_k have measure at most $\epsilon_k + \eta_k$ in each tower A_{k+1}, B_{k+1}, C_{k+1}, D_{k+1}, hence in the whole space.

Thus we can build a measure-theoretic isomorphism between the rank one system (X', T', μ') with towers A'_k and (X, T) equipped with some invariant probability measure μ_1 which we retrieve from μ', and μ_1 is ergodic as μ' is. We do the same for the rank one system with towers D'_k, defining an ergodic μ_2. Then the tower A_1 has measure greater than $\frac{1}{2}$ for μ_1 and smaller than $\frac{1}{2}$ for μ_2, thus they are different, and it is known [23] [37] that T has at most two invariant ergodic probabilities. □

Note that the two convergent series conditions are exactly the one needed in the definition of rank one systems.

Theorem 5.5. *One can construct recursively sequences m, n such that the corresponding transformation T is not uniquely ergodic, topologically weakly mixing, weakly mixing for one of its invariant ergodic measures, while for the other one it has $\frac{1}{2}$ as an eigenvalue.*

Proof
We fix $M > 5$ such that for all $y \geq M$ there exist a prime number between $6y/10$ and $9y/10$ and a prime number between $11y/10$ and $14y/10$. This is possible as a consequence of the prime numbers theorem.

At the beginning, note that a_1, b_1 are even; we choose n_1 and m_1 such that c_1 is even, d_1 is odd, a_1 and d_1 are coprime.

Given a_k, b_k, c_k, d_k, some p_k to be specified later, and the assumptions that a_k and d_k are coprime, a_k, b_k, c_k are even and d_k is odd, we choose the next partial quotients as follows. Let z_k be the greatest common divisor of p_k and d_k, with $d_k = d'_k z_k$, $p_k = p'_k z_k$. Note that $d'_k > 2$ as d_k is odd. Let $p"_k$ be an inverse of p'_k mod d_k. We choose first a unit u'_k of $\mathbb{Z}/d'_k\mathbb{Z}$ such that

$$u'_k \not\equiv p'_k(a_k - c_k) + t' \mod d'_k \quad \text{for any} \quad -d'_k/2M < t' < d'_k/2M.$$

This is possible: if $d'_k \geq M$, we take the class modulo d'_k of one of the two prime numbers defined above for $y = d'_k$ (the first one if the class of $p'_k(a_k - c_k)$ is between 0 and $d'_k/2$, the second one otherwise), while if $2 < d'_k < M$ this forbids at most one unit. We choose now a unit u_k of $\mathbb{Z}/d_k\mathbb{Z}$ such that

$$u_k \equiv p"_k u'_k \mod d'_k$$

(this is possible as, to be a unit, u_k has just to be coprime with the prime factors of d_k which are not factors of d'_k). Now we choose n_{k+1} large enough for the first condition of Proposition 5.4 and such that

$$n_{k+1}a_k \equiv u_k - a_k - b_k - c_k \mod d_k,$$

thus a_{k+1} is coprime with d_k; and we choose then m_{k+1} large enough for the second condition of Proposition 5.4 and such that

$$(m_{k+1} + 1)d_k + c_k - a_k \quad \text{is invertible modulo} \quad a_{k+1}.$$

Thus our assumptions are satisfied for $k + 1$ (note that m_{k+1} has to be even).

We explain now how to choose the p_k. When m_{k+1} and n_{k+1} are fixed, for any $0 < p < d_k$ there is at most one integer $0 < l < d_{k+1}$ such that

$$\left|\frac{p}{d_k} - \frac{l}{d_{k+1}}\right| < \frac{1}{2Md_{k+1}}.$$

We call this integer $l = \phi_{k+1}(p)$, when it exists. Now, our choice of partial quotients ensures that $p_k d_{k+1} \equiv p_k u_k + p_k(c_k - a_k)$ mod d_k, and $p_k u_k \not\equiv p_k(a_k - c_k) + t$ mod d_k for any $-d_k/2M < t < d_k/2M$ (by multiplying by z_k the relation satisfied by u'_k, and $p'_k u_k$, mod d'_k), thus $p_k d_{k+1} \not\equiv t$ mod d_k for any $-d_k/2M < t < d_k/2M$; this means exactly that $\phi_{k+1}(p_k)$ *does not exist*. Starting from $k_1 = 1$, we define inductively a sequence of integers k_j; at stage j we put $p_{k_j} = 1$; then $p_{k_j+1} = \phi_{k_j+1}(2)$ if it exists, otherwise $p_{k_j+1} = \phi_{k_j+1}(3)$ if it exists, and so on... If no $\phi_{k_j+1}(p)$ exists anymore, we put $k_{j+1} = k_j + 1$, otherwise p_{k_j+1} will be some $\phi_{k_j+1}(p)$, and for p_{k_j+2} we try first $\phi_{k_j+2}\phi_{k_j+1}(p + 1)$ if it exists, then $\phi_{k_j+2}\phi_{k_j+1}(p + 2)$ and so on... If no $\phi_{k_j+2}\phi_{k_j+1}(q)$ exists anymore, we put $k_{j+1} = k_j + 1$, otherwise p_{k_j+2} is some $\phi_{k_j+2}\phi_{k_j+1}(q)$, and for p_{k_j+3} we try first $\phi_{k_j+3}\phi_{k_j+2}\phi_{k_j+1}(q + 1)$ if it exists, and so on... After at most $d_{k_j} - 1$ steps, we have defined k_{j+1}

