Proof Nets as Concurrent Processes

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Logic and Interaction ™

Links between LL and concurrency are obvious from the start:

- Proof nets as a parallel syntax
- Abramsky 90's: Computational Interpretations of LL ...
- Bellin & Scott 1994: LL and π-calculus

Polarisation is a crucial notion:

- Focalisation, LLP ...
- Interactive interpretations (games, ludics)

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The calculus of solos

The agents of the calculus of solos are defined as

$$
\begin{array}{ll}
P := \mathbf{0} & \text{empty process} \\
\quad \mid (P \mid P) & \text{parallel composition} \\
\quad \mid \bar{u}(\tilde{x}) & \text{emission} \\
\quad \mid u(\tilde{x}) & \text{reception} \\
\quad \mid \nu x.P & \text{scoping} \\
\quad \mid !P & \text{replication}
\end{array}
$$

The reduction works by fusion as:

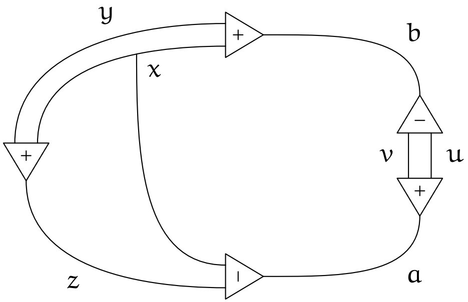
$$u(\tilde{x}) \mid \bar{u}(\tilde{y}) \mid P \rightarrow P[\tilde{x} = \tilde{y}]$$

Graphical syntax

The process

$$\nu xyz.\big(a(yx) \mid \bar{z}(xy) \mid \bar{b}(yx) \mid b(uv) \mid \bar{a}(uv)\big)$$

is represented as



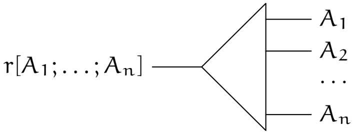
Forms

The set of forms is defined by the following grammar:

$$A ::= x \mid r[A; \ldots ; A] \qquad \text{forms}$$

$$r ::= s \mid \cdot \mid \sharp \qquad \text{replicability flags}$$

Forming rule:



If u crosses a box, its form is $\sharp[\cdots]$

LL with explicit polarities

The formulas of MELL$\updownarrow$ are defined as:

$$\mathsf{P} := \mathsf{X} \mid \mathsf{P} \otimes \mathsf{P} \mid !\mathsf{N} \mid \downarrow\mathsf{N}$$

$$\mathsf{N} := \mathsf{X}^{\perp} \mid \mathsf{N} \parr \mathsf{N} \mid ?\mathsf{P} \mid \uparrow\mathsf{P}$$

Exponentials are (intuitively) decomposed into

$$!\mathsf{N} = \downarrow\sharp\mathsf{N} \qquad \text{and dually} \qquad ?\mathsf{P} = \uparrow\flat\mathsf{P}$$

where $\sharp$ represents replicability and $\flat$ represents multiplicity.

MELL$\updownarrow$ — inference rules

axiom and cut

$$\frac{}{\vdash X , X^\perp} \qquad \frac{\vdash \Gamma , A \qquad \vdash A^\perp , \Delta}{\vdash \Gamma , \Delta}$$

multiplicatives

$$\frac{\vdash \Gamma , P \qquad \vdash Q , \Delta}{\vdash \Gamma , P \otimes Q , \Delta} \qquad \frac{\vdash \Gamma , M , N}{\vdash \Gamma , M \parr N}$$

exponentials

$$\frac{\vdash ?\Gamma , N}{\vdash ?\Gamma , !N} \qquad \frac{\vdash \Gamma , P}{\vdash \Gamma , ?P}$$

contraction
and weakening

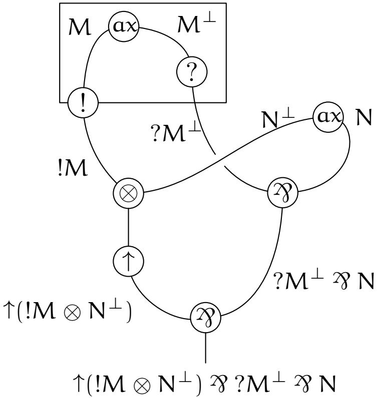
$$\frac{\vdash \Gamma , ?P , ?P}{\vdash \Gamma , ?P} \qquad \frac{\vdash \Gamma}{\vdash \Gamma , ?P}$$

shifts

$$\frac{\vdash \Gamma , P}{\vdash \Gamma , \uparrow P} \qquad \frac{\vdash \Gamma , N}{\vdash \Gamma , \downarrow N}$$

