An algebraic process calculus

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Abstract

We present an extension of the πI-calculus with formal sums of terms. The study of the properties of this sum reveals that its neutral element can be used to make assumptions about the behaviour of the environment of a process. Furthermore, the formal sum appears as a fundamental construct that can be used to decompose both internal and external choice. From these observations, we derive an enriched calculus that enjoys a confluent reduction which preserves the testing semantics of processes. This system is shown to be strongly normalising for terms without replication, and the study of its normal forms provides a fully abstract trace semantics for testing of πI processes.

1. Introduction

The point of this paper is to define a meaningful notion of normalisation for process calculi. Normalisation is not an obvious idea in the context of concurrency because of the non-determinism that is present in most process calculi, and it makes it crucial to distinguish two related notions for term languages: *execution*, which is a relation that describes the intended dynamics of a term considered as a program in a given model of computation, and *evaluation*, which is a relation that preserves the computational meaning of terms while simplifying them (in some sense).

In the λ-calculus, the standard notion of evaluation is β-reduction, which is confluent and strongly normalising for typed terms, while execution refers to particular evaluation strategies, or particular abstract machines. In the π-calculus, the dynamics of terms is usually given either as a labelled transition system or as a reduction relation up to some structural congruence; we refer to the latter form as execution, since it represents the way a process actually runs.

A problem in the search for semantics for processes is that there is no related notion of evaluation. For instance, consider the most straightforward case of non-determinism: $p := (\nu a)(a.p_1|a.p_2|\bar{a})$. This process obviously has two possible reductions: $p \to p_1$ and $p \to p_2$, assuming a does not occur in p_1 or p_2. We cannot say that the *value* of p is that of p_1 or p_2, since this would either lose information about p or imply that p_1 and p_2 have the same value; the most we can say is that the value of p is "either that of p_1 or that of p_2". This cannot be expressed in general in the language of processes, therefore we introduce in the language a formal sum, so that we can formally get the equation $p = p_1 + p_2$.

This sum has pleasant properties like bilinearity of parallel composition (i.e. $(p + q) \mid r = p \mid r + q \mid r$) and linearity of hiding and even prefixes. However, the real gain in expressiveness comes when considering the neutral element of this sum, that we will write 0. This element has to satisfy the equation $p \mid 0 = 0$, which is unusual in a process calculus. As we will see, it has a meaningful interpretation in terms of testing semantics, where an occurrence of 0 at top level means success. Moreover, a study of the properties of 0 shows that standard actions are not actually linear, since $\alpha.0 \neq 0$, but rather affine, and we can extract a notion of purely linear action that does satisfy the rule $\hat{\alpha}.0 = 0$. The introduction of 0 and linear actions provides enough expressiveness so that we can define for our calculus a system of reduction rules that is normalising and preserves the testing semantics.

Contributions After recalling the syntax and operational semantics of the πI-calculus, in section 3 we define the algebraic πI-calculus and provide intuition on the meaning of the new constructs. Execution is presented in two equivalent forms, namely as a reduction up to structural congruence, and as a labelled transition system, and the definition of fair testing is deduced. In section 4, we introduce the system of evaluation rules over processes and we prove its soundness with respect to the fair testing semantics. Examples show that these rules can be used to effectively compute process equivalences. In section 5 we prove that this system is locally confluent and that it is strongly normalising for terms without replication, and in section 6 we study the

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normal forms of processes for this reduction. It appears that normal forms are sums of traces augmented with *inaction* information, i.e. an extended trace is a sequence of actions followed by the information that some actions could have been performed but will not. This provides a simple trace semantics that is proved to be fully abstract for finite terms. Deciding the observational preorder over traces is straightforward, and this provides an effective way to decide the testing preorder over finite terms. In section 7, we discuss how our approach can be extended to handle a wider class of processes and different forms of equivalence.

Related work The relevance of linearity in the study of processes has been known and studied for some time. It was first stressed by Kobayashi, Pierce and Turner [10] who showed that linear typing of processes leads to significant improvements in the theory of the calculus, in particular by providing a form of partial confluence. Yoshida, Berger and Honda [17] propose a type system for strong normalisation that uses linearity information to guarantee normalisation using an "extended reduction" that allows processes to be reduced in a more liberal way than by standard execution. The use of linear actions, i.e. actions that will be performed exactly once, is a crucial element in these studies, as in the present paper. In a sense, our approach allows to keep the refinement of linearity without the need for typing.

Using a formal sum to express non-determinism without losing information is a natural idea. It is used for instance in Ehrhard and Regnier's differential λ-calculus [6]: in order to get a framework in which differential operators can be defined, the λ-calculus is extended with linear combinations of terms, which appear as a natural and semantically justified way of representing non-determinism without renouncing to denotational semantics. This idea is further justified by the possibility of embedding a finitary π-calculus in differential interaction nets [5]. Our algebraic process calculus should provide insight on the relationships between this use of differential nets and process calculi. One can also expect that extant work on the links between the λ-calculus and π-calculi could be extended to the differential case in our formalism.

The study of such connections is deferred to further works, since the focus of this paper is the study of the formal sum itself in the context of process calculi. Moreover, establishing a formal link with the differential λ-calculus requires the introduction of linear coefficients in our processes. Vaux's study of linear combinations of λ-terms [16] shows that one should be careful even in the very regular setting of the λ-calculus, so extra complications are to be expected here, notably when seeking confluence. Besides, the meaning of coefficients is unclear from the point of view of semantics of concurrency, hence no reference semantics is there yet to guide us in making linear combinations more than a formal construction. On the other hand, the sum and zero were justified by the study of behaviours in testing semantics, as shown in particular in the author's work on concurrent realisability [1, 2].

Giving proper status to 0 is necessary if the formal sum needs to have the actual structure of a sum; its interpretation as a testing token and the fact that it greatly improves the theory of the calculus should not come as a surprise, for instance a similar approach lead to the definition of ludics [8] from the sequent calculus of linear logic.

The trace semantics we derive from the evaluation rules has a simple formulation, and the interpretation of "inactions" that make it fully abstract for fair and must testing is new, as far as the author knows. However, several forms of extended trace semantics exist in the literature. Among those presented in van Glabbeek's survey [15] of process semantics, our semantics, when restricted to standard terms (as of definition 4) is equivalent to the so-called *readiness semantics* of Olderog and Hoare [11]: a readiness trace is a sequence of actions followed by the set of actions that can be performed next, and inactions do provide this information, at least when considering standard processes.

Other axiomatisations of various equivalences over the π-calculus have been proposed in previous works (see [14] for instance). In particular, Hennessy's fully abstract denotational semantics of the π-calculus [9] contains an axiomatisation of testing equivalences that is comparable to the present work. Indeed, many of the equations in Hennessy's axiomatisation of the testing preorder can be deduced from equivalence or reduction rules in our system. His approach is based on the distinction between internal choice (which is much like our algebraic sum) and external choice; we believe our system to be more fundamental since our sum and its neutral element enjoy simpler algebraic properties while being expressive enough to represent both kinds of choices, as discussed in section 7.1. A supporting argument in this respect is the fact that we need less axioms because we do not have to distinguish two sums.

Note on notation The notations used here are in conflict with the tradition of process calculi, but we believe this conflict to be justified. The algebraic sum that we introduce is written $p+q$, because + is the only sensible symbol for this. It is significantly different from the external choice operator usually written +; this choice operator will be written & here by analogy with linear logic. The neutral element of the sum, or "null process" is naturally written 0 for coherence. The process that does nothing is called "neutral process" and is written 1, since it is the neutral element of parallel composition, which is a multiplicative operation (that distributes over the sum, in particular).

2. The πI-calculus

We start from Sangiorgi's polyadic π-calculus with internal mobility, or πI-calculus [13], without choice, and we use fair testing as the observational equivalence over processes. We briefly recall the definitions of this calculus here for reference. The decision not to include choice in the main part of our study is for simplicity; as shown in section 7.1, including it would be harmless.

Definition 1. Let $\boldsymbol{N}$ be an infinite set of names. Let $\boldsymbol{P} = \{\downarrow, \uparrow\}$ be the set of polarities, let ε range over $\boldsymbol{P}$. Terms are generated by the following grammar:

$$
\begin{array}{lll}
p, q := u^{\varepsilon}(x_1 \dots x_n).p & & \text{action} \\
\, !u^{\varepsilon}(x_1 \dots x_n).p & & \text{replicated action} \\
\, (\nu x)p & & \text{hiding} \\
\, 1 \qquad p \mid q & & \text{parallel composition}
\end{array}
$$

In $u^{\varepsilon}(x_1 \dots x_n)$ we have $u, x_1, \dots, x_n \in \boldsymbol{N}$ and these names are pairwise distinct. $(\nu x)p$ is a binder for x, $u^{\varepsilon}(x_1 \dots x_n).p$ is a binder for $x_1, \dots, x_n$, and terms are considered up to injective renaming of bound names.

We often write $(\nu x_1 \dots x_n)p$ for $(\nu x_1) \dots (\nu x_n)p$, and $\vec{x}$ for $x_1 \dots x_n$. The ε in actions is the polarity, by convention $\downarrow$ is positive and corresponds to input, $\uparrow$ is negative and corresponds to output. Subsequently $u(\vec{x})$ stands for $u^{\downarrow}(\vec{x})$ and $\bar{u}(\vec{x})$ stands for $u^{\uparrow}(\vec{x})$. If α is an action, $\bar{\alpha}$ stands for the action with opposite polarity.

Definition 2. Structural congruence is generated by the following rules, where z does not occur free in p:

$$
\begin{array}{lll}
p \mid q \equiv q \mid p & p \mid (q \mid r) \equiv (p \mid q) \mid r & p \mid 1 \equiv p \\
(\nu xy)p \equiv (\nu yx)p & (\nu z)(p \mid q) \equiv p \mid (\nu z)q & (\nu x)1 \equiv 1 \\
!\alpha.p \equiv \alpha.(p \mid !\alpha.p) & &
\end{array}
$$

Terms up to structural congruence are called processes. Execution is the smallest relation over processes that is stable under parallel composition and hiding and contains the rule

$$\bar{u}(\vec{x}).p \mid u(\vec{x}).q \to (\nu\vec{x})(p \mid q)$$

The rule above is written with the same sequence of names $\vec{x}$ in both actions; this is not a restriction, since terms are considered up to renaming of bound names.

Definition 3. Assume $\boldsymbol{N}$ contains an element ω that is always used as the action $\omega^{\uparrow}()$, written $\bar{\omega}$. A term p is *accepting* if there is a p' such that $p \equiv \bar{\omega} \mid p'$. For terms p and q, *p passes the test q* if for any execution $p \mid q \to^* r$ there is an execution $r \to^* s$ with s accepting. The testing preorder is defined as $p \sqsubseteq_I q$ if for all r, if p passes the test r then q passes the test r.

This preorder is written $\sqsubseteq_I$ to distinguish it from the preorder $\sqsubseteq$ defined below. Their equivalence is theorem 4.

3. Introducing the sum

We now redefine the calculus, with its operational semantics and testing, extending it with sum and zero.

Definition 4. Let $\boldsymbol{N}$ be the infinite set of names. Let $\boldsymbol{P} = \{\uparrow, \downarrow\}$ be the set of polarities, let ε range over $\boldsymbol{P}$. Terms are generated by the following grammar:

$$
\begin{array}{lll}
p, q := u^{\varepsilon}(x_1 \dots x_n).p & & \text{affine action} \\
\, !u^{\varepsilon}(x_1 \dots x_n).p & & \text{replicated action} \\
\, \hat{u}^{\varepsilon}(x_1 \dots x_n).p & & \text{linear action} \\
\, (\nu x)p & & \text{hiding} \\
\, 1 \qquad p \mid q & & \text{parallel composition} \\
\, 0 \qquad p + q & & \text{formal sum}
\end{array}
$$

In $u^{\varepsilon}(x_1 \dots x_n)$ and $\hat{u}^{\varepsilon}(x_1 \dots x_n)$ we have $u, x_1, \dots, x_n \in \boldsymbol{N}$ and these names are pairwise distinct. $(\nu x)p$ is a binder for x, $u^{*\varepsilon}(x_1 \dots x_n).p$ is a binder for $x_1, \dots, x_n$, terms are considered up to injective renaming of bound names. We write $\mathrm{fn}(p)$ for the set of free names of the term p.

A term is *quasi-standard* if it has no occurrence of + or linear actions, it is *standard* if has no occurrence of 0 either.

The same notations as above are used for actions and name sequences. We use Greek letters $\alpha, \beta, \gamma \dots$ to represent actions. For cases where linearity is ignored, $\mathring{\alpha}$ stands for either α or $\hat{\alpha}$.

3.1. Execution and observation

Definition 5. Structural congruence is that of definition 2 extended with the following rules:

$$p \mid (q + r) \equiv p \mid q + p \mid r \qquad (\nu x)(p + q) \equiv (\nu x)p + (\nu x)q$$

Note that these structural equivalence rules do not provide the structure one could expect from a sum: it is neither associative nor commutative, and 0 is neither neutral for the sum nor absorbing for composition, since there is no rule that involves it. Indeed, structural congruence contains the minimum required for defining the execution relation.

Definition 6. Execution is the smallest relation over processes that is stable under sum, parallel composition and hiding and contains the rule

$$\mathring{\bar{u}}(\vec{x}).p \mid \mathring{u}(\vec{x}).q \to (\nu\vec{x})(p \mid q)$$

Non-standard processes are not considered as proper computational objects, i.e. their execution is not intended to be seen as computation in any realistic model. However, it does make sense to interpret each term in a sum as a possible state of a given system, this way an execution step of the sum is an execution step of one of its possible states. These "possible states" are formally called *slices*:

Definition 7. A *slice* is a term where any sum occurs under an action. The set $\mathfrak{s}(p)$ of the slices of a term p is defined inductively as

$$\begin{aligned} \mathfrak{s}(p) &:= \{p\} \qquad \text{if } p \text{ is a constant or an action} \\ \mathfrak{s}((\nu x)p) &:= \{ (\nu x)s \mid s \in \mathfrak{s}(p) \} \\ \mathfrak{s}(p+q) &:= \mathfrak{s}(p) \cup \mathfrak{s}(q) \\ \mathfrak{s}(p \mid q) &:= \{ (s \mid t) \mid s \in \mathfrak{s}(p), t \in \mathfrak{s}(q) \} \end{aligned}$$

Interpreting the sum as pure non-deterministic choice is adequate. The meaning of 0 and linear actions requires more explanation. A sum of terms $p_1 + \ldots + p_n$ can be understood as the fact that, after some execution steps of a process p, this process may have reached either of the states $p_1, \ldots, p_n$, depending on non-deterministic choices that have occurred. In this context, 0 and the linearity annotations are used to make such choices ahead of time, by making assumptions on what will happen:

- a linear action $\hat{\alpha}.p$ is the action $\alpha.p$ with the assumption that it will actually be consumed at some point of the execution,
- 0 at top level in a term means that contradicting assumptions were made.

This interpretation is the key of the present study, and we will see that it allows to decompose standard actions as $\alpha.p \simeq \hat{\alpha}.p + \alpha.0$. In words, this simply means that a standard action either will be consumed at some point, or will never be consumed. From a formal point of view, this justifies that standard actions are *affine*, $\hat{\alpha}.p$ being the linear part and $\alpha.0$ being the constant.

The definition of slices does not consider sums that occur under an action, because these are non-deterministic choices that will be made when the action is consumed. Therefore the execution of a slice does not in general lead to a slice, for this reason we define slicing execution:

Definition 8. *Slicing execution* is the relation $\Rightarrow$ over slices such that $s \Rightarrow t$ if there is a p such that $s \to p$ and $t \in \mathfrak{s}(p)$. A *run* is a sequence of slicing executions. A *maximal run* of s is either a finite sequence $(s_0, \ldots, s_n)$ such that $s_0 = s$, $s_i \Rightarrow s_{i+1}$ for all $i < n$, and s_n is irreducible, or an infinite sequence $(s_i)_{i \in \mathbb{N}}$ such that $s_0 = p$ and $s_i \Rightarrow p_{i+1}$ for all i.

