we finally obtain

$$S_N(x) = N\frac{U_{2N-2}(x) - 2N + 1}{T_{2N}(x) - 1}. \tag{12}$$

A straightforward calculation shows that $(*)$ is true for $a = 0$. Since $\cos(\pi - x) = -\cos x$, it is easily seen that the value of S_N does not change if a is replaced by $-a$. Hence, it suffices to prove $(*)$ for $0 < a < 1$. In (12) we take $x = (1 + a^2)/(2a)$. Then we have $S_N(x) = 4a^2 S_N$. From (1) and (2) it follows that

$$T_{2N}(x) = \tfrac{1}{2}(a^{-2N} + a^{2N}) \quad \text{and} \quad U_{2N-2}(x) = \tfrac{a}{1-a^2}(a^{-(2N-1)} - a^{2N-1}).$$

Now, $(*)$ easily follows from (12).

It should be noted that, except for $x \in \{\cos(k\pi/N) : k = 1, \ldots, N-1\}$, (12) is valid for all complex numbers . Taking $x = \cos\phi$ in (12), using (3) and $T_{2N}(\cos\phi) = \cos(2N\phi)$, a simple calculation yields

$$\sum_{k=1}^{N-1} \frac{\sin^2\left(\frac{k\pi}{N}\right)}{\left(\cos\phi - \cos\left(\frac{k\pi}{N}\right)\right)^2} = N\Big(N\csc^2(N\phi) - \cot(N\phi)\cot\phi - 1\Big),$$

valid for all ϕ with $\cos\phi \notin \{\cos(k\pi/N) : k = 1, \ldots, N-1\}$. In particular, if N is odd, we may take $\phi = \pi/2$ to get

$$\sum_{k=1}^{N-1} \tan^2\left(\tfrac{k\pi}{N}\right) = N(N-1) \quad (N = 3, 5, 7, \ldots).$$

There is no record of any other solution being received.

Everywhere Unimodal

10243 [1992, 675]. *Proposed by Michel Balazard, Université Bordeaux I, Talence, France.*

Define a sequence of functions $f_k(t)$ for $t > k$ recursively by

$$f_1(t) = 1$$

$$f_{k+1}(t) = \int_k^{t-1} f_k(u)\tfrac{du}{u}$$

Prove that, for every real number $t > 1$, the sequence $\langle f_k(t) : 1 \le k < t\rangle$ is unimodal.

Solution by Robin J. Chapman, University of Exeter, U. K. It is clear that $f_k(t) > 0$ for all $t > k$. We claim that for each $k \ge 1$ there exists t_k in $(k+1, \infty]$ such that $f_k(t) > f_{k+1}(t)$ if $k + 1 < t < t_k$ and $f_k(t) < f_{k+1}(t)$ if $t > t_k$. To establish this by induction on k, first observe that $f_2(t) = \log(t-1)$, so $t_1 = 1+e$. Then assume that $k > 1$ and that the inductive hypothesis holds for $k-1$. Consider the function g_k defined by $g_k(t) = f_k(t) - f_{k+1}(t)$. We have

$$g_k(t) = \int_{k-1}^{k} f_{k-1}(u)\tfrac{du}{u} + \int_k^{t-1} [f_{k-1}(u) - f_k(u)]\tfrac{du}{u}.$$

The function $g_k(t)$ is increasing for $k+1 < t < t_{k-1}+1$ and decreasing when $t > t_{k-1}+1$. Since it is positive for $k+1 < t < t_{k-1}+1$, it has at most one zero, at a value $t_k > t_{k-1}+1$. (If g_k has no zero, then we put $t_k = \infty$ and $t_\ell = \infty$ for all $\ell \ge k$.) This proves the claim and also that $t_{k+1} \ge t_k + 1$ for all k.

To show unimodality it suffices to show "once a decrease always a decrease." Consider a fixed t. If $f_k(t) \ge f_{k+1}(t)$, then $t \le t_k$, and hence $t < t_{k+1}$. Thus $f_{k+1}(t) > f_{k+2}(t)$, and the result follows.

Editorial comment. The proposer's solution worked with the function r_k defined by $r_k(t) = f_{k+1}(t)/f_k(t)$, and showed that $r_k(t)$ is strictly increasing in t for $t > k+1$. This led to $f_{k+1}^2(t) > f_k(t) f_{k+2}(t)$ for $t > k+2$, hence the sequence $f_k(t)$ is logarithmically concave for $1 \le k < t$.

The proposer introduced these functions to study the unimodality of the distribution of the number of prime divisors of