Proof-nets with shifts

This is a proof of $\downarrow(!M \multimap N) \multimap !M \multimap N$:



The principle

The translation is **polarised**:

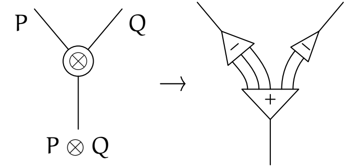
$$\text{positive formulas} \rightarrow \text{emissions}$$
$$\text{negative formulas} \rightarrow \text{receptions}$$

and **typed**:

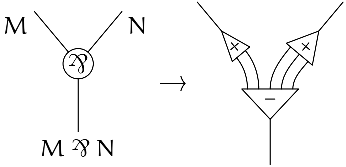
$$\text{link of type } A \rightarrow \text{channel of form } [\![A]\!]$$

$$
\begin{aligned}
[\![A^{\perp}]\!] &= [\![A]\!] \\
[\![P \otimes Q]\!] &= [\![P]\!] \times [\![Q]\!] \\
[\![\downarrow N]\!] &= [\,[\![N]\!]\,] \\
[\![!N]\!] &= [\,\sharp[\![N]\!]\,]
\end{aligned}
$$

The translation: multiplicatives

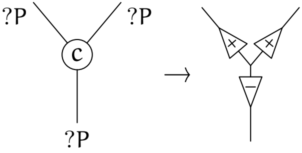


$$\frac{A \vdash \Gamma\,,\, a : P \qquad B \vdash b : Q\,,\, \Delta}{\nu ab.(A \mid B \mid \otimes_{abc}) \vdash \Gamma\,,\, c : P \otimes Q\,,\, \Delta}$$

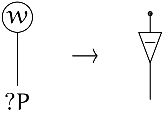


$$\frac{A \vdash \Gamma\,,\, a : M\,,\, b : N}{\nu ab.(A \mid \parr_{abc}) \vdash \Gamma\,,\, c : M \parr N}$$

The translation:
contraction and weakening

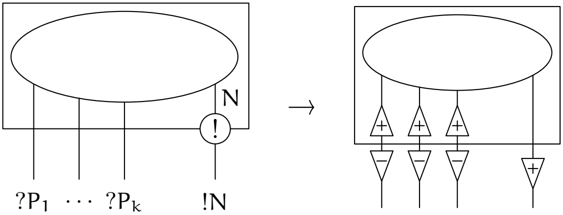


$$\frac{A \vdash \Gamma\, , a : ?P\, , b : ?P}{\nu ab.(A \mid c_{abc}) \vdash \Gamma\, , c : ?P}$$



$$\frac{A \vdash \Gamma}{A \mid \nu x.\bar{a}(x) \vdash \Gamma\, , a : ?P}$$

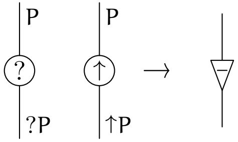
The translation:
exponential boxes



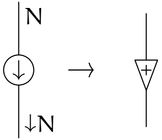
$$\frac{A \vdash a_1 : ?P_1 , \ldots , a_n : ?P_n , b : N}{\text{big stuff} \vdash a'_1 : ?P_1 , \ldots , a'_n : ?P_n , c : !N}$$

The translation:

derelictions and shifts

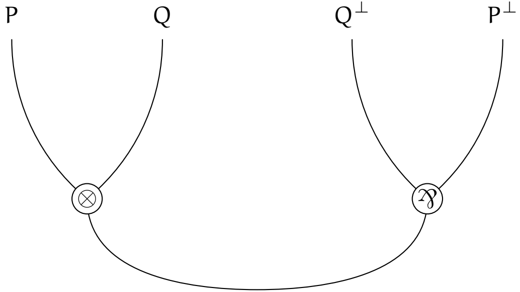


$$\frac{A \vdash \Gamma\,,\, a : P}{\nu a.(A \mid b(a)) \vdash \Gamma\,,\, b : ?P}$$