As mentioned above, linear actions and 0 are used to express assumptions on the execution of a process. This is formalised as the following notion of consistency:

Definition 9. A slice s is *null* if there is an s' such that $s \equiv 0 \mid s'$. A run (s_i) is *consistent* if none of the s_i is null and for any linear action occurring at top level in some s_i there is an execution step $s_j \Rightarrow s_{j+1}$ where it is consumed. A slice is consistent if it has a consistent maximal run. A term or process p is consistent if it has a consistent slice.

Lemma 1. *Inconsistency is preserved by structural congruence, execution and slicing execution.*

Consistency is what we need to get a meaningful operational interpretation of terms with sums: an execution in the standard sense corresponds to a consistent run. Of course this is not computationally realistic, since it amounts to having an oracle that could decide if a given process is consistent. This is not really a problem since standard terms, which are the actual object of study, are always consistent and execution of standard terms leads to standard terms.

Definition 10. A slice s is *successful* if for any run $s \Rightarrow^* s'$, there is a run $s' \Rightarrow^* s''$ with s'' inconsistent. Slices s and t are orthogonal, written $s \perp t$, if $s \mid t$ is successful. Processes p and q are orthogonal if all slices of $p \mid q$ are successful.

For a set A of processes, let $A^\perp = \{ q \mid \forall p \in A, p \perp q \}$. A behaviour is a set of processes A such that $A^{\perp\perp} = A$. The testing preorder is defined as $p \sqsubseteq q$ if $\{p\}^\perp \subseteq \{q\}^\perp$. The associated testing equivalence is written $p \simeq q$.

The term "successful" might seem in contradiction with the definition that requires inconsistency. We consider it as success because it corresponds to the case where a pair of processes are in the orthogonality relation. It can also be considered as success on the part of the environment, i.e. the opponent, in game semantics terms. This is all a matter of conventions, anyway.

This definition of testing by orthogonality is not fundamentally new, we use it because it provides a good structure to the set of processes. The power of this simple technique was used for instance in the functional framework in Krivine's realisability [3, 4], it was also employed in a different way by Rathke, Sassone and Sobociński [12] as a way of defining observables in process algebra.

Lemma 2. *Let p and q be two terms. $p \sqsubseteq q$ if and only if for all slices s, if $s \perp p$ then $s \perp q$.*

3.2. Labelled transition system

We now redefine the operational semantics of slices as a labelled transition system, strictly defined on terms without structural congruence. This will simplify proofs of observational equivalence by allowing us to reason only on sequences of transitions, without worrying about the precise syntax of terms.

Definition 11. The set L of labels consists of τ and actions $u^\varepsilon(x_1 \ldots x_n)$, with $u, x_1, \ldots, x_n$ pairwise distinct. Define:

$$\begin{array}{lll} \text{names:} & \mathrm{n}(\tau) = \emptyset & \mathrm{n}(u^\varepsilon(\vec{x})) = \{u, x_1, \ldots, x_n\} \\ \text{bound names:} & \mathrm{bn}(\tau) = \emptyset & \mathrm{bn}(u^\varepsilon(\vec{x})) = \{x_1, \ldots, x_n\} \end{array}$$

The labelled transition relation is the relation between slices and terms, labelled over L, defined by the rules of table 1

$$\frac{}{\mathring{\alpha}.p \xrightarrow{\alpha} p} \qquad \frac{}{!\alpha.p \xrightarrow{\alpha} p \mid !\alpha.p} \qquad \frac{p \xrightarrow{\bar{u}^{\varepsilon}(\vec{x})} p' \qquad q \xrightarrow{u^{\varepsilon}(\vec{x})} q'}{p \mid q \xrightarrow{\tau} (\nu\vec{x})(p' \mid q')}$$

$$\frac{p \xrightarrow{\ell} p' \qquad \mathrm{bn}(\ell) \cap \mathrm{fn}(q) = \emptyset}{p \mid q \xrightarrow{\ell} p' \mid q} \qquad \frac{p \xrightarrow{\ell} p' \qquad x \notin \mathrm{n}(\ell)}{(\nu x)p \xrightarrow{\ell} (\nu x)p'}$$

Table 1. Transition rules for slices

(and the symmetric rule for parallel composition). The slicing transition relation over slices is defined as $s \overset{a}{\Rightarrow} t$ if there is a term p such that $s \xrightarrow{a} p$ and $t \in \mathfrak{s}(p)$.

Definition 12. Affine contexts are defined by the grammar

$$C ::= [],\ C \mid p,\ p \mid C,\ C + p,\ p + C,\ (\nu x)C,\ \alpha.C,\ \hat{\alpha}.C$$

Execution contexts are contexts made of parallel compositions and hidings. Replacing [ ] by p in C is written $C[p]$.

Proposition 1. *For any slices s and t, there is a slicing execution $s \Rightarrow t$ if and only if there is a slice t' such that $t \equiv t'$ and the transition $s \overset{\tau}{\Rightarrow} t'$ holds.*

The proof can be found in the appendix. Thanks to this proposition, we can consider interchangeably slicing execution steps or slicing τ transitions.

Proposition 2. *The testing preorder is preserved by affine contexts.*

Proof. Let p and q be two terms such that $p \sqsubseteq q$, i.e. such that for any term s, if $p|s$ is successful then $q|s$ is successful.

Let r be a term. Let s be a term such that $(p + r) \mid s$ is successful. Then for all slices $t \in \mathfrak{s}(p) \cup \mathfrak{s}(r)$ and $s' \in \mathfrak{s}(s)$, $t \mid s'$ is successful. This implies that $p \mid s'$ is successful, so $q|s'$ is successful, i.e. for all slice $t \in \mathfrak{s}(q)$, $t|s'$ is successful. Therefore $(q + r) \mid s$ is successful, so $p + r \sqsubseteq q + r$.

Let r be a term, then for any term s, if $(p|r)|s$ is successful, then $p \mid (r \mid s)$ is successful (since success is preserved by structural congruence), so $q \mid (r \mid s)$ is successful (by hypothesis), so $(q \mid r) \mid s$ is successful. Hence $p \mid r \sqsubseteq q \mid r$.

Let x be a name. Let s be a term. Let y be a fresh name, that does not occur in p, q or s. Assume $(\nu x)p \mid s$ is successful, then by structural congruence $(\nu y)(p[y/x] | s)$ is successful. By the definition of success it is clear that $p[y/x] \mid s$ is successful. From $p \sqsubseteq q$ we can deduce $p[y/x] \sqsubseteq q[y/x]$, so $q[y/x] \mid s$ is successful, and so are $(\nu y)(q[y/x] \mid s)$, and $(\nu x)q \mid s$. Therefore $(\nu x)p \sqsubseteq (\nu x)q$.

Let α be an action. Let s be a slice such that $\alpha.p \mid s$ is successful. Consider a run $\alpha.q|s \Rightarrow^* r$. There are two cases:

- Either the action $\alpha.q$ is consumed, then there is a sequence of transitions $s \overset{\tau*}{\Rightarrow} s' \overset{\bar{\alpha}}{\Rightarrow} s''$ such that the run is decomposed as $\alpha.q \mid s \Rightarrow^* \alpha.q \mid s' \Rightarrow (\nu\vec{x})(q' \mid s'') \Rightarrow^* r$ with $q' \in \mathfrak{s}(q)$. For any slice $p' \in \mathfrak{s}(p)$ The run $\alpha.p \mid s \Rightarrow^* \alpha.p \mid s' \Rightarrow (\nu\vec{x})(p' \mid s'')$ holds too, and by hypothesis $\alpha.p \mid s$ is successful, so $p \mid s''$ is successful, and since $p \sqsubseteq q$ the term $q \mid s''$ is successful, so $q' \mid s''$ is successful and r has an inconsistent reduct.
- Or the action is not consumed, and the run can be written $\alpha.q|s \Rightarrow^* \alpha.q|s'$ with $s \Rightarrow^* s'$. If there is a sequence of transitions $s' \overset{\tau*}{\Rightarrow} s'' \overset{\bar{\alpha}}{\Rightarrow} t$ then we can extend the considered run into one that does consume the action and we are back to the previous case. Otherwise $\alpha.p|s'$ has an inconsistent reduct of the form $\alpha.p \mid s''$, and $\alpha.q \mid s''$ is inconsistent too.

Hence $\alpha.q|s$ is successful, from which we deduce $\alpha.p \sqsubseteq \alpha.q$. We prove $\hat{\alpha}.p \sqsubseteq \hat{\alpha}.q$ by the same argument, except that in the second case, if s' has no transition labelled $\bar{\alpha}$, then $\hat{\alpha}.q|s'$ is inconsistent because $\hat{\alpha}$ can never be consumed. ◻

4. Reduction rules

In the following sections, we focus mainly on the finite fragment of the calculus, in particular the reduction defined below says nothing about replicated actions. Nevertheless, the formulated results hold for terms with replications, unless stated. We implicitly consider replicated terms up to structural congruence, i.e. $!\alpha.p$ is the infinite term q such that $q = \alpha.(p \mid q)$. The question of extending this work to handle replication explicitly is discussed in section 7.2.

Definition 13. The equivalence relation $\cong$ over terms and the relation $\succ$ over terms up to $\cong$ are defined by the rules of table 2, in any affine context, except that no reduction rule applies in a subterm p such that $p \cong 0$.

The congruence $\cong$ allows every term to be written either as 0 or as a sum of slices with no 0 in active position. Such a presentation of a term is called an *expanded form*. The side conditions are easy to check on these forms.

In definition 13 and rules A and L2, the clumsy treatment of 0 is needed to avoid trivial loops like $0 \cong 0 \mid p \succ 0 \mid p' \cong 0$ (whenever $p \succ p'$) or $\alpha.0 \succ \hat{\alpha}.0 + \alpha.0 \succ 0 + \alpha.0 \cong \alpha.0$ or $\hat{\alpha}.p \cong \hat{\alpha}.(p + 0) \succ \hat{\alpha}.p + \hat{\alpha}.0 \succ \hat{\alpha}.p + 0 \cong \hat{\alpha}.p$. Another possibility would be to turn the related structural equivalence rules for 0 into reduction rules, but this would make the system heavier. The condition in N3 reflects the fact that actions of different arities cannot interact. If the terms were typed so that each name is used with a fixed arity, the rule could be simplified into $(\nu u)u^{\varepsilon}(\vec{x}).0 \succ 1$.

It is clear that if a reduction $p \succ q$ holds, then a similar reduction holds in the expanded form, possibly duplicated

Structural equivalence, with $z \notin \text{fn}(p)$:

$$(\nu xy)p \cong (\nu yx)p \qquad (\nu x)1 \cong 1$$
$$(\nu z)(p \mid q) \cong p \mid (\nu z)q \qquad (\nu x)(p + q) \cong (\nu x)p + (\nu x)q$$
$$p \mid q \cong q \mid p \qquad p \mid (q \mid r) \cong (p \mid q) \mid r \qquad p \mid 1 \cong p$$
$$p + q \cong q + p \qquad p + (q + r) \cong (p + q) + r \qquad p + 0 \cong p$$
$$p \mid (q + r) \cong p \mid q + p \mid r \qquad p \mid 0 \cong 0$$

Normalisation:

S	$p + p \succ p$	
A	$\alpha.p \succ \hat{\alpha}.p + \alpha.0$	if $p \not\cong 0$
L1	$\hat{\alpha}.0 \succ 0$	
L2	$\hat{\alpha}.(p + q) \succ \hat{\alpha}.p + \hat{\alpha}.q$	if $p \not\cong 0$ and $q \not\cong 0$
I1	$\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q \succ \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q) + (\nu\vec{x})(p \mid q)$	where $\vec{x} = \text{bn}(\alpha)$, if $p \not\cong 0$ and $q \not\cong 0$
I2	$\hat{\alpha}.p \mid \hat{\beta}.q \succ \hat{\alpha}.(p \mid \hat{\beta}.q) + \hat{\beta}.(\hat{\alpha}.p \mid q)$	if $\beta \neq \bar{\alpha}$
I3	$\hat{\alpha}.p \mid \beta.0 \succ \hat{\alpha}.(p \mid \beta.0)$	
I4	$\alpha.0 \mid \alpha.0 \succ \alpha.0$	
I5	$\alpha.0 \mid \bar{\alpha}.0 \succ 0$	
N1	$(\nu v)\hat{u}^{\varepsilon}(\vec{x}).p \succ \hat{u}(\vec{x}).(\nu v)p$	if $u \neq v$
N2	$(\nu u)\hat{u}^{\varepsilon}(\vec{x}).p \succ 0$	
N3	$(\nu u)n \succ 1$	if n is a composition of terms of the form $u^{\varepsilon}(\vec{x}).0$ that contains no dual actions

Table 2. Rules for normalisation

because of the distribution rule. Therefore we can restrict to expanded forms without loss of generality.

Theorem 1 (soundness)**.** *Let p and q be two terms. If $p \cong q$ or $p \succ q$ then $p \simeq q$.*

Proof. Let us sketch the soundness proof of rule A. We have to show that for any slice q, $\alpha.p \perp q$ if and only if $\alpha.0 \perp q$ and $\hat{\alpha}.p \perp q$. First assume that $\alpha.p \perp q$. Consider a run $\hat{\alpha}.p \mid q \Rightarrow^* r$. We can assume that the action of $\hat{\alpha}.p$ is consumed, i.e. that the considered run is $\hat{\alpha}.p|q \Rightarrow^* \hat{\alpha}.p|q_1 \Rightarrow (\nu\vec{x})(p \mid q_2) \Rightarrow^* r$, with $\vec{x} = \text{bn}(\alpha)$. Thus there is a run $\alpha.p \mid q \Rightarrow^* \alpha.p \mid q_1 \Rightarrow (\nu\vec{x})(p \mid q_2) \Rightarrow^* r$ and by hypothesis there is a run $r \Rightarrow^* s$ with s inconsistent. Therefore $\hat{\alpha}.p \perp q$. Now consider a run $\alpha.0 \mid q \Rightarrow^* r$. If this run consumes the action α, then the reduct r has 0 in active position and it is inconsistent, otherwise the run is $\alpha.0 \mid q \Rightarrow^* \alpha.0 \mid q'$. If there is an run of $\alpha.0 \mid q'$ that consumes α then the considered run can be extended to an inconsistent state. Otherwise any maximal run of $\alpha.0|q'$ is a maximal run of q' in parallel with $a.0$. From this we deduce a maximal run of $\alpha.p|q'$, which is inconsistent by hypothesis. Since $\alpha.p$ is left untouched, the source of inconsistency in $\alpha.p|q'$ also applies in $\alpha.0|q'$, and $\alpha.0 \mid q'$ is inconsistent too. Therefore we also have $\alpha.0 \perp q$.

Now assume that $\hat{\alpha}.p \perp q$ and $\alpha.0 \perp q$, and consider a run $\alpha.p|q \Rightarrow^* r$. If this run consumes $\alpha.p$ or can be extended into a run that does, then there is a run of $\hat{\alpha}.p \mid q$ that leads to r by the same reduction steps, by hypothesis $\hat{\alpha}.p \mid q$ is successful so r has an inconsistent reduct. Otherwise we have $r = \alpha.p \mid q'$, and the run $\alpha.0 \mid q \Rightarrow^* \alpha.0 \mid q'$ holds, and $\alpha.0 \mid q'$ reduces to an inconsistent term $\alpha.0 \mid q''$ without ever consuming $\alpha.0$. This proves that the source of inconsistency is q'', and $\alpha.p \mid q''$ is inconsistent too, therefore $\alpha.p \perp q$.