and ensured that for any $0 < p < d_{k_j}$, there exists $t \leq k_{j+1}$ such that $\phi_t \dots \phi_{k_j+1}(p)$ does not exist; then we start again from $p_{k_{j+1}} = 1$.

We apply Proposition 5.4 to get μ_1 and μ_2. As a_k is always even, (X, T, μ_1) has an eigenvalue $\frac{1}{2}$ as in Theorem 4.7. Now, let θ be an eigenvalue for (X, T, μ_2): by Proposition 1.2 we must have $m_{k+1}||d_k\theta|| < \frac{1}{4M}$ for k large enough, which implies $|\theta - \frac{t_k}{d_k}| < \frac{1}{3Md_{k+1}}$ for k large enough and some $0 < t_k < d_k$; this implies that for all k large enough $\phi_{k+1}(t_k)$ exists and is t_{k+1}, and this is impossible by the choice of p_k. Thus (X, T, μ_2) is weakly mixing, which implies the topological weak mixing of T (which we could also have ensured directly as in Proposition 4.4). □

As in [26] we can choose the vector of lengths (among a segment of possible solutions), so that μ_1 is the Lebesgue measure, or so that μ_2 is the Lebesgue measure, or so that neither μ_1 nor μ_2 is the Lebesgue measure. Note that (X, T, μ_1) and (X, T, μ_2) are rigid by Proposition 1.1.

6. Questions and comments

Among examples we would have liked to build are transformations T with two (or more) rationally independent irrational eigenvalues; a similar result has been claimed by Parreau and Guenais (still unpublished) for $d = 3$ intervals, by very different methods which do not generalize to $d > 3$; the methods of the present paper being based on rank one, what we would need is an explicit rank one construction for rotations of $\mathbb{T}^2$, and this in itself is an interesting open problem.

Very interesting also would be a transformation T with a continuous eigenfunction; this does not exist for $d = 3$ intervals [30]; for every $d \geq 4$ nontrivial examples have been derived by Hmili [22] (in answer to a question asked in a preliminary version of the present paper): these examples are semi-conjugate, in a rather straightforward way, to rotations of $\mathbb{T}_1$. Older examples have been built by Arnoux and Yoccoz [2] for some permutation on $d = 7$ intervals: they are semi-conjugate, in a non-straightforward way, to rotations of $\mathbb{T}_2$. No example we know of has total irrationality.

The condition of alternate discontinuities simplifies the situation but can be dispensed with, see [21]. The generalization of our methods to build examples on d intervals should not introduce any fundamental difficulty but the computations become horrendous; as for other permutations than the symmetric one, while our self-dual induction is not defined in the general case, it can be made to work on classes of examples as in [21]; but the case $d = 4$ for one non-trivial permutation is representative of the whole problem, as happens for Keane's [26] non-uniquely ergodic examples which were not extended beyond that until a recent course of Yoccoz [42].

We recall that Veech's question on simplicity is far from solved; another question is to define a set $\mathcal{M}$" as in the introduction by requiring T to be *topologically strongly mixing*, that is, for every Borelian A and B, $T^nA \cap B$ is nonempty for n large enough. Boshernitzan (unpublished) has proved that $\mathcal{M}$" is empty for $d = 3$ intervals, but, after many computer simulations, conjectures that $\mathcal{M}$" is of full measure for $d \geq 4$ intervals. Again during the process of refereeing the present paper, this question has been mostly solved for $d = 4$ by Chaika [8] (he gets a residual set, though not necessarily of full measure).

As for the specificity of the self-dual induction: it is possible that these or similar examples could have been obtained via other well-known induction methods, by first building a parametrized family of examples, and then manipulating the parameters. Indeed this approach was used by

Chaika to answer a related question [8], starting from the family of examples in [26] which was built by using a variant of the Rauzy induction (actually anterior to Rauzy).

However, the authors found the self-dual induction developed herein to be well suited for this task. In particular, one can stress the role of the quantities we call partial quotients, which appear naturally as the number of consecutive times a given loop is followed in a castle graph, and which share some of the arithmetic properties of the usual partial quotients in the Euclid algorithm; indeed, in the simpler case of $d = 3$ intervals, they are used to define a multiplicative self-dual induction [20], though this is less obvious for $d = 4$.

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