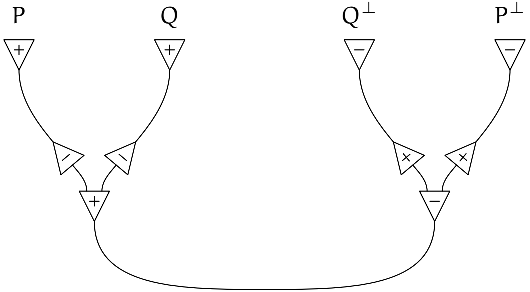


$$\frac{A \vdash \Gamma\,,\, a : N}{\nu a.(A \mid \bar{b}(a)) \vdash \Gamma\,,\, b : \downarrow N}$$

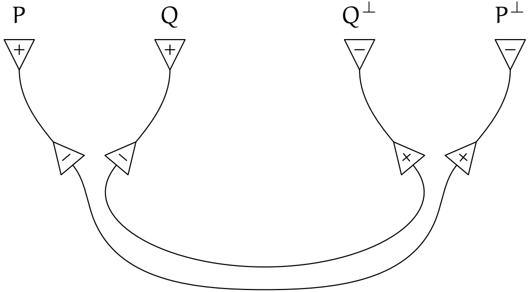
Simulation: $\otimes / \parr$



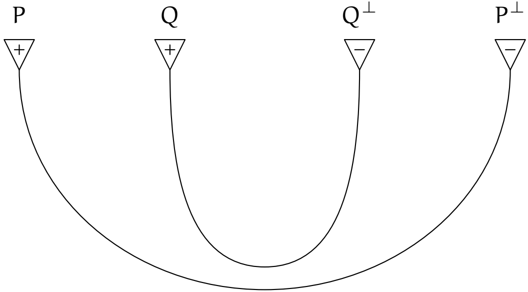
Simulation: $\otimes / \parr$



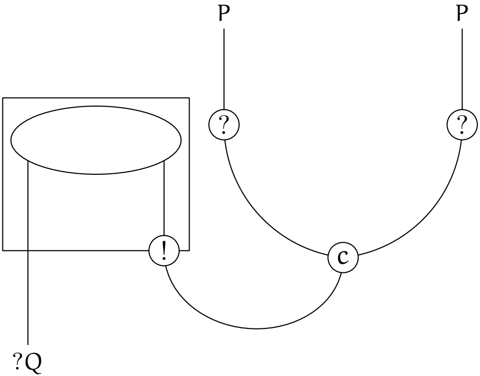
Simulation: $\otimes / \parr$



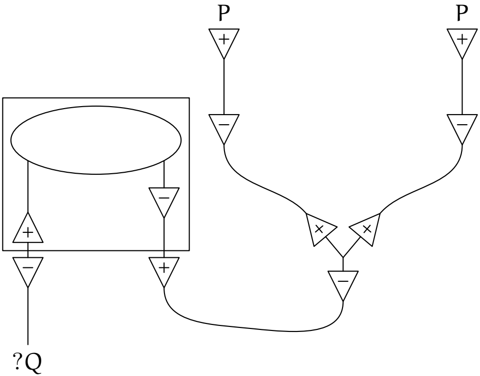
Simulation: $\otimes / \parr$



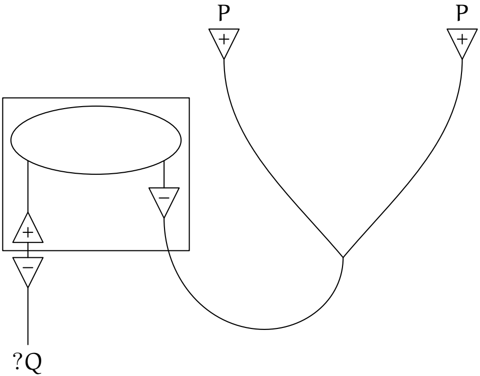
Simulation: exponentials



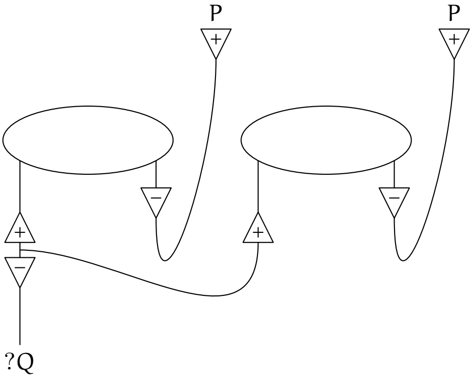
Simulation: exponentials



Simulation: exponentials



Simulation: exponentials

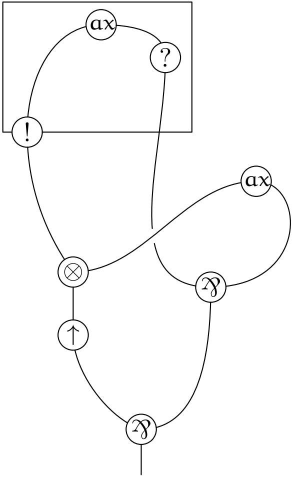


Correctness criterion

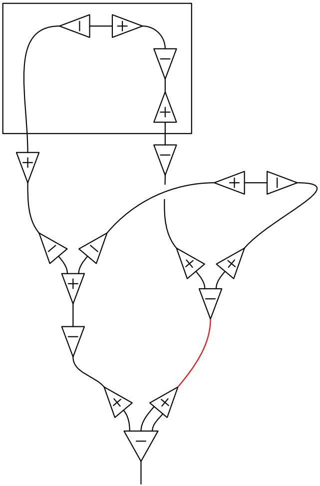
A process P is reduced from an MELL$\updownarrow$ proof structure iff

- P has a form f
- every non-binary channel is $\sharp$