So we split the runs of $\alpha.p \mid s$ into those that consume α or may do it, and those that do not. Similar arguments are used for the other rules. For instance, in the case of I1, we split the runs of $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q \mid s$ according to the first transition of $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q$, which can be α, $\bar{\alpha}$ or τ. For I2, the same applies without the τ transition. The full proof for all rules can be found in the appendix. □

As this proof illustrates, the combined use of linearity and the formal sum allows our calculus to admit many more simplifications than other calculi. This is because the algebraic sum enjoys the good properties of both internal choice (like the distributivity $p \mid (q + r) \simeq p \mid q + p \mid r$) and external choice (like the interleaving rules I1 and I2), together with other regularity properties, like linearity of actions in the rule $\hat{\alpha}.(p + q) \simeq \hat{\alpha}.p + \hat{a}.q$.

Examples The soundness of our rewrite system shows that it can be used to prove observational equivalence of processes by purely computational methods. For instance we can formulate the interference between two identical actions in a natural way:

$$\begin{aligned}\alpha.p \mid \alpha.q &\simeq \hat{\alpha}.p \mid \hat{\alpha}.q + \hat{\alpha}.p \mid \alpha.0 + \alpha.0 \mid \hat{\alpha}.q + \alpha.0 \mid \alpha.0 \\ &\simeq \hat{\alpha}.(p \mid \hat{\alpha}.q + \hat{\alpha}.p \mid q + p \mid \alpha.0 + \alpha.0 \mid q) + \alpha.0 \\ &\simeq \hat{\alpha}.(p \mid (\hat{\alpha}.q + \alpha.0) + (\hat{\alpha}.p \mid \alpha.0) \mid q) + \alpha.0 \\ &\simeq \alpha.(p \mid \alpha.q + \alpha.p \mid q)\end{aligned}$$

Interaction without interference can also be computed:

$$\begin{aligned}&(\nu u)(u(\vec{x}).p \mid \bar{u}(\vec{x}).q) \\ &\simeq (\nu u)(\hat{u}(\vec{x}).p \mid \hat{\bar{u}}(\vec{x}).q) + (\nu u)(\hat{u}(\vec{x}).p \mid \bar{u}(\vec{x}).0) \\ &\quad + (\nu u)(u(\vec{x}).0 \mid \hat{\bar{u}}(\vec{x}).q) + (\nu u)(u(\vec{x}).0 \mid \bar{u}(\vec{x}).0)\end{aligned}$$

$$\begin{aligned}&\simeq (\nu u)\hat{u}(\vec{x}).(p \mid \hat{\bar{u}}(\vec{x}).q) + (\nu u)\hat{u}(\vec{x}).(\hat{\bar{u}}(\vec{x}).p \mid q) + (\nu u\vec{x})(p \mid q) \\ &\quad + (\nu u)\hat{u}(\vec{x}).(p \mid \bar{u}(\vec{x}).0) + (\nu u)\hat{\bar{u}}(\vec{x}).(u(\vec{x}).0 \mid q) + 0 \\ &\simeq 0 + 0 + (\nu u\vec{x})(p \mid q) + 0 + 0 + 0 \simeq (\nu u\vec{x})(p \mid q)\end{aligned}$$

From this we can show that the formal sum is equivalent to pure internal choice as follows, where u is not free in p or q:

$$\begin{aligned}(\nu u)(u.p \mid u.q \mid \bar{u}) &\simeq (\nu u)(u.(p \mid u.q + u.p \mid q) \mid \bar{u}) \\ &\simeq p \mid (\nu u)u.q + (\nu u)u.p \mid q \simeq p \mid 1 + 1 \mid q \simeq p + q\end{aligned}$$

where $(\nu u)u.p \simeq 1$ is computed as $(\nu u)u.p \simeq (\nu u)\hat{u}.p + (\nu u)u.0 \simeq 0 + 1 \simeq 1$.

The rewrite rules can also be used to detect deadlocks. Consider the symptomatic term $p = a.\bar{b} \mid b.\bar{a}.q$. We can easily show the general formula

$$\alpha.p \mid \beta.q \simeq \alpha.0 \mid \beta.0 + \hat{\alpha}.(p \mid \beta.q) + \hat{\beta}.(\alpha.p \mid q)$$

so we get $p \simeq a.0 \mid b.0 + \hat{a}.(\bar{b} \mid b.\bar{a}.q) + \hat{b}.(a.\bar{b} \mid \bar{a}.q)$. Therefore $(\nu ab)p$ is equal to the same sum with (νab) at top level in each summand, but every summand except the first one starts with a linear action in a or b so we get

$$(\nu ab)p \simeq (\nu ab)(a.0 \mid b.0) \simeq (\nu a)a.0 \mid (\nu b)b.0 \simeq 1 \mid 1 \equiv 1$$

which proves purely by computation that $(\nu ab)p$ is a deadlock. If we only restrict the name a we also deduce something about the process:

$$\begin{aligned}(\nu a)p &\simeq (\nu a)a.0 \mid b.0 + (\nu a)\hat{b}.(a.\bar{b} \mid \bar{a}.q)\\ &\simeq b.0 + \hat{b}.(\nu a)(a.\bar{b} \mid \bar{a}.q)\\ &\simeq b.0 + \hat{b}.(\bar{b} \mid (\nu a)q)\\ &\simeq b.(\bar{b} \mid (\nu a)q)\end{aligned}$$

5. Normalisation

The purpose of this section is to prove strong normalisation of the relation $\succ$ over terms without replication. In the following developments, an n-ary sum (up to $\cong$) is written using the $\sum$ notation, and an n-ary parallel composition is written using the $\prod$ notation.

Local confluence The first step of our normalisation proof is to prove local confluence of the relation $\succ$. For this we need the following lemma:

Lemma 3. *For all actions α and all terms p and q, $\hat{\alpha}.p \mid q$ and $\hat{\alpha}.p \mid q + \hat{\alpha}.(p \mid q)$ have a common reduct for $\succ$.*

The detailed proof can be found in the appendix. The idea is that q can be reduced to a parallel composition of actions $\hat{\alpha}_i.q_i$ or actions $\alpha_i.0$, and using the rules I1, I2 and I3 we reduce $\hat{\alpha}.p \mid q$ to a term that contains a slice $\hat{\alpha}.(p \mid q)$, then we conclude using rule S.

This lemma is the reason we introduced the rule S in our system. It also explains why introducing scalar coefficients, to get linear combinations instead of sums, seems problematic in the current state of things.

Theorem 2. *The relation $\succ$ is locally confluent.*

Proof. We proceed by studying all possible conflicts between the rules. Not counting rule S, there are 22 critical pairs, all developed in the appendix. Most cases are easily solved, the most difficult case is that of two interfering applications of rule I1, and we detail it here. The term $p_0 = \hat{\alpha}.p \mid \hat{\alpha}.q \mid \hat{\bar{\alpha}}.r$ reduces into two terms:

$$\begin{aligned}p_1 &= \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r) \mid \hat{\alpha}.q + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid r) \mid \hat{\alpha}.q + (\nu\vec{x})(p \mid r) \mid \hat{\alpha}.q\\ p_2 &= \hat{\alpha}.(q \mid \hat{\bar{\alpha}}.r) \mid \hat{\alpha}.p + \hat{\bar{\alpha}}.(\hat{\alpha}.q \mid r) \mid \hat{\alpha}.p + (\nu\vec{x})(q \mid r) \mid \hat{\alpha}.p\end{aligned}$$

We reduce p_1 as follows (the reduced terms are underlined):

$$\begin{aligned}p_1 &= \underline{\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r) \mid \hat{\alpha}.q} + \underline{\hat{\bar{\alpha}}.(\hat{\alpha}.p \mid r) \mid \hat{\alpha}.q} + (\nu\vec{x})(p \mid r) \mid \hat{\alpha}.q\\ &\succ^2 \hat{\alpha}.(p \mid \underline{\hat{\alpha}.q \mid \hat{\bar{\alpha}}.r}) + \hat{\alpha}.(\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r) \mid q)\\ &\quad + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid \hat{\alpha}.q \mid r) + \hat{\alpha}.(\hat{\bar{\alpha}}.(\hat{\alpha}.p \mid r) \mid q)\\ &\quad + \hat{\alpha}.p \mid (\nu\vec{x})(r \mid q) + (\nu\vec{x})(p \mid r) \mid \hat{\alpha}.q\\ &\succ^3 \hat{\alpha}.(p \mid \hat{\alpha}.(q \mid \hat{\bar{\alpha}}.r)) + \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.(\hat{\alpha}.q \mid r))\\ &\quad + \hat{\alpha}.(p \mid (\nu\vec{x})(q \mid r)) + \hat{\alpha}.(\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r) \mid q)\\ &\quad + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid \hat{\alpha}.q \mid r) + \hat{\alpha}.(\hat{\bar{\alpha}}.(\hat{\alpha}.p \mid r) \mid q)\\ &\quad + \hat{\alpha}.p \mid (\nu\vec{x})(q \mid r) + \hat{\alpha}.q \mid (\nu\vec{x})(p \mid r)\end{aligned}$$

The last step is $\succ^3$ because it is one step of I1 followed by two steps of L2 to expand the sum. If we do the same in p_2, we get the same term with p and q swapped. Define

$$\begin{aligned}S = {}&\hat{\alpha}.(p \mid \hat{\alpha}.(q \mid \hat{\bar{\alpha}}.r)) + \hat{\alpha}.(\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r) \mid q) + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid \hat{\alpha}.q \mid r)\\ &+ \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.(\hat{\alpha}.q \mid r)) + \hat{\alpha}.(\hat{\bar{\alpha}}.(\hat{\alpha}.p \mid r) \mid q)\end{aligned}$$

then swapping p and q does not change S and we have

$$\begin{aligned}p_1 &\succ^* S + \hat{\alpha}.q \mid (\nu\vec{x})(p \mid r)\\ &\qquad + \hat{\alpha}.p \mid (\nu\vec{x})(q \mid r) + \hat{\alpha}.(p \mid (\nu\vec{x})(q \mid r))\\ p_2 &\succ^* S + \hat{\alpha}.p \mid (\nu\vec{x})(q \mid r)\\ &\qquad + \hat{\alpha}.q \mid (\nu\vec{x})(p \mid r) + \hat{\alpha}.(q \mid (\nu\vec{x})(p \mid r))\end{aligned}$$

By lemma 3, the terms $\hat{\alpha}.q \mid (\nu\vec{x})(p \mid r)$ and $\hat{\alpha}.q \mid (\nu\vec{x})(p \mid r) + \hat{\alpha}.(q \mid (\nu\vec{x})(p \mid r))$ have a common reduct r_1, so

$$\begin{aligned}p_1 &\succ^* S + r_1 + \hat{\alpha}.p \mid (\nu\vec{x})(q \mid r) + \hat{\alpha}.(p \mid (\nu\vec{x})(q \mid r))\\ p_2 &\succ^* S + \hat{\alpha}.p \mid (\nu\vec{x})(q \mid r) + r_1\end{aligned}$$

and by the same lemma $\hat{\alpha}.p \mid (\nu\vec{x})(q \mid r) + \hat{\alpha}.(p \mid (\nu\vec{x})(q \mid r))$ and $\hat{\alpha}.p \mid (\nu\vec{x})(q \mid r)$ have a common reduct r_2, hence p_1 and p_2 both reduce to $S + r_1 + r_2$. □

Termination We now prove that the relation $\succ$ is well founded on finite terms. For this we make a relationship between evaluation and execution, first by considering reductions at top level.

Definition 14. For a term p, let $\|p\|_t$ be the maximum length of the runs of slices of p, or ∞ if it is undefined. Let $\succ_t$ be the top-level reduction, i.e. $p \succ_t q$ if $p \succ q$ by applying a reduction rule not under an action.

Lemma 4. *If $p \succ_t q$ then $\|p\|_t \geqslant \|q\|_t$.*

Lemma 5. *The restriction of $\succ_t$ that does not use the rule* I1 *is terminating.*

Lemma 4 is a simple case analysis on the rules, and lemma 5 uses a measure that essentially counts the number of actions in parallel in each slice (details are in the appendix). This shows that the only possible source of infinite reductions is rule I1, indeed it is the one that extends the execution relation. By combining the measure used in lemma 5 and the norm $\|p\|_t$, which strictly decreases along execution, we can prove the following equivalence:

Lemma 6. *For all terms p, there is an infinite sequence of $\succ_t$ if and only if a slice of p has an infinite run.*

The next step is to generalise this to arbitrary reductions. For this we introduce another norm, which takes into account the maximum length of sequences of transitions, instead of execution steps:

Definition 15. For a slice s, let $\|s\|_a$ be the maximum length of sequences of transitions of s, of ∞. For a term p, let $\|p\|_a = \max\{\, \|s\|_a \mid s \in \mathfrak{s}(p)\,\}$. Let $\|p\|$ be the maximum of $\|\cdot\|_a$ over all subterms of p.

Lemma 7. *If $p \succ q$ then $\|p\| \geqslant \|q\|$.*

Proposition 3. *p has no infinite reduction if $\|p\|$ is finite.*

The proof (detailed in appendix) is by induction on $\|p\|$. The idea is that any sequence of reductions can be reordered so that all top-level reductions come first, followed by all reductions of subterms prefixed by actions; such subterms are strictly smaller than $\|p\|$. A proof by induction on the height terms would not work since some rules make the height of terms strictly grow. As a direct consequence we finally get the expect result, remarking that terms without replication always have a finite norm $\|p\|$:

Theorem 3. *The relation $\succ$ is strongly normalising on terms without replication.*

6. A trace semantics

The soundness and strong normalisation of relation $\succ$ imply that every term without replication is equivalent to an irreducible term. The point of this section is to study the properties of these normal forms.

Definition 16. An *inaction set n* is a finite set of terms $\alpha.0$ that does not contain dual actions, i.e. such that for all $\alpha.0 \in n$ and $\beta.0 \in n$, $\beta \neq \bar{\alpha}$. Inaction sets are identified with the parallel composition of their elements. A *trace* is a term of the form $t = \hat{\alpha}_1.\hat{\alpha}_2 \ldots \hat{\alpha}_k.n$ where n is an inaction set. k is called the *length* of t and is written $|t|$, the sequence $\hat{\alpha}_1 \ldots \hat{\alpha}_k$ is the *action part* of t, n is the *inaction part* of t.

Proposition 4. *Normal forms are sums of distinct traces.*

The proof (in appendix) consists in showing that all affine actions, parallel compositions and hidings are eliminated by reduction. By theorem 3, this shows that each term is equivalent to a sum of traces, and reduction provides a way to compute a set of traces equivalent to a given term. Thus we get a trace semantics, and the purpose of this section is to study some properties of this semantics.

Definition 17. Let $\boldsymbol{T}$ be the set of all traces. For any set $B \subseteq \boldsymbol{T}$, define $B^* = \{\, t \mid t \in \boldsymbol{T}, \forall u \in B, t \perp u \,\}$. A trace behaviour is a set $B \subseteq \boldsymbol{T}$ such that $B = B^{**}$.

Proposition 5. *The set of term behaviours and the set of trace behaviours, ordered by inclusion, are isomorphic complete lattices. The isomorphism is the function* t *defined as* $\mathsf{t}(A) = A \cap \boldsymbol{T}$, *with* $\mathsf{t}^{-1}(B) = B^{\perp\perp}$.

Proof. Firstly, remark that for any term behaviour A we have $A^\perp = \mathsf{t}(A)^\perp$: it is obvious that $\mathsf{t}(A) \subseteq A$, so $A^\perp \subseteq \mathsf{t}(A)^\perp$. For the reverse inclusion consider a term $p \perp \mathsf{t}(A)$ and a term $q \in A$. By theorem 3 and proposition 4, there is a decomposition $q \simeq \sum_{i=1}^n t_i$ where the t_i are traces, therefore $p \perp q$ if and only if $p \perp t_i$ for all i. By definition of observational equivalence we have $\{\, t_i \mid 1 \leqslant i \leqslant n \,\}^{\perp\perp} = \{q\}^{\perp\perp} \subseteq A$ so for each i we have $t_i \in A$, so $t_i \in \mathsf{t}(A)$, therefore $p \perp t_i$. This proves $p \perp q$, and subsequently $p \perp A$ so $p \in A^\perp$.

This implies that for any term behaviour A we have $A = \mathsf{t}(A)^{\perp\perp}$, moreover by definition for any set of traces B we have $B^* = \mathsf{t}(B^\perp)$ hence $B^{**} = \mathsf{t}(\mathsf{t}(B^\perp)^\perp) = \mathsf{t}(B^{\perp\perp})$, so t is a bijection. It is clearly increasing, and it also clear that it commutes with lower bounds, since lower bounds are intersections in both sets. For upper bounds, if $(A_i)_{i\in I}$ is a family of term behaviours, we have $\mathsf{t}(\bigvee_{i\in I} A_i) = \mathsf{t}((\bigcup_{i\in I} A_i)^{\perp\perp}) = \mathsf{t}((\bigcup_{i\in I} \mathsf{t}(A_i)^{\perp\perp})^{\perp\perp}) = \mathsf{t}((\bigcup_{i\in I} \mathsf{t}(A_i))^{\perp\perp}) = (\bigcup_{i\in I} \mathsf{t}(A_i))^{**} = \bigvee_{i\in I} \mathsf{t}(A_i)$. Therefore t is a lattice isomorphism. □

This proposition implies that the behaviour of a process p is completely described by the set of traces contained in $\{p\}^{\perp\perp}$, i.e. by the trace behaviour of p. So it is enough to consider trace behaviours when studying the testing semantics of processes. This is especially interesting because orthogonality of traces is straightforward to characterise syntactically (the proof can be found in appendix):

Proposition 6. *Let $t = \hat{\alpha}_1 \ldots \hat{\alpha}_k.m$ and $u = \hat{\beta}_1 \ldots \hat{\beta}_\ell.n$ be two traces, then*

- *$t \perp u$ unless $k = \ell$, $\beta_i = \bar{\alpha}_i$ for all i, and $m \cap \bar{n} = \emptyset$,*
- *$t \sqsubseteq u$ if and only if $k = \ell$, $\beta_i = \alpha_i$ for all i, and $m \subseteq n$.*

In the sequel we write $[\![p]\!]$ to represent the trace behaviour of p. This characterisation of the testing preorder shows that it is actually an order over traces: syntactically distinct traces are always distinguishable. It also implies that the preorder is decidable for terms without replication: if p and q are normal forms, i.e. finite sets of traces, then $p \sqsubseteq q$ if and only if $[\![q]\!] \subseteq [\![p]\!]$, if and only if for all traces $t \in [\![q]\!]$ there is a trace $u \in [\![p]\!]$ such that $u \sqsubseteq t$. If we add a reduction rule $t + u \succ t$ if t and u are traces and $t \sqsubseteq u$ (the condition is easy to check), then strong normalisation is obviously preserved, and normal forms are finite sets of pairwise incomparable traces. Then equivalence of processes coincides with the equality of normal forms.

We thus have an isomorphism between behaviours, i.e. sets of terms closed upwards for the observational preorder, and trace behaviours, i.e. sets of traces closed upwards for the inclusion of inaction sets. The trace behaviour of a term can be expressed inductively as follows:

$$
\begin{aligned}
&[\![p+q]\!] = [\![p]\!] \cup [\![q]\!] && [\![0]\!] = \emptyset \\
&[\![p \mid q]\!] = \textstyle\bigcup_{t\in[\![p]\!],u\in[\![q]\!]}[t \mid u] && [\![1]\!] = \{\, t \mid t \in \boldsymbol{T}, |t| = 0 \,\} \\
&[\![\alpha.p]\!] = \{\, \hat{\alpha}.t \mid t \in [\![p]\!] \,\} \cup \{\alpha.0\} \\
&[\![\hat{\alpha}.p]\!] = \{\, \hat{\alpha}.t \mid t \in [\![p]\!] \,\} \\
&[\![(\nu x)p]\!] = \left\{ \hat{\alpha}_1 \dots \hat{\alpha}_k.(n \setminus x) \;\middle|\; \begin{array}{l} \hat{\alpha}_1 \dots \hat{\alpha}_k.n \in [\![p]\!] \\ \forall i, x \notin \mathrm{n}(\alpha_i) \end{array} \right\}
\end{aligned}
$$

where $n \setminus x$ is the set n without the inactions with x as the subject name. The notation $[t \mid u]$ stands for the set of traces obtained by normalising $t \mid u$. We can define a kind of abstract machine to compute it: say a state is a set of names $X \subseteq \boldsymbol{N}$ and a configuration is a triple $c \in \boldsymbol{T}\times\mathcal{P}(\boldsymbol{N})\times\boldsymbol{T}$. Define three kinds of actions: productions $(t, X, u) \xrightarrow{\alpha} (t', X', u')$, silent steps $(t, X, u) \to (t', X', u')$ and terminations $(t, X, u) \to n$. Define execution as

$$
\begin{aligned}
(\hat{\alpha}.t, X, u) &\xrightarrow{\alpha} (t, X \cup \mathrm{bn}(\alpha), u) && \text{if } \mathrm{n}(\alpha) \cap X = \emptyset \\
(t, X, \hat{\beta}.u) &\xrightarrow{\beta} (t, X \cup \mathrm{bn}(\beta), u) && \text{if } \mathrm{n}(\beta) \cap X = \emptyset \\
(\hat{\alpha}.t, X, \hat{\bar{\alpha}}.u) &\to (t, X \cup \mathrm{bn}(\alpha), u) \\
(m, X, n) &\to (m \cup n) \setminus X && \text{if } m \cap \bar{n} = \emptyset
\end{aligned}
$$

Then $[t \mid u]$ is the set of traces $\hat{\alpha}_1 \dots \hat{\alpha}_k.n$ such that there is a run $(t, \emptyset, u) \to^* \xrightarrow{\alpha_1} \to^* \xrightarrow{\alpha_2} \to^* \cdots \xrightarrow{\alpha_k} \to^* n$.

To get back to our starting point, that is the testing preorder in the πI-calculus, the final question we have to ask is the relationship between this testing, from definition 3, and the one we studied in the algebraic setting. This relationship is easy: the preorders are equal. To prove this, first remark that using 0 as the testing token is equivalent to using an action on a special channel:

Lemma 8. *For any term p, let $[p]_\omega$ be the term obtained by replacing each occurrence of* 0 *by the action $\bar{\omega}$. If p is a quasi-standard term such that $\omega \notin \mathrm{fn}(p)$ then p is successful if and only if for any run $[p]_\omega \Rightarrow^* q$ there is a run $q \Rightarrow^* \bar{\omega}|r$.*

The details of this proof are in the appendix, it consists in remarking that 0 and $\bar{\omega}$ play the same role in testing. From this we can finally deduce our full abstraction theorem:

Theorem 4. *For all standard terms p and q, $p \sqsubseteq q$ if and only if $p \sqsubseteq_I q$ if and only if $[\![p]\!] \subseteq [\![q]\!]$.*

Proof. The first step is to remark that every term is observationally equivalent to a quasi-standard term. To prove this we translate each extended syntactic construction using the equivalences of table 2. In section 4 we have shown that for any terms p and q and any fresh name u we have $p + q \simeq (\nu u)(u.p \,|\, u.q \,|\, \bar{u})$. For linear actions, consider a term $\hat{\alpha}.p$ and a fresh name u. Remark that $u.0|\bar{u} \simeq u.0|\bar{u}.0 + u.0|\hat{\bar{u}} \simeq \hat{\bar{u}}.u.0$, so we have

$$
\begin{aligned}
&(\nu u)(\alpha.u.p \mid u.0 \mid \bar{u}) \\
&\simeq (\nu u)((\hat{\alpha}.\hat{u}.p + \hat{\alpha}.u.0 + \alpha.0) \mid \hat{\bar{u}}.u.0) \\
&\simeq (\nu u)(\hat{\alpha}.\hat{u}.p \mid \hat{\bar{u}}.u.0) + (\nu u)(\hat{\alpha}.u.0 \mid \hat{\bar{u}}.u.0) + \alpha.0 \mid (\nu u)\hat{\bar{u}}.u.0 \\
&\simeq \hat{\alpha}.(\nu u)(p \mid u.0) + (\nu u)(\hat{\alpha}.(u.0 \mid \hat{\bar{u}}.u.0)) + 0 \\
&\simeq \hat{\alpha}.(p \mid (\nu u)u.0) + \hat{\alpha}.(\nu u)\hat{\bar{u}}.(u.0 \mid u.0) \\
&\simeq \hat{\alpha}.(p \mid 1) + \hat{\alpha}.0 \simeq \hat{\alpha}.p
\end{aligned}
$$

This proves that two standard terms p and q are distinguishable in the algebraic setting if and only if they are distinguishable by a quasi-standard term. By lemma 8 this is the same as being distinguishable according to definition 3. □

Note that this implies that the testing preorder over finite terms is decidable by the purely axiomatic system given by the rules of table 2 and the characterisation of proposition 6.

7. Extensions

7.1. Choice

When introducing the sum operator, it is natural to ask which relationship there is between this sum and the traditional choice operator of process calculi. Assume we have a construct $\&_{i\in I}\, \alpha_i.p_i$ whose operational semantics is defined in the labelled transition system as

$$\mathop{\&}_{i\in I} \alpha_i.p_i \xrightarrow{\alpha_i} p_i$$

for each $i \in I$. The argument justifying rule A applies here as follows: either one of the α_i will be consumed, or none of them will. Formally, assuming that the set $\{\, \alpha_i \mid i \in I \,\}$ does not contain dual actions, we get the equation

$$\&_{i\in I}\, \alpha_i.p_i \simeq \textstyle\sum_{i\in I} \hat{\alpha}_i.p_i + \prod_{i\in I} \alpha_i.0$$

where the product stands for a parallel composition. Thus external choice is indeed related to the algebraic sum, and our extensions already provide enough power to express it, even mixed choice, the only exception being that of a choice between dual actions.

This decomposition does not hold when two dual actions occur in the same choice, because of the rule I5:

$\alpha.0 \mid \bar{\alpha}.0 \simeq 0$. If we were to extend our formalism to allow this kind of choice, the only extra material needed would be terms of the form $\alpha.0 \mathbin{\&} \bar{\alpha}.0$, which intuitively require that the environment will never perform $\bar{\alpha}$ nor α. One can easily check that this simple extension does not compromise the results exposed in this paper.

Using the rule above with the rules of table 2, we can easily derive usual laws like the following, where $\beta \neq \bar{\alpha}$:

$$\alpha.p \mid \beta.q \simeq \alpha.(p \mid \beta.q) \mathbin{\&} \beta.(\alpha.p \mid q)$$
$$\alpha.p \mathbin{\&} \alpha.q \simeq \alpha.(p + q)$$

Among standard examples used to describe the properties of process calculi is the comparison of $a.(b \mathbin{\&} c)$ and $a.b \mathbin{\&} a.c$ (with our notations). These terms are not bisimilar although they have the same traces (in the usual sense), and indeed they are equivalent for may-testing but not for must-testing. The use of inactions in our traces solves this problem:

$$\begin{aligned} a.(b \mathbin{\&} c) &\simeq \hat{a}.(\hat{b} + \hat{c} + b.0 \mid c.0) + a.0 \\ &\simeq \hat{a}.\hat{b} + \hat{a}.\hat{c} + \hat{a}.(b.0 \mid c.0) + a.0 \\ a.b \mathbin{\&} a.c &\simeq \hat{a}.(\hat{b} + b.0) + \hat{a}.(\hat{c} + c.0) + a.0 \\ &\simeq \hat{a}.\hat{b} + \hat{a}.\hat{c} + \hat{a}.(b.0 + c.0) + a.0 \end{aligned}$$

In the first case there is one trace $\hat{\alpha}.(b.0 \mid c.0)$ while in the second case there are two traces $\hat{\alpha}.b.0$ and $\hat{a}.c.0$. The first case reveals that the process may have two available actions after performing α (although 0 assumes none of them will happen) while the second case states that only one action is available. The syntactic characterisation of the observational preorder also implies $a.b \mathbin{\&} a.c \sqsubseteq a.(b \mathbin{\&} c)$.

Composing these terms with, say, $\bar{a}.\bar{b}.d$ makes the difference in behaviours explicit: in the first case we get

$$\begin{aligned} &(\nu abc)(a.(b \mathbin{\&} c) \mid \bar{a}.\bar{b}.d) \\ &\simeq (\nu abc)(a.(\hat{b} + \hat{c} + b.0 \mid c.0) \mid \bar{a}.(\hat{\bar{b}}.d + \bar{b}.0)) \\ &\simeq (\nu bc)((\hat{b} + \hat{c} + b.0 \mid c.0) \mid (\hat{\bar{b}}.d + \bar{b}.0)) \\ &\simeq (\nu bc)(\hat{b} \mid \hat{\bar{b}}.d) + (\nu bc)(\hat{c} \mid \hat{\bar{b}}.d) + (\nu bc)(b.0 \mid c.0 \mid \hat{\bar{b}}.d) \\ &\quad + (\nu bc)(\hat{b} \mid \bar{b}.0) + (\nu bc)(\hat{c} \mid \bar{b}.0) + (\nu bc)(b.0 \mid c.0 \mid \bar{b}.0) \\ &\simeq d + 0 + 0 + 0 + 0 + 0 \simeq d \end{aligned}$$

In the second case we get

$$\begin{aligned} &(\nu abc)(a.b \mathbin{\&} a.c \mid \bar{a}.\bar{b}.d) \\ &\simeq (\nu abc)(a.(\hat{b} + \hat{c} + b.0 + c.0) \mid \bar{a}.(\hat{\bar{b}}.d + \bar{b}.0)) \\ &\simeq (\nu bc)((\hat{b} + \hat{c} + b.0 + c.0) \mid (\hat{\bar{b}}.d + \bar{b}.0)) \\ &\simeq (\nu bc)(\hat{b} \mid \hat{\bar{b}}.d) + (\nu bc)(\hat{c} \mid \hat{\bar{b}}.d) + (\nu bc)(b.0 \mid \hat{\bar{b}}.d) \\ &\quad + (\nu bc)(c.0 \mid \hat{\bar{b}}.d) + (\nu bc)(\hat{b} \mid \bar{b}.0) + (\nu bc)(\hat{c} \mid \bar{b}.0) \\ &\quad + (\nu bc)(b.0 \mid \bar{b}.0) + (\nu bc)(c.0 \mid \bar{b}.0) \\ &\simeq d + 0 + 0 + 0 + 0 + 0 + 0 + 1 \simeq d + 1 \end{aligned}$$

This shows that in the second case, there is the possibility of a deadlock, since the final sum contains 1, while in the first case there is no such possibility because d is the only term in the result.

7.2. Infinity

The main developments in this paper focus on terms without replication, and the trace semantics is not valid anymore in the presence of infinite behaviours. Consider for instance the term $!u$. The recursive definition states $!u \equiv u.!u$, so if U is the trace behaviour of $!u$ we must have $U = \{u.0\} \cup \{\, \hat{u}.t \mid t \in U \,\}$. This has one solution, namely $U = \{\, \hat{u}^n.u.0 \mid n \in \mathbb{N} \,\}$, where $\hat{u}^n.t$ stands for a term $\hat{u}. \cdots .\hat{u}.t$ with n occurrences of $\hat{u}$. Now if we check if U and $\bar{U}$ are orthogonal, we have to check that $\hat{u}^i.u.0 \perp \hat{\bar{u}}^j.\bar{u}.0$ for all i and j, an by proposition 6 this always holds, hence $U \perp \bar{U}$. But clearly $!u$ and $!\bar{u}$ are not orthogonal, since $!u \mid !\bar{u}$ has exactly one maximal run, namely the infinite run $!u \mid !\bar{u} \to !u \mid !\bar{u} \to \cdots$, which is consistent.

This can be solved in several ways, as explained below. We defer the detailed study of these possible solutions to an extended version of this paper.

Cheating Consider infinite runs as inconsistent. This is the easiest way but it is an unusual kind of test which amounts to considering that all the terms we ever consider terminate. This might be satisfactory in a context where termination is guaranteed, for instance by typing. However, restricting to a typed calculus may have important consequences when defining testing semantics, for instance if testing is defined against typed environments only. Besides, our motivation here is to study an untyped calculus.

Partial traces This approach consists in adapting the concept of Böhm trees to our calculus. By introducing a new process constant Ω to represent undefinedness, we can define partial traces, i.e. traces ending with Ω. Then trace behaviours will be sets of traces downwards closed by a refinement order. An equivalent formulation is to extend the language with infinite traces, as limits of partial traces. Since the reduction rules produce traces in an incremental way, we can even compute approximations of a term.

This approach is appropriate if we want to extend our fully abstract trace semantics. On the other hand, it does not provide a way to extend our decidability result for the testing preorder to a class of processes with infinite behaviour.

New rules While these solutions keep the same rewriting rules, a different approach is to extend the rules with normalisation of replicated actions. The extended rules should probably allow special cases like $(\nu u)(!u(x).p \mid !v(y).q) \succ^* !v(y).(\nu u)(!u(x).p \mid q)$ (if u does not occur in p).

This kind of rule for simplifying replication has already been proposed in fragments of the π-calculus (an *extended reduction* with comparable properties is proposed by Yoshida, Berger and Honda in their typed π-calculus [17]). Because of the undecidability of most equivalences on infinitary processes, the possibility of defining a reduction that could normalise arbitrary terms is unlikely, but partial solutions could exist, for instance in reasonable restrictions of the calculus. This approach is appropriate in order to study embeddings of λ-calculi in process calculi, as it would provide an extension and decomposition of β-reduction.

7.3. Other forms of tests

All results here are derived while specifically considering the fair testing semantics, and a similar work could be done for other forms of testing. Standard must testing is defined as $p \perp q$ if every maximal execution of $p \mid q$ eventually reaches a state where the action $\bar{\omega}$ is available; in our algebraic setting it simply means that $p \mid q$ is inconsistent.

This is equivalent to fair testing in the finitary calculus, and the difference appears with divergence. Although the definition of orthogonality is simpler than in fair testing, the semantics is a bit more complicated. Now we need a new constant Ω to represent pure divergence, and we also need a constant $\tau.0$ to represent a process that is not null but may always reduce to 0. Their respective execution rules are $\Omega \to \Omega$ and $\tau.0 \to 0$. Then we get the following rules:

	$\Omega \mid p \cong \Omega$	if $p \not\cong 0$
I3′	$\hat{\alpha}.p \mid c \succ \hat{\alpha}.(p \mid c)$	if $c = \tau.0$ or $c = \Omega$
I5	$\alpha.0 \mid \bar{\alpha}.0 \succ \tau.0$	
I6	$\alpha.0 \mid \tau.0 \succ \tau.0$	

Now traces may end not with an inaction set but with $\tau.0$ or Ω, and the syntactic characterisation of the testing preorder has to be extended. The discussion in section 7.2 about infinite traces also applies in this case.

Doing the same work for may testing, on the other hand, requires a radical change in the definition of the observation. While consistency is unchanged, we now define that $s \perp t$ if there is a run $s \mid t \Rightarrow^* r$ with r inconsistent. We did not study the precise consequences of this, but the resulting reduction can be expected to be very different, since most of the soundness arguments use here become invalid. Nevertheless, carrying the same study for other forms of tests would surely provide insight into the genericity of our construction; such a generalisation will be discussed in an extended version of the present work.

Finally, it is natural to ask whether a similar approach could be used to get axiomatisations of some forms of bisimulation. The answer to this question is likely to be negative: the decomposition of processes into sums of traces seems to lose all branching information while branching is crucial in bisimulations.

7.4. Name passing

The present work studies the introduction of an algebraic sum in the πI-calculus instead of the full π-calculus because this fragment is simpler and allows us to focus specifically on the new elements introduced by algebraicity. The next step in the search for normalisation in process calculi is communication of free names, as in standard π-calculus.

Adapting our approach to this context does not lead to fundamental changes, but quite a few things have to be adjusted. First of all, in order to make the testing preorder contextual, we have to break the symmetry between processes and tests: now a test is a pair (q, σ) where q is a process and σ is a name substitution, and a process p passes the test (q, σ) if $p\sigma \mid q$ is successful. The equivalence rules are kept unchanged except for I1 and I5, which are the ones where interaction may occur. The rule I1 is extended as follows, with the side condition that $|\vec{x}| = |\vec{y}|$:

$$\hat{\bar{u}}\langle\vec{x}\rangle.p \mid \hat{v}(\vec{y}).q \succ \hat{\bar{u}}\langle\vec{x}\rangle.(p \mid \hat{v}(\vec{y}).q) + \hat{v}(\vec{y}).(\hat{\bar{u}}\langle\vec{x}\rangle.p \mid q) + [u \hat{=} v] \mid p \mid q[\vec{x}/\vec{y}]$$

In the reduct, $[u \hat{=} v]$ is the linear content of the usual match operator, which is decomposed as

$$[u{=}v].p \simeq [u \hat{=} v] \mid p + [u \hat{\neq} v]$$

i.e. using also a linear mismatch operator. The meaning of $[u \hat{=} v]$ is intuitively "when I am in head position, u and v will have been unified by substitution", while $[u \hat{\neq} v]$ means that they will not. Incidentally, this proves that if the match operator is present, then the mismatch operator will automatically be available too.

Since tests contain name substitutions, these operators can be simplified only if they are are trivial or under binders:

$$[u \hat{=} u] \succ 1 \qquad (\nu u)[u \hat{=} v] \succ 0$$
$$[u \hat{\neq} u] \succ 0 \qquad (\nu u)[u \hat{\neq} v] \succ 1$$

if u and v are distinct names.

Normalisation requires other equivalence rules, including something like $[u \hat{=} v] \mid p \simeq [u \hat{=} v] \mid p[u/v]$, which is reminiscent of the structural rule for explicit fusions in Gardner and Wischik's calculus [7]. After introducing all the required rules, we will finally get normal forms that are sums of extended traces of the form

$$c_1 \mid \hat{\alpha}_1.(c_2 \mid \hat{\alpha}_2 \ldots (c_k \mid \hat{\alpha}_k.(c_{k+1} \mid n)) \ldots)$$

where the c_i are *conditions*, i.e. parallel compositions of linear matches and mismatches, the α_i are transition labels of the form $u(\vec{x})$ or $(\nu\vec{z})\bar{u}\langle\vec{y}\rangle$ with $\vec{z} \subseteq \vec{y} \setminus \{u\}$, and where n is a parallel composition of inactions.

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A. Technicalities

A.1. Labelled transition system

Proof of proposition 1. For the "only if" part, consider a slicing execution $s \Rightarrow t$. By definition, there is a slice $s' = E[\mathring{\bar{u}}(\vec{x}).p \,|\, \mathring{u}(\vec{x}).q]$, where E is an execution context, and a slice $t' \in \mathfrak{s}(p \mid q)$ such such that $s \equiv s'$ and $t \equiv E[(\nu\vec{x})t']$. The transition $\mathring{\bar{u}}(\vec{x}).p|\mathring{u}(\vec{x}).q \xrightarrow{\tau} (\nu\vec{x})(p|q)$ holds by definition of the LTS, hence the transition $s' \xrightarrow{\tau} E[(\nu\vec{x})(p \mid q)]$ holds. Since $t' \in \mathfrak{s}(p \mid q)$ we have $s \equiv s' \overset{\tau}{\Rightarrow} E[(\nu\vec{x})t'] \equiv t$.

For the "if" part, consider a transition $s \overset{\tau}{\Rightarrow} t$. By definition of the LTS there is a an action $\alpha = u^{\varepsilon}(\vec{x})$ and a decomposition $s = E[E_1[\alpha.p] \mid E_2[\bar{\alpha}.q]]$ (where E, E_1 and E_2 are execution contexts) such that the transition is derived as

$$\dfrac{\dfrac{\dfrac{\mathring{\alpha}.p \xrightarrow{\alpha} p}{E_1[\mathring{\alpha}.p] \xrightarrow{\alpha} E_1[p]} \qquad \dfrac{\mathring{\bar{\alpha}}.q \xrightarrow{\bar{\alpha}} q}{E_2[\mathring{\bar{\alpha}}.q] \xrightarrow{\bar{\alpha}} E_2[q]}}{E_1[\mathring{\alpha}.p] \mid E_2[\mathring{\bar{\alpha}}.q] \xrightarrow{\tau} (\nu\vec{x})(E_1[p] \mid E_2[q])}}{s \xrightarrow{\tau} E[(\nu\vec{x})(E_1[p] \mid E[q])]}$$

and $t \in \mathfrak{s}(E[E_1[p] \mid E[q]])$. Thus there are slices $t_1 \in \mathfrak{s}(p)$ and $t_2 \in \mathfrak{s}(q)$ such that $t = E[E_1[t_1] \mid E_2[t_2]]$. By an easy induction on contexts, one can show that there is a sequence of names $\vec{y}$ and a term r such that the following holds:

$$\begin{aligned} s = E[E_1[\mathring{\alpha}.p] \mid E[\mathring{\bar{\alpha}}.q]] &\equiv (\nu\vec{y})(r \mid (\mathring{\alpha}.p \mid \mathring{\bar{\alpha}}.q)) \\ &\to (\nu\vec{y})(r \mid (\nu\vec{x})(p \mid q)) \equiv E[(\nu\vec{x})(E_1[p] \mid E_2[q])] \end{aligned}$$

thus there is a slicing execution $s \Rightarrow t$. □

A.2. Soundness

The proof of the soundness theorem is mostly decomposed into the various propositions in this section. The soundness of $\cong$ derives from the fact that testing equivalence contains the structural equivalence $\equiv$ and from proposition 7. The soundness of rule S is obvious since p and $p + p$ have the same slices. Rule A is proposition 8. Rules L1 and L2 are proposition 9. Rule I1 is proposition 10. Rule I2 is proposition 11. Rules I3, I4 and I5 are proposition 12. Rules N1, N2 and N3 are proposition 13.

Proposition 7. *For any terms p, q, r, the following holds:*

$$p + q \simeq q + p \qquad p + (q + r) \simeq (p + q) + r$$
$$p + 0 \simeq p \qquad p \mid 0 \simeq 0$$

Proof. By definition 10, for terms p and q, $p \simeq q$ means that for any slice s, all slices of $p \mid s$ are successful if and only if all slices of $q \mid s$ are successful. By definition $\mathfrak{s}(p + q) = \mathfrak{s}(p) \cup \mathfrak{s}(q)$, so associativity and commutativity of the sum are immediate.

Any slice that contains 0 at top level is inconsistent, hence $0 \mid s$ is inconsistent. Since $\mathfrak{s}(p + 0) = \mathfrak{s}(p) \cup \{0\}$ and $0 \mid s$ is inconsistent, we have $p + 0 \perp s$ if and only if $p \perp s$, hence $p + 0 \simeq 0$.

All slices of $(0 \mid p) \mid s$ are inconsistent, as well as all those of $0 \mid s$, therefore $0 \mid p \simeq 0$. □

Lemma 9. *A slice $\hat{\alpha}.p|q$ is successful if and only if for every run $\hat{\alpha}.p \mid q \overset{\tau*}{\Rightarrow} r$ that consumes the action $\hat{\alpha}$ there is a run $r \overset{\tau*}{\Rightarrow} s$ such that s is inconsistent.*

Proof. If $\hat{\alpha}.p \mid q$ is successful, then for any run $\hat{\alpha}.p \mid q \overset{\tau*}{\Rightarrow} r$ there is a run $r \overset{\tau*}{\Rightarrow} s$ such that s is inconsistent, so this holds in particular for the runs that consume the action $\hat{\alpha}$.

Now assume that for every run $\hat{\alpha}.p|q \overset{\tau*}{\Rightarrow} r$ that consumes the action $\hat{\alpha}$ there is a run $r \overset{\tau*}{\Rightarrow} s$ such that s is inconsistent. Consider an arbitrary run $\hat{\alpha}.p \mid q \overset{\tau*}{\Rightarrow} r'$. If this run does consume $\hat{\alpha}$, then r' has an inconsistent reduct by hypothesis. Otherwise the run is $\hat{\alpha}.p|q \overset{\tau*}{\Rightarrow} \hat{\alpha}.p|q'$. If there is a sequence of transitions $q' \overset{\tau*}{\Rightarrow} q'' \overset{\bar{\alpha}}{\Rightarrow} t$, then there is a run $\hat{\alpha}.p|q \overset{\tau*}{\Rightarrow} \hat{\alpha}.p| q'' \overset{\tau}{\Rightarrow} (\nu\vec{x})(p \mid t)$ where $\vec{x}$ is the sequence of bound names of α. Then this is a run that consumes $\hat{\alpha}$ so it can be extended into a run that reaches an inconsistent term, by hypothesis. Finally, if there is no sequence of transitions $q' \overset{\tau*}{\Rightarrow} q'' \overset{\bar{\alpha}}{\Rightarrow} t$, then every maximal run of $\hat{\alpha}.p \mid q'$ leaves $\hat{\alpha}$ untouched, which means that $\hat{\alpha}.p \mid q'$ is inconsistent. Therefore $\hat{\alpha}$ is successful. □

Proposition 8. *For any action α and any term p, we have $\alpha.p \simeq \hat{\alpha}.p + \alpha.0$.*

Proof. Let $u^{\varepsilon}(\vec{x})$ be the action α. We have to show that for any slice q, $\alpha.p \perp q$ if and only if $\alpha.0 \perp q$ and $\hat{\alpha}.p \perp q$.

First assume that $\alpha.p \perp q$. Consider a run $\hat{\alpha}.p \mid q \overset{\tau*}{\Rightarrow} r$. Using lemma 9, we assume that the action of $\hat{\alpha}.p$ is consumed in this run by a reduct of q, i.e. there are transitions $q \overset{\tau*}{\Rightarrow} q_1 \overset{\bar{\alpha}}{\Rightarrow} q_2$ such that the considered run is $\hat{\alpha}.p \mid q \overset{\tau*}{\Rightarrow} \hat{\alpha}.p \mid q_1 \overset{\tau}{\Rightarrow} (\nu\vec{x})(p \mid q_2) \overset{\tau*}{\Rightarrow} r$. Thus there is a run $\alpha.p \mid q \overset{\tau*}{\Rightarrow} \alpha.p \mid q_1 \overset{\tau}{\Rightarrow} (\nu\vec{x})(p \mid q_2) \overset{\tau*}{\Rightarrow} r$ and by hypothesis there is a run $r \overset{\tau*}{\Rightarrow} s$ with s inconsistent. Therefore $\hat{\alpha}.p \perp q$.

Now consider a run $\alpha.0 \mid q \overset{\tau*}{\Rightarrow} r$. Two cases may occur:

- Either the action of $\alpha.0$ is consumed by a reduct of q, i.e. there are transitions $q \overset{\tau*}{\Rightarrow} q_1 \overset{\bar{\alpha}}{\Rightarrow} q_2$ such that the considered run is $\alpha.0 \mid q \overset{\tau*}{\Rightarrow} \alpha.0 \mid q_1 \overset{\tau}{\Rightarrow} (\nu\vec{x})(0 \mid q_2) \overset{\tau*}{\Rightarrow} r$. Then r is null thus inconsistent.

- Or the action is not consumed and the run is actually a run of q: $\alpha.0|q \overset{\tau*}{\Rightarrow} \alpha.0|q_1 = r$. Then $\alpha.p|q \overset{\tau*}{\Rightarrow} \alpha.p|q_1$ is a valid run, and by hypothesis there is a run $\alpha.p|q_1 \stackrel{\tau*}{\Rightarrow} s$ with s inconsistent. If this run consumes the action in $\alpha.p$ then there is a run of $\alpha.0 \mid q_1$ which consumes the action in $\alpha.0$, and the reduct is null thus inconsistent. Otherwise the action is not consumed and the run can be written $\alpha.p \mid q_1 \stackrel{\tau*}{\Rightarrow} \alpha.p \mid q_2$ with $\alpha.p \mid q_2$ inconsistent. This means that $\alpha.p \mid q_2$ has no consistent maximal run. Consider a maximal run of $\alpha.0 \mid q_2$: if it consumes the action of $\alpha.0$, then it is inconsistent since it reaches a null state. Otherwise $\alpha.0$ is left untouched and the run is a maximal run of q_2, from which we can deduce a maximal run of $\alpha.p \mid q_2$, which is inconsistent by hypothesis. This means that either it reaches a null state, then it is also the case for the run $\alpha.0 \mid q_2$, or it leaves some active linear action untouched, then this action is produced by a reduct of q_2, and the same action is never consumed in the run of $\alpha.0 \mid q_2$. Therefore any maximal run of $\alpha.0 \mid q_2$ is inconsistent, and $\alpha.0 \mid q_2$ is inconsistent.

Therefore $\alpha.0 \perp q$.

Now assume that $\hat{\alpha}.p \perp q$ and $\alpha.0 \perp q$. Consider a run $\alpha.p \mid q \stackrel{\tau*}{\Rightarrow} r$.

- If this run consumes $\alpha.p$ then there is a run of $\hat{\alpha}.p|q$ that leads to r by the same reduction steps. By hypothesis $\hat{\alpha}.p \mid q$ is successful so r has an inconsistent reduct.
- If this run does not consume $\alpha.p$, then we have $r = \alpha.p|q'$ with $q \stackrel{\tau*}{\Rightarrow} q'$. By hypothesis $\hat{\alpha}.p|q$ and $\alpha.0|q$ are successful, and so $\hat{\alpha}.p \mid q'$ and $\alpha.0 \mid q'$ have inconsistent reducts.
  - If r has a run that consumes $\alpha.p$, then this run is $\alpha.p \mid q' \stackrel{\tau*}{\Rightarrow} \alpha.p \mid q'' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(p \mid t)$, and $(\nu\vec{x})(p \mid t)$ is also a reduct of $\hat{\alpha}.p \mid q'$, hence it has an inconsistent reduct.
  - If there is no run of r that consumes $\alpha.p$, then we consider $\alpha.0 \mid q'$: by hypothesis there is a run $\alpha.0 \mid q' \stackrel{\tau*}{\Rightarrow} \alpha.0 \mid q''$ where $\alpha.0 \mid q''$ is inconsistent. This does not depend on the 0 that is prefixed with α since this α is never consumed, so $\alpha.p \mid q''$ is inconsistent too.

Therefore r has an inconsistent reduct, and $\alpha.p \perp q$. □

Proposition 9. *For all actions α and terms p and q,*

$$\hat{\alpha}.0 \simeq 0 \qquad\qquad \hat{\alpha}.(p+q) \simeq \hat{\alpha}.p + \hat{\alpha}.q$$

Proof. In order to prove $\hat{\alpha}.0 \simeq 0$ it suffices to prove that, for any slice s, $\hat{\alpha}.0 \perp s$. Let s be an arbitrary slice, consider a run $\hat{\alpha}.0 \mid s \stackrel{\tau*}{\Rightarrow} r$. By lemma 9, assume this run consumes $\hat{\alpha}$, then 0 occurs at top level in r so r is inconsistent.

Let s be a slice orthogonal to $\hat{\alpha}.(p+q)$, consider a run $\hat{\alpha}.p \mid s \stackrel{\tau*}{\Rightarrow} r$ that consumes $\hat{\alpha}$. Thus the run it is $\hat{\alpha}.p \mid s \stackrel{\tau*}{\Rightarrow} \hat{\alpha}.p \mid s' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(t \mid s'') \stackrel{\tau*}{\Rightarrow} r$ where $\vec{x}$ is the set of bound names of α, s'' is a term such that $s' \stackrel{\bar{\alpha}}{\Rightarrow} s''$ and t is a slice of p. Then t is a slice of $p+q$ thus the execution $\hat{\alpha}.(p+q) \mid s \stackrel{\tau*}{\Rightarrow} \hat{\alpha}.(p+q) \mid s' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(t \mid s'') \stackrel{\tau*}{\Rightarrow} r$ holds, and by hypothesis r has an inconsistent reduct. Thus $\hat{\alpha}.p \perp s$, and by the same argument we get $\hat{\alpha}.q \perp s$, so $\hat{\alpha}.p + \hat{\alpha}.q \perp s$.

Now let s be a slice orthogonal to $\hat{\alpha}.p$ and $\hat{\alpha}.q$, consider a run $\hat{\alpha}.(p+q) \mid s \stackrel{\tau*}{\Rightarrow} r$ that consumes $\hat{\alpha}$. Then the run is $\hat{\alpha}.(p+q) \mid s \stackrel{\tau*}{\Rightarrow} \hat{\alpha}.(p+q) \mid s' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(t \mid s') \stackrel{\tau*}{\Rightarrow} r$ with $s' \stackrel{\bar{\alpha}}{\Rightarrow} s''$ and $t \in \mathfrak{s}(p+q) = \mathfrak{s}(p) \cup \mathfrak{s}(q)$. If $t \in \mathfrak{s}(p)$ then the run $\hat{\alpha}.p \mid s \stackrel{\tau*}{\Rightarrow} \hat{\alpha}.p \mid s' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(t \mid s'') \stackrel{\tau*}{\Rightarrow} r$ holds, and by hypothesis r has an inconsistent reduct. If $t \in \mathfrak{s}(q)$, we conclude by a similar argument, therefore $\hat{\alpha}.(p+q) \perp s$. □

Proposition 10. *For all actions $\alpha = u^{\varepsilon}(\vec{x})$ and terms p and q,*

$$\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q \simeq \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q) + (\nu\vec{x})(p \mid q)$$

Proof. Let s be a slice that is orthogonal to $\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q)$, $\hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q)$ and $(\nu\vec{x})(p \mid q)$. Consider a run $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q \mid s \stackrel{\tau*}{\Rightarrow} r$.

- If the first transition of $\hat{\alpha}.p|\hat{\bar{\alpha}}.q$ is labelled α, then there is a sequence of transitions $s \stackrel{\tau*}{\Rightarrow} s' \stackrel{\bar{\alpha}}{\Rightarrow} t$ such that the run is $\hat{\alpha}.p|\hat{\bar{\alpha}}.q|s \stackrel{\tau*}{\Rightarrow} \hat{\alpha}.p|\hat{\bar{\alpha}}.q|s' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(p|\hat{\bar{\alpha}}.q|t) \stackrel{\tau*}{\Rightarrow} r$. From this we can deduce the execution $\hat{\alpha}.(p|\hat{\bar{\alpha}}.q)|s \stackrel{\tau*}{\Rightarrow} \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) \mid s' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(p \mid \hat{\bar{\alpha}}.q \mid t) \stackrel{\tau*}{\Rightarrow} r$, so by hypothesis r has an inconsistent reduct.
- The case where the first transition of $\hat{\alpha}.p|\hat{\bar{\alpha}}.q$ is labelled $\bar{\alpha}$ is similar.
- If the first transition of $\hat{\alpha}.p|\hat{\bar{\alpha}}.q$ is labelled τ, then there is a sequence of transitions $s \stackrel{\tau*}{\Rightarrow} s'$ such that the run is $\hat{\alpha}.p|\hat{\bar{\alpha}}.q|s \stackrel{\tau*}{\Rightarrow} \hat{\alpha}.p|\hat{\bar{\alpha}}.q|s' \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(p|q)|s' \stackrel{\tau*}{\Rightarrow} r$. From this we can deduce the execution $(\nu\vec{x})(p|q)|s \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(p|q)|s' \stackrel{\tau*}{\Rightarrow} r$, so by hypothesis r has an inconsistent reduct.
- If the considered run does not contain any transition of $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q$ then the run can be extended by one step that reduces $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q$ into $(\nu\vec{x})(p \mid q)$ and we get back to the previous case.

Therefore s is orthogonal to $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q$.

Now let s be a slice orthogonal to $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q$. Using lemma 9, consider a run $\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) \mid s \stackrel{\tau*}{\Rightarrow} r$ that consumes $\hat{\alpha}$, then r is a reduct of $\hat{\alpha}.p|\hat{\bar{\alpha}}.q|s$ so it has an inconsistent reduct, therefore we have $\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) \perp s$. By a similar argument we get $\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) \perp s$. Finally, consider a run $(\nu\vec{x})(p|q)|s \stackrel{\tau*}{\Rightarrow} r$: the execution $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q \mid s \stackrel{\tau}{\Rightarrow} (\nu\vec{x})(p \mid q) \mid s \stackrel{\tau*}{\Rightarrow} r$ is valid so

r has an inconsistent reduct by hypothesis. Therefore s is orthogonal to $\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q) + (\nu\vec{x})(p \mid q)$. □

Proposition 11. *For any non dual actions α and β and terms p and q,*

$$\hat{\alpha}.p \mid \hat{\beta}.q \simeq \hat{\alpha}.(p \mid \hat{\beta}.q) + \hat{\beta}.(\hat{\alpha}.p \mid q)$$

Proof. The proof is essentially the same as that of proposition 10 except that the cases for the interaction between α and $\bar{\alpha}$ are irrelevant here. □

Proposition 12. *For any actions α and β and any term p, we have $\hat{\alpha}.p|\beta.0 \simeq \hat{\alpha}.(p|\beta.0)$, $\alpha.0|\alpha.0 \simeq \alpha.0$ and $\alpha.0|\bar{\alpha}.0 \simeq 0$.*

Proof. Assume $\alpha = u^{\varepsilon}(\vec{x})$. Let s be a slice orthogonal to $\hat{\alpha}.p|\beta.0$. Using lemma 9, consider a run $\hat{\alpha}.(p|\beta.0)|s \stackrel{\tau*}{\Longrightarrow} \hat{\alpha}.(p| \beta.0) \mid s' \stackrel{\tau}{\Longrightarrow} (\nu\vec{x})(p \mid \beta.0 \mid s'') \stackrel{\tau*}{\Longrightarrow} r$ with s'' such that $s' \stackrel{\bar{\alpha}}{\Longrightarrow} s''$, then the run $\hat{\alpha}.p|\beta.0|s \stackrel{\tau*}{\Longrightarrow} \hat{\alpha}.p|\beta.0|s' \stackrel{\tau}{\Longrightarrow} (\nu\vec{x})(p|\beta.0|s'') \stackrel{\tau*}{\Longrightarrow} r$ holds, and r has an inconsistent reduct.

Now let s be a slice orthogonal to $\hat{\alpha}.(p \mid \beta.0)$. Consider a run $\hat{\alpha}.p \mid \beta.0 \mid s \stackrel{\tau*}{\Longrightarrow} r$, and using lemma 9 assume that this run consumes $\hat{\alpha}$. The transition that consumes $\hat{\alpha}$ is either produced by $\beta.0$, which may happen if $\beta = \bar{\alpha}$, or by a reduct of s. In the first case, 0 occurs at top level in r, which is then inconsistent. In the second case, r is a reduct of $\hat{\alpha}.(p \mid \beta.0) \mid s$ so it has an inconsistent reduct by hypothesis. Thus $\hat{\alpha}.p \mid \beta.0 \perp s$.

The equivalence of $\alpha.0$ and $\alpha.0 \mid \alpha.0$ comes from the fact that both terms have exactly one transition labelled α, leading to a null term, 0 in one case and $0 \mid \alpha.0$ in the other.

For the equivalence of $\alpha.0 \mid \bar{\alpha}.0$ and 0, consider an arbitrary slice s and run $\alpha.0 \mid \bar{\alpha}.0 \mid s \stackrel{\tau*}{\Longrightarrow} r$. If this run contains a transition of $\alpha.0 \mid \bar{\alpha}.0$, then 0 necessarily occurs at top level in r, which is thus inconsistent. Otherwise we have $r = \alpha.0 \mid \bar{\alpha}.0 \mid s'$ for some run $s \stackrel{\tau*}{\Longrightarrow} s'$, and the transition $\alpha.0 \mid \bar{\alpha}.0 \mid s' \stackrel{\tau}{\Longrightarrow} 0 \mid 0 \mid s'$ holds. Therefore $\alpha.0 \mid \bar{\alpha}.0 \perp s$ for any s, so $\alpha.0 \mid \bar{\alpha}.0 \simeq 0$. □

Proposition 13. *For any action $u^{\varepsilon}(\vec{x})$, any term p and any name $v \neq u$, we have $(\nu v)\hat{u}^{\varepsilon}(\vec{x}).p \simeq \hat{u}^{\varepsilon}(\vec{x}).(\nu v)p$, $(\nu u)\hat{u}^{\varepsilon}(\vec{x}).p \simeq 0$ and $(\nu u)u^{\varepsilon}(\vec{x}).0 \simeq 1$.*

Proof. The first equivalence derives from the fact that $(\nu v)\hat{u}^{\varepsilon}(\vec{x}).p$ and $\hat{u}^{\varepsilon}(\vec{x}).(\nu v)p$ both have exactly one transition, labelled $u^{\varepsilon}(\vec{x})$, that leads to the same term $(\nu v)p$ (assuming without loss of generality that $v \notin \vec{x}$).

For the second equality, let s be an arbitrary slice. Any run $(\nu u)\hat{u}^{\varepsilon}(\vec{x}).p \mid s \stackrel{\tau*}{\Longrightarrow} r$ leaves $\hat{u}^{\varepsilon}(\vec{x})$ untouched, since no occurrence of u may occur in s, therefore $(\nu u)\hat{u}^{\varepsilon}(\vec{x})$ is inconsistent, thus r is inconsistent.

The third equality derives from the fact that $(\nu u)u^{\varepsilon}(\vec{x}).0$ has no free name, no transition, and no linear action or 0 at top level that could produce inconsistencies. □

A.3. Confluence

Proof of lemma 3. We first prove, by induction, that if a slice q has no occurrence of 0 or actions $\beta.q'$ with $q' \not\cong 0$ at top level, then for all $\hat{\alpha}.p$ there is an r such that $\hat{\alpha}.p \mid q \succ^* \hat{\alpha}.(p|q)+r$. If $q = \hat{\bar{\alpha}}.q_1$, rule I1 gives $\hat{\alpha}.p|\hat{\bar{\beta}}.q_1 \succ \hat{\alpha}.(p|\hat{\bar{\beta}}.q_1) + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q_1) + (\nu\vec{x})(p \mid q_1)$ where $\vec{x} = \mathrm{bn}(\alpha)$, so we can choose $r = \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q_1) + (\nu\vec{x})(p \mid q_1)$. If $q = \hat{\beta}.q_1$ with $\beta \neq \bar{\alpha}$, rule I2 gives $\hat{\alpha}.p|\hat{\beta}.q_1 \succ \hat{\alpha}.(p|\hat{\beta}.q_1) + \hat{\beta}.(\hat{\alpha}.p|q_1)$, so we can choose $r = \hat{\beta}.(\hat{\alpha}.p|q_1)$. If $q = \beta.0$, rule I3 gives $\hat{\alpha}.p|\beta.0 \succ \hat{\alpha}.(p|\beta.0)$ so we can choose $r = 0$. If $q = q_1 \mid q_2$, then by induction hypothesis on q_1 we get $\hat{\alpha}.p \mid q_1 \succ^* \hat{\alpha}.(p \mid q_1) + r_1$, so $\hat{\alpha}.p \mid q \succ^* \hat{\alpha}.(p \mid q_1) \mid q_2 + r_1 \mid q_2$. By induction hypothesis on q_2 we get $\hat{\alpha}.(p \mid q_1) \mid q_2 \succ^* \hat{\alpha}.(p \mid q_1 \mid q_2) + r_2$, hence $\hat{\alpha}.p \mid q \succ^* \hat{\alpha}.(p \mid q) + r_1 \mid q_2 + r_2$. If $q = (\nu x)q'$, the induction hypothesis gives $\hat{\alpha}.p \mid q' \succ^* \hat{\alpha}.(p \mid q') + r'$, hence we have $\hat{\alpha}.p \mid (\nu x)q' \cong (\nu x)(\hat{\alpha}.p \mid q') \succ^* (\nu x)\hat{\alpha}.(p \mid q') + (\nu x)r' \succ \hat{\alpha}.(p \mid (\nu x)q')$ by rule N1. The case of 1 is trivial and completes this proof.

Now consider an arbitrary term q. By applications of rule A we get $q \succ^* q'$, where q' only has actions of the form $\hat{\alpha}.p$ or $\alpha.0$ at top level. Up to $\cong$, q' is 0 or a sum of terms of the form used in the first part of the proof. For 0 we get $\hat{\alpha}.p \mid 0 \cong 0$ and $\hat{\alpha}.p \mid 0 + \hat{\alpha}.(p \mid 0) \cong 0 + \hat{\alpha}.0 \succ 0$ by rule L1, hence 0 is a common reduct. If q' is a sum $q' = \sum_{i \in I} q_i$, we have $\hat{\alpha}.p \mid q \succ^* \sum_{i \in I} \hat{\alpha}.p \mid q_i$, and by rule L2 we get $\hat{\alpha}.(p \mid q) \succ^* \sum_{i \in I} \hat{\alpha}.(p \mid q_i)$, so $\hat{\alpha}.p \mid q + \hat{\alpha}.(p \mid q) \succ^* \sum_{i \in I}(\hat{\alpha}.p \mid q_i + \hat{\alpha}.(p \mid q_i))$. For each i, by the first part of the proof, there is an r_i such that $\hat{\alpha}.p \mid q_i \succ^* \hat{\alpha}.(p \mid q_i) + r_i$, so we have $\hat{\alpha}.p \mid q_i + \hat{\alpha}.(p \mid q_i) \succ^* \hat{\alpha}.(p \mid q_i) + r_i + \hat{\alpha}.(p \mid q_i)$, which reduces to $\hat{\alpha}.(p \mid q_i) + r_i$ by rule S. So we finally get that $\hat{\alpha}.p \mid q$ and $\hat{\alpha}.p \mid q + \hat{\alpha}.(p \mid q)$ have a common reduct. □

For the proof of theorem 2, let us examine all possible critical pairs:

- A conflict on S is a reduction $p + p \succ p' + p$, versus a reduction $p + p \succ p$, and p' is a reduct of both, through $p' + p \succ p' + p' \succ p'$ in the first case.
- Rules A and L1 have no conflict (because of the side-conditions).
- L2/I1:

  $$\hat{\alpha}.(p + q) \mid \hat{\bar{\alpha}}.r \succ (\hat{\alpha}.p + \hat{\alpha}.q) \mid \hat{\bar{\alpha}}.r$$
  $$\hat{\alpha}.(p + q) \mid \hat{\bar{\alpha}}.r \succ \hat{\alpha}.((p + q) \mid \hat{\bar{\alpha}}.r) + \hat{\bar{\alpha}}.(\hat{\alpha}.(p + q) \mid r) + (\nu\vec{x})((p + q) \mid r)$$

  Then the first reduct is structurally equivalent to $\hat{\alpha}.p \mid \hat{\bar{\alpha}}.r + \hat{\alpha}.q \mid \hat{\bar{\alpha}}.r$, and the following reductions hold:

  $$\hat{\alpha}.p \mid \hat{\bar{\alpha}}.r \succ \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r) + \hat{\bar{\alpha}}.(\alpha.p \mid r) + (\nu\vec{x})(p \mid r)$$
  $$\hat{\alpha}.q \mid \hat{\bar{\alpha}}.r \succ \hat{\alpha}.(q \mid \hat{\bar{\alpha}}.r) + \hat{\bar{\alpha}}.(\alpha.q \mid r) + (\nu\vec{x})(q \mid r)$$

The summands of the second reduct can be reduced as

$$\begin{aligned}\hat{\alpha}.((p+q) \mid \hat{\bar{\alpha}}.r) &\cong \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r + q \mid \hat{\bar{\alpha}}.r)\\ &\succ \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.r) + \hat{\alpha}.(q \mid \hat{\bar{\alpha}}.r)\\ \hat{\bar{\alpha}}.(\hat{\alpha}.(p+q) \mid r) &\succ \hat{\bar{\alpha}}.((\hat{\alpha}.p + \hat{\alpha}.q) \mid r)\\ &\succ \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid r) + \hat{\bar{\alpha}}.(\hat{\alpha}.q \mid r)\\ (\boldsymbol{\nu}\vec{x})((p+q) \mid r) &\cong (\boldsymbol{\nu}\vec{x})(p \mid r) + (\boldsymbol{\nu}\vec{x})(p \mid r)\end{aligned}$$

The sums in both cases are equal.

- The L2/I2 and L2/I3 conflicts are solved the same way.
- L2/N1:

$$\begin{aligned}(\boldsymbol{\nu}v)\hat{u}^{\varepsilon}(\vec{x}).(p+q) &\succ (\boldsymbol{\nu}v)\hat{u}^{\varepsilon}(\vec{x}).p + (\boldsymbol{\nu}v)\hat{u}^{\varepsilon}(\vec{x}).q\\ &\succ^2 \hat{u}^{\varepsilon}(\vec{x}).(\boldsymbol{\nu}v)p + \hat{u}^{\varepsilon}(\vec{x}).(\boldsymbol{\nu}v)q\\ (\boldsymbol{\nu}v)\hat{u}^{\varepsilon}(\vec{x}).(p+q) &\succ \hat{u}^{\varepsilon}(\vec{x}).((\boldsymbol{\nu}v)p + (\boldsymbol{\nu}v)q)\\ &\succ \hat{u}^{\varepsilon}(\vec{x}).(\boldsymbol{\nu}v)p + \hat{u}^{\varepsilon}(\vec{x}).(\boldsymbol{\nu}v)q\end{aligned}$$

- L2/N2:

$$\begin{aligned}&(\boldsymbol{\nu}u)\hat{u}^{\varepsilon}(\vec{x}).(p+q) \succ (\boldsymbol{\nu}u)\hat{u}^{\varepsilon}(\vec{x}).p + (\boldsymbol{\nu}v)\hat{u}^{\varepsilon}(\vec{x}).q \succ^2 0 + 0 \cong 0\\ &(\boldsymbol{\nu}v)\hat{u}^{\varepsilon}(\vec{x}).(p+q) \succ 0\end{aligned}$$

- I1/I1: This rule is solved in the main part of the text.
- The I1/I2 conflict is resolved in a similar manner.
- I1/I3:

$$\begin{aligned}&\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q \mid \beta.0\\ &\succ \hat{\alpha}.(p \mid \beta.0) \mid \hat{\bar{\alpha}}.q\\ &\succ \hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q \mid \beta.0) + \hat{\bar{\alpha}}.(\hat{\alpha}.(p \mid \beta.0) \mid q) + (\boldsymbol{\nu}\vec{x})(p \mid q) \mid \beta.0\\ &\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q \mid \beta.0\\ &\succ \hat{a}.(p \mid \hat{\bar{\alpha}}.q) \mid \beta.0 + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q) \mid \beta.0 + (\boldsymbol{\nu}\vec{x})(p \mid q) \mid \beta.0\\ &\succ^2 \hat{a}.(p \mid \hat{\bar{\alpha}}.q \mid \beta.0) + \hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q \mid \beta.0) + (\boldsymbol{\nu}\vec{x})(p \mid q) \mid \beta.0\\ &\succ \hat{a}.(p \mid \hat{\bar{\alpha}}.q \mid \beta.0) + \hat{\bar{\alpha}}.(\hat{\alpha}.(p \mid \beta.0) \mid q) + (\boldsymbol{\nu}\vec{x})(p \mid q) \mid \beta.0\end{aligned}$$

- I1/N1: if $u \notin \mathrm{n}(\alpha)$,

$$\begin{aligned}&(\boldsymbol{\nu}u)\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q\\ &\succ (\boldsymbol{\nu}u)\hat{\alpha}.(p \mid \hat{\bar{\alpha}}.q) + (\boldsymbol{\nu}u)\hat{\bar{\alpha}}.(\hat{\alpha}.p \mid q) + (\boldsymbol{\nu}u\vec{x})(p \mid q)\\ &\succ^3 \hat{\alpha}.((\boldsymbol{\nu}u)p \mid \hat{\bar{\alpha}}.q) + \hat{\bar{\alpha}}.(\hat{\alpha}.(\boldsymbol{\nu}u)p \mid q) + (\boldsymbol{\nu}\vec{x})((\boldsymbol{\nu}u)p \mid q)\\ &(\boldsymbol{\nu}u)\hat{\alpha}.p \mid \hat{\bar{\alpha}}.q\\ &\succ \hat{\alpha}.(\boldsymbol{\nu}u)p \mid \hat{\bar{\alpha}}.q\\ &\succ \hat{\alpha}.((\boldsymbol{\nu}u)p \mid \hat{\bar{\alpha}}.q) + \hat{\bar{\alpha}}.(\hat{\alpha}.(\boldsymbol{\nu}u)p \mid q) + (\boldsymbol{\nu}\vec{x})((\boldsymbol{\nu}u)p \mid q)\end{aligned}$$

- For I2/I2, assume we have two reductions

$$\begin{aligned}&\hat{\alpha}.p \mid \hat{\beta}.q \mid \hat{\gamma}.r \succ \hat{\alpha}.(p \mid \hat{\beta}.q) \mid \hat{\gamma}.r + \hat{\beta}.(\hat{\alpha}.p \mid q) \mid \hat{\gamma}.r\\ &\hat{\alpha}.p \mid \hat{\beta}.q \mid \hat{\gamma}.r \succ \hat{\alpha}.(p \mid \hat{\gamma}.r) \mid \hat{\beta}.q + \hat{\gamma}.(\hat{\alpha}.p \mid r) \mid \hat{\beta}.q\end{aligned}$$

then by several applications of the R2 and L2 rules we can reduce both terms to

$$\begin{aligned}\hat{\alpha}.(p \mid \hat{\beta}.q \mid \hat{\gamma}.r) + \hat{\beta}.(\hat{\alpha}.(p \mid \hat{\gamma}.r) \mid q) + \hat{\beta}.(\hat{\gamma}.(\hat{\alpha}.p \mid r) \mid q)\\ + \hat{\gamma}.(\hat{\alpha}.(p \mid \hat{\beta}.q) \mid r) + \hat{\gamma}.(\hat{\beta}.(\hat{\alpha}.p \mid q) \mid r)\end{aligned}$$

- I2/I3 is resolved like I1/I3.
- I2/N1 is resolved like I1/N1.
- I2/N2: if u is the subject name of α,

$$\begin{aligned}(\boldsymbol{\nu}u)\hat{\alpha}.p \mid \hat{\beta}.q &\succ (\boldsymbol{\nu}u)\hat{\alpha}.(p \mid \hat{\beta}.q) + (\boldsymbol{\nu}u)\hat{\beta}.(\hat{\alpha}.p \mid q)\\ &\succ^2 0 + \hat{\beta}.((\boldsymbol{\nu}u)\hat{\alpha}.p \mid q) \succ \hat{\beta}.(0 \mid q) \succ 0\\ (\boldsymbol{\nu}u)\hat{\alpha}.p \mid \hat{\beta}.q &\succ 0 \mid \hat{\beta}.q \cong 0\end{aligned}$$

- I3/I3 can be of two kinds: either the linear action is involved in two reductions, or it is the inaction. For the first case, we have

$$\begin{aligned}&\hat{\alpha}.p \mid \beta.0 \mid \gamma.0 \succ \hat{\alpha}.(p \mid \beta.0) \mid \gamma.0 \succ \hat{\alpha}.(p \mid \beta.0 \mid \gamma.0)\\ &\hat{\alpha}.p \mid \beta.0 \mid \gamma.0 \succ \hat{\alpha}.(p \mid \gamma.0) \mid \beta.0 \succ \hat{\alpha}.(p \mid \beta.0 \mid \gamma.0)\end{aligned}$$

For the second case we have

$$\begin{aligned}\hat{\alpha}.p \mid \hat{\beta}.q \mid \gamma.0 &\succ \hat{\alpha}.(p \mid \gamma.0) \mid \hat{\beta}.q\\ &\succ \hat{\alpha}.(p \mid \hat{\beta}.q \mid \gamma.0) + \hat{\beta}.(\hat{\alpha}.(p \mid \gamma.0) \mid q)\\ &\succ \hat{\alpha}.(p \mid \hat{\beta}.(q \mid \gamma.0)) + \hat{\beta}.(\hat{\alpha}.(p \mid \gamma.0) \mid q)\\ \hat{\alpha}.p \mid \hat{\beta}.q \mid \gamma.0 &\succ \hat{\beta}.(q \mid \gamma.0) \mid \hat{\alpha}.p\\ &\succ \hat{\alpha}.(p \mid \hat{\beta}.(q \mid \gamma.0)) + \hat{\beta}.(\hat{\alpha}.p \mid q \mid \gamma.0)\\ &\succ \hat{\alpha}.(p \mid \hat{\beta}.(q \mid \gamma.0)) + \hat{\beta}.(\hat{\alpha}.(p \mid \gamma.0) \mid q)\end{aligned}$$

This assumes $\beta \neq \bar{\alpha}$, but the case $\beta = \bar{\alpha}$ is similar with an extra $(\boldsymbol{\nu}\vec{x})(p \mid q) \mid \gamma.0$ in each reduction.

- I3/I4:

$$\begin{aligned}\hat{\alpha}.p \mid \beta.0 \mid \beta.0 &\succ \hat{\alpha}.(p \mid \beta.0) \mid \beta.0\\ &\succ \hat{\alpha}.(p \mid \beta.0 \mid \beta.0) \succ \hat{\alpha}.(p \mid \beta.0)\\ \hat{\alpha}.p \mid \beta.0 \mid \beta.0 &\succ \hat{\alpha}.p \mid \beta.0 \succ \hat{\alpha}.(p \mid \beta.0)\end{aligned}$$

- I3/I5:

$$\begin{aligned}\hat{\alpha}.p \mid \beta.0 \mid \bar{\beta}.0 &\succ \hat{\alpha}.(p \mid \beta.0) \mid \bar{\beta}.0\\ &\succ \hat{\alpha}.(p \mid \beta.0 \mid \bar{\beta}.0) \succ \hat{\alpha}.(p \mid 0) \succ 0\\ \hat{\alpha}.p \mid \beta.0 \mid \bar{\beta}.0 &\succ \hat{\alpha}.p \mid 0 \cong 0\end{aligned}$$

- I3/N1: if $u \notin \mathrm{n}(\alpha)$:

$$\begin{aligned}&(\boldsymbol{\nu}u)\hat{\alpha}.p \mid \beta.0 \succ (\boldsymbol{\nu}u)\hat{\alpha}.(p \mid \beta.0) \succ \hat{\alpha}.((\boldsymbol{\nu}u)p \mid \beta.0)\\ &(\boldsymbol{\nu}u)\hat{\alpha}.p \mid \beta.0 \succ \hat{\alpha}.(\boldsymbol{\nu}u)p \mid \beta.0 \succ \hat{\alpha}.((\boldsymbol{\nu}u)p \mid \beta.0)\end{aligned}$$

- I3/N2: if u is the subject name of α,

$$(\nu u)\hat{\alpha}.p \mid \beta.0 \succ (\nu u)\hat{\alpha}.(p \mid \beta.0) \succ 0$$
$$(\nu u)\hat{\alpha}.p \mid \beta.0 \succ 0 \mid \beta.0 \cong 0$$

- I3/N3: if u is the subject name of β,

$$\hat{\alpha}.p \mid (\nu u)\beta.0 \succ (\nu u)\hat{\alpha}.(p \mid \beta.0)$$
$$\succ \hat{\alpha}.(p \mid (\nu u)\beta.0) \succ \hat{\alpha}.(p \mid 1) \cong \hat{\alpha}.p$$
$$\hat{\alpha}.p \mid (\nu u)\beta.0 \succ \hat{\alpha}.p \mid 1 \cong \hat{\alpha}.p$$

- I4/I4 and I5/I5 are obvious
- I4/I5:

$$\alpha.0 \mid \alpha.0 \mid \bar{\alpha}.0 \succ \alpha.0 \mid \bar{\alpha}.0 \succ 0$$
$$\alpha.0 \mid \alpha.0 \mid \bar{\alpha}.0 \succ \alpha.0 \mid 0 \cong 0$$

This solves all the possible conflicts between the rules.

A.4. Termination

Proof of lemma 4. We reason by case analysis on the rewriting rules. We can assume without loss of generality that p is a slice. For rule I1 we have $p \cong (\nu\vec{x})(\hat{\alpha}.p_1 | \hat{\bar{\alpha}}.p_2 | p_3)$ and q is the sum of $q_1 = (\nu\vec{x})(\hat{\alpha}.(p_1 \mid \hat{\bar{\alpha}}.p_2) \mid p_3)$, $q_2 = (\nu\vec{x})(\hat{\bar{\alpha}}.(\hat{\alpha}.p_1 | p_2) | p_3)$ and $q_3 = (\nu\vec{x}\vec{y}).(p_1 | p_2 | p_3)$, where $\vec{y}$ is the sequence of bound names of α. Consider a run of q_1: if it consumes $\hat{\alpha}$, then it begins with $(\nu\vec{x})(\hat{\alpha}.(p_1 \mid \hat{\bar{\alpha}}.p_2) \mid p_3) \stackrel{\tau*}{\Rightarrow} (\nu\vec{x})(\hat{\alpha}.(p_1 | \hat{\bar{\alpha}}.p_2) | p_3') \stackrel{\tau}{\Rightarrow} (\nu\vec{x}\vec{y})(p_1 | \hat{\bar{\alpha}}.p_2 | p_3')$, from which we can deduce a run of the same length in p: $(\nu\vec{x})(\hat{\alpha}.p_1 \mid \hat{\bar{\alpha}}.p_2 \mid p_3) \stackrel{\tau*}{\Rightarrow} (\nu\vec{x})(\hat{\alpha}.p_1 \mid \hat{\bar{\alpha}}.p_2 \mid p_3') \stackrel{\tau}{\Rightarrow} (\nu\vec{x}\vec{y})(p_1 \mid \hat{\bar{\alpha}}.p_2 \mid p_3')$, which proves that the considered run has length at most $\|p\|_t$. If the considered run does not touch α, it is a run of p_3 and we deduce a run of the same length in p, hence $\|q_1\|_t \leqslant \|p\|_t$. By similar arguments we get $\|q_2\|_t \leqslant \|p\|_t$. If we consider a run of q_3, we can deduce a run of p of the same length plus one by prefixing it by the transition $p \succ q_3$, hence $\|q_3\|_t \leqslant \|p\|_t$, and finally $\|q\|_t \leqslant \|p\|_t$. The result is proved for the other rules in a similar way. □

Proof of lemma 5. We define a measure $N(p)$ on terms. A term p, up to $\cong$, can always be written $\sum_{i\in I}(\nu\vec{x}_i)\prod_{j\in J_i}\mathring{\alpha}_{ij}.p_{ij}$, in a unique way up to permutation, assuming $\vec{x}_i$ is minimal. Then let $N(p)$ be the multiset $\left\{ |\vec{x}_i| + \sum_{j\in J_i}\sharp\mathrm{s}(p_{ij}) + k_{ij} \;\middle|\; i \in I \right\}$, where $k_{ij} = 1$ if $\mathring{\alpha}_{ij}$ is affine and $p_{ij} \not\cong 0$ and $k_{ij} = 0$ otherwise. Then each rule except I1 makes N decrease strictly, for the multiset order, when applied at top level: Rule A is $\alpha.p \succ_t \hat{\alpha}.p + \alpha.0$ if $p \not\cong 0$, so it replaces $\{a + \sharp\mathrm{s}(p) + 1\}$, where a is the contribution of the context, with $\{a + \sharp\mathrm{s}(p), a + 1\}$, and this is strictly decreasing since $\sharp\mathrm{s}(p) \geqslant 1$ for any term p. Rule L2 is $\hat{\alpha}.(p + q) \succ_t \hat{\alpha}.p + \hat{\alpha}.q$, it replaces $\{a + \sharp\mathrm{s}(p) + \sharp\mathrm{s}(q)\}$ with $\{a + \sharp\mathrm{s}(p), a + \sharp\mathrm{s}(q)\}$. Rule I2 is $\hat{\alpha}.p|\hat{\beta}.q \succ_t \hat{\alpha}.(p|\hat{\beta}.q)+\hat{\beta}.(\hat{\alpha}.p|q)$, it replaces $\{a+\sharp\mathrm{s}(p)+\sharp\mathrm{s}(q)\}$ with $\{a + \sharp\mathrm{s}(p), a + \sharp\mathrm{s}(q)\}$. Rule I3 is $\hat{\alpha}.p \mid \beta.0 \succ_t \hat{\alpha}.(p \mid \beta.0)$, it replaces $\{a + \sharp\mathrm{s}(p) + 1$ with $\{a + \sharp\mathrm{s}(p)\}$. Rules I4, N2 and N3 remove an action in a slice so the decrease a value in N. Rules S, L1 and I5 removes a term of the sum so they remove an element of N. Rule N1 removes a bound name from the top level of a slice, so it decreases a value in N. Since the multiset order is well founded, this proves that there is no infinite sequence of $\succ_t$ without the rule I1. □

Proof of lemma 6. It is clear that a slice p has a τ transition if and only if the rule I1 can be applied at top level in p, so if p has an infinite run then there is an infinite sequence of I1 reductions, hence an infinite sequence of $\succ_t$.

Now assume p has an infinite sequence of $\succ_t$ reductions. If rule S is used at some point, we can remove it from the reduction and still get a valid infinite reduction, with extra summands at each step, so we assume the sequence does not use S. Let $p_0 = p$. Let i be an integer, assume p_i has an infinite reduction. Then some slice s_i of p_i has an infinite reduction. Define r_i and p_{i+1} such that $p_i = s_i + r_i$, $s_i \succ_t p_{i+1}$ and p_{i+1} has an infinite reduction. For all i, let $q_i = p_i + \sum_{j<i} r_j$, then $q_0 \succ_t q_1 \succ_t q_2 \ldots$ is an infinite reduction of p.

Assume that p has no infinite run, i.e. that $\|p\|_t$ is finite. Let i be an integer. As in lemma 5, if a rule other than I1 is used to define $s_i \succ_t p_{i+1}$, then $\|p_{i+1}\|_t \leqslant \|s_i\|_t$ and $N(p_{i+1}) < N(s_i)$, and since s_{i+1} is a slice of p_{i+1} we have $\|s_{i+1}\|_t \leqslant \|s_i\|_t$ and $N(s_{i+1}) < N(s_i)$. If I1 is used but s_{i+1} is not the slice where the $(\nu\vec{x})(p|q)$ occurs, we also have $\|s_{i+1}\|_t \leqslant \|s_i\|_t$ and $N(s_{i+1}) < N(s_i)$. If I1 is used and s_{i+1} is the slice where the $(\nu\vec{x})(p \mid q)$ occurs, then we have $s_i \stackrel{\tau}{\Rightarrow} s_{i+1}$, so the maximal runs of s_{i+1} are strictly shorter than those of s_i, i.e. $\|s_{i+1}\|_t < \|s_i\|_t$. Therefore the pair $(\|s_i\|_t, N(s_i))$ is strictly decreasing, thus finite, which is contradictory. □

Proof of lemma 7. It is easy to prove that $\|\mathring{\alpha}.p\| = \|p\| + 1$, $\|p + q\| = \max(\|p\|, \|q\|)$, $\|p \mid q\| = \|p\| + \|q\|$, $\|(\nu x)p\| = \|p\|$ and $\|0\| = \|1\| = 0$, from which we deduce that, for any affine context C, if $\|p\| \geqslant \|q\|$ then $\|C[p]\| \geqslant \|C[q]\|$. So it is enough to prove the result for $p \succ_t q$. This derives from the fact that all sequences of transitions in slices of q are also present in some slice of p. The only exception is when $p \succ_t q$ uses rule I1, in which case for a sequence of transitions of q there is the same sequence in p, possibly with an extra τ at the beginning. □

Proof of proposition 3. Suppose p has an infinite sequence of reductions. As above, we assume without loss of generality that rule S is not used. For any reductions $p \succ q \succ_t r$ where $p \succ q$ is not at top level, there is a term q' such that $p \succ_t q' \succ^* r$, with no reduction at top level in $q' \succ^* r$ (this is proved by case analysis on the reduction $q \succ_t r$, we have to use $q' \succ^* r$ and not $q' \succ r$ because several rules duplicate or erase subterms using sums). Therefore p also has an infinite sequence of reductions where all reductions at top level are done first. Since $\|p\|$ is finite, $\|p\|_t$ is finite, and by lemma 6 we know that it has no infinite sequence of top-level reductions. Therefore we have $p >_t^* p'$ where p' has an infinite sequence of reductions not at top level, so there is a slice $s \in \mathfrak{s}(p)$ that has this property. If we write $s = (\nu\vec{x}) \prod_{i\in I} \mathring{\alpha}_i.p_i$, since there the considered infinite sequence has no reduction at top level, all reduction steps occur in one of the p_i, and reductions in different p_i's are independent. Therefore there is a p_i that has an infinite sequence of reductions. We have $\|p_i\| = \|\mathring{\alpha}_i.p_i\| - 1 \leqslant \|p'\| - 1$, and by lemma 7 we have $\|p'\| \leqslant \|p\|$, hence $\|p_i\| < \|p\|$. We conclude by induction on $\|p\|$. □

A.5. Trace semantics

Proof of proposition 4. Up to structural equivalence, every term can be written $\sum_{i=1}^{a} (\nu\vec{x}_i) \prod_{j=1}^{b_i} s_{ij}$ where the s_{ij} have the form $\alpha.p$, $\hat{\alpha}.p$ or 0 (the product stands for parallel composition). Assume an irreducible term t is a parallel composition $\prod_{j=1}^{b} s_j$ of such terms. None of the s_j is an affine action with a continuation other than 0, otherwise the A rule would apply. If $j = 0$ then $s = 1$ and it is the trace of length 0 with the empty inaction part. If $j \geqslant 2$, then none of the s_j is a linear action, otherwise one of the I rules could apply. Thus t is either one linear action with an irreducible continuation p, in this case p is not a sum or 0 because of the L rules, or t is a parallel composition of inactions. This shows that irreducible terms are sums of terms of the form $(\nu\vec{x})t$. All terms of the form $(\nu v)\hat{u}^{\varepsilon}(\vec{x}).p$ are reducible by one of the N rules, therefore an irreducible term without sums has the form $\hat{\alpha}_1 \ldots \hat{\alpha}_k.(\nu\vec{x})n$ where n is an irreducible parallel composition of inactions. If n contains several occurrences of a given $\alpha.0$ then it is reducible by I4, and if it contains an $\alpha.0$ and the dual $\bar{\alpha}.0$ then it is reducible by I5, therefore n is actually an inaction set. If $u \notin \mathrm{fn}(n)$ then $(\nu u)n \cong n$, and if $u \in \mathrm{fn}(n)$, then all inactions $u^{\varepsilon}(\vec{x}).0$ reduce to 0 by N3, after what (νu) can be removed. Therefore $\vec{x}$ is empty if the term is irreducible. The fact that S does not apply proves that the terms in an irreducible sum are pairwise distinct. □

Proof of proposition 6. Assume t and u are two traces that are not orthogonal. We show by induction on $\min(|t|, |u|)$ that $|t| = |u|$. If the minimum length is 0, if t or u has not length 0 (say it is t, without loss of generality), then $t \mid u$ has the form $\hat{\alpha}.t' \mid n$ where n is an inaction set. Then a maximal run of $t|u$ either consumes $\hat{\alpha}$ using an inaction, which makes 0 appear at top level, or it does not consume $\hat{\alpha}$ and it is inconsistent, so $t \perp u$. Therefore $|t| = |u| = 0$. By hypothesis $t \mid u$ has a consistent maximal run, so it has no execution, since consuming any action in it would make 0 appear at top level. Therefore $t \mid u$ does not contain dual actions, which is the expected property.

If $\min(|t|, |u|) \geqslant 1$, then $t|u$ has the form $\hat{\alpha}.t' | \hat{\beta}.u'$. If $\beta \neq \bar{\alpha}$ then $t \mid u$ has no execution, so it is inconsistent since the linear actions at top level cannot be consumed. Therefore $\beta = \bar{\alpha}$, and every maximal run starts with the execution step $\hat{\alpha}.t' \mid \hat{\bar{\alpha}}.u' \overset{\tau}{\Rightarrow} (\nu\vec{x})(t' \mid u')$, with $\vec{x} = \mathrm{bn}(\alpha)$. This implies that t' and u' are not orthogonal, and we conclude by induction hypothesis.

Reciprocally, it is clear that two traces $t = \hat{\alpha}_1 \ldots \hat{\alpha}_k.m$ and $u = \hat{\bar{\alpha}}_1 \ldots \hat{\bar{\alpha}}_k.n$ with $m \cap \bar{n} = \emptyset$ are not orthogonal, since $t|u$ has a run of length k that leads to $m|n$ which is consistent and irreducible.

Now let $t = \hat{\alpha}_1 \ldots \hat{\alpha}_k.m$ and $u = \hat{\beta}_1 \ldots \hat{\beta}_\ell.n$ be traces such that $t \sqsubseteq u$, i.e. $\{t\}^{\perp} \subseteq \{u\}^{\perp}$. Let γ be an action such that $\gamma.0 \notin n$. Let $v = \hat{\bar{\beta}}_1 \ldots \hat{\bar{\beta}}_\ell.\bar{\gamma}.0$, then by the characterisation above we get $u \not\perp v$, hence $t \not\perp v$, so we have $|t| = |\bar{u}| = |u|$, $\bar{\alpha}_i = \bar{\beta}_i$ for all i, and $\gamma.0 \notin m$. So t and u have the same action part, and an inaction that is not in n cannot be in m, hence $m \subseteq n$.

Finally, assume $u = \hat{\alpha}_1 \ldots \hat{\alpha}_k.n$ with $m \subseteq n$. A trace that is not in $\{u\}^{\perp}$ has the form $\hat{\bar{\alpha}}_1 \ldots \hat{\bar{\alpha}}_k.n'$ with $n' \cap \bar{n} = \emptyset$, so by inclusion $n' \cap \bar{m} = \emptyset$ and this trace is also not orthogonal to t, hence $\{t\}^{\perp} \subseteq \{u\}^{\perp}$, i.e. $t \sqsubseteq u$. □

Proof of lemma 8. For any term q define $[q]_0$ as the term obtained by replacing each occurrence of the action $\bar{\omega}$ in q by 0. First assume p is successful, and consider a run $[p]_\omega \Rightarrow^* q$. By hypothesis $\omega \notin \mathrm{fn}(p)$ so an occurrence of $\bar{\omega}$ in $[p]_\omega$ can never be consumed, so we can deduce a run $p \Rightarrow^* [q]_0$. Since p is successful we can deduce a run $[q]_0 \Rightarrow^* r$ such that r is inconsistent. Quasi-standard terms are stable under slicing execution, so r is quasi-standard, so the only way for r to be inconsistent is to have an occurrence of 0 at top level: $r \equiv 0 \mid r'$. From this we deduce a run $q \Rightarrow^* [r]_\omega \equiv \bar{\omega} \mid [r']_\omega$.

Now assume that for any run $[p]_\omega \Rightarrow^* q$ there is a run $q \Rightarrow^* \bar{\omega} \mid r$, and consider a run $p \Rightarrow^* q$. Then the run $[p]_\omega \Rightarrow^* [q]_\omega$ holds, thus there is a run $[q]_\omega \Rightarrow^* \bar{\omega} \mid r$. By the same argument as above, an occurrence of $\bar{\omega}$ can never be consumed in a run of $[q]_\omega$ therefore replacing $\bar{\omega}$ by 0 preserves the run, and we can deduce a run $q \Rightarrow^* [\bar{\omega} \mid r]_0 = 0 \mid [r]_0$. Thus p is successful. □