- every channel is connected to one negative node
- if P has an action $u(x)$ and $f(u)$ is $\sharp$, then u crosses a box that contains $u(x)$

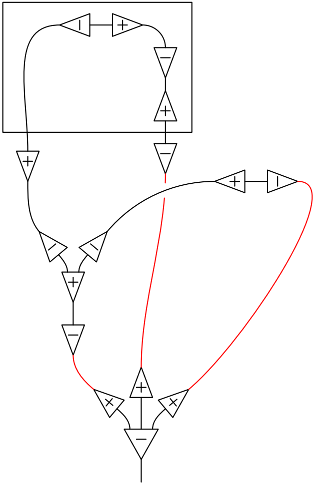
Reducing inside proofs



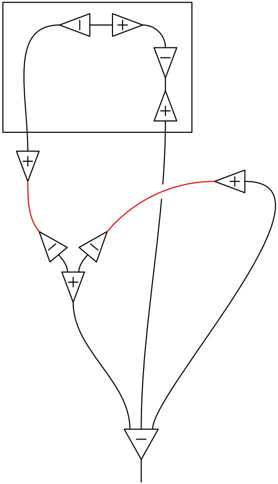
Reducing inside proofs



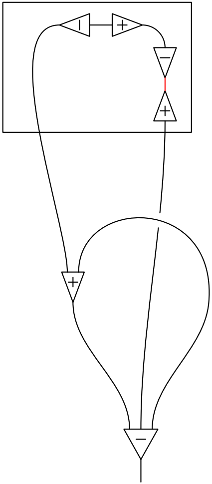
Reducing inside proofs



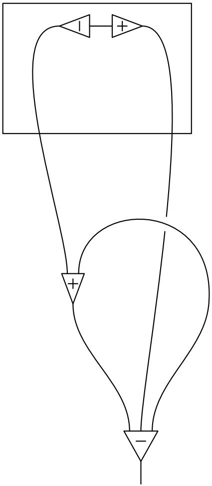
Reducing inside proofs



Reducing inside proofs

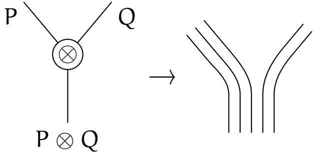


Reducing inside proofs



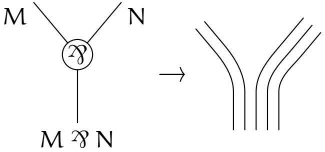
Reduced translation:
multiplicatives

$\otimes$ is the parallel composition:



$$\frac{A \vdash \Gamma\,,\, \tilde{a} : P \qquad B \vdash \tilde{b} : Q\,,\, \Delta}{A \mid B \vdash \Gamma\,,\, \tilde{a}\tilde{b} : P \otimes Q\,,\, \Delta}$$

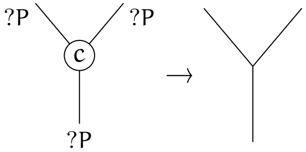
$\parr$ has no effect:



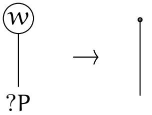
$$\frac{A \vdash \Gamma\,,\, \tilde{a} : M\,,\, \tilde{b} : N}{A \vdash \Gamma\,,\, \tilde{a}\tilde{b} : M \parr N}$$

Reduced translation:

contraction and weakening



$$\frac{A \vdash \Gamma\,,\, a : ?P\,,\, b : ?P}{A[a/b] \vdash \Gamma\,,\, a : ?P}$$

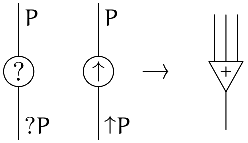


$$\frac{A \vdash \Gamma}{A \vdash \Gamma\,,\, a : ?P} \quad (a \text{ is fresh})$$

Reduced translation:

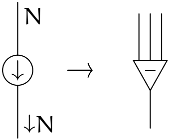
derelictions and shifts

Negative shifts become emissions:



$$\frac{A \vdash \Gamma\,,\, \tilde{a} : P}{\nu\tilde{a}.(A \mid b(\tilde{a})) \vdash \Gamma\,,\, b : ?P}$$

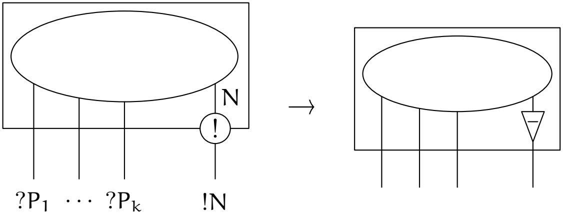
Positive shifts become receptions:



$$\frac{A \vdash \Gamma\,,\, \tilde{a} : N}{\nu\tilde{a}.(A \mid \bar{b}(\tilde{a})) \vdash \Gamma\,,\, b : \downarrow N}$$

Reduced translation:
exponential boxes

$!N$ is now translated as $\sharp\downarrow N$:



$$\frac{A \vdash a_1 : ?P_1 , \ldots , a_n : ?P_n , \tilde{b} : N}{!\nu\tilde{b}.(A \mid c(\tilde{b})) \vdash a_1 : ?P_1 , \ldots , a_n : ?P_n , c : !N}$$

One word about ludics

A proof

$$\dfrac{\dfrac{\dfrac{}{\vdash \uparrow P_1 , \downarrow P_1^\perp} \qquad \dfrac{}{\vdash \uparrow P_2 , \downarrow P_2^\perp}}{\vdash \uparrow P_1 , \uparrow P_2 , \downarrow P_1^\perp \otimes \downarrow P_2^\perp} \qquad \dfrac{}{\vdash \uparrow P_3 , \downarrow P_3^\perp}}{\dfrac{\vdash \uparrow P_1 , \uparrow P_2 , \uparrow P_3 , \downarrow P_1^\perp \otimes \downarrow P_2^\perp \otimes \downarrow P_3^\perp}{\vdash \uparrow P_1 \parr \uparrow P_2 \parr \uparrow P_3 \parr \uparrow (\downarrow P_1^\perp \otimes \downarrow P_2^\perp \otimes \downarrow P_3^\perp)}}$$

A dessin

$$\dfrac{\dfrac{\dfrac{\vdots}{\xi 01 \vdash \xi 1} \qquad \dfrac{\vdots}{\xi 02 \vdash \xi 2} \qquad \dfrac{\vdots}{\xi 03 \vdash \xi 3}}{\vdash \xi 0 , \xi 1 , \xi 2 , \xi 3}}{\xi \vdash}$$

A dessein

$$\dfrac{\dfrac{\mathfrak{Fax}_{\xi 01 \vdash \xi 1} \qquad \mathfrak{Fax}_{\xi 02 \vdash \xi 2} \qquad \mathfrak{Fax}_{\xi 03 \vdash \xi 3}}{(+ , \xi 0 , \{1 , 2 , 3\})}}{(- , \xi , \{0 , 1 , 2 , 3\})}$$

What we have

A pleasant translation of proof nets

- that knows about focalisation
- reminiscent of ludics and game semantics

A natural notion of typing that

- captures proof structures
- ensures strong normalisation

What comes next

Several topics not discussed here:

- The additives (not so nice)
- Typed λ-calculus
- Realisability and phase semantics

Ongoing work and directions:

- More links with ludics and games
- What about LLP and $\lambda\mu$-calculus ?
- Questions of sequentialisation

Thanks for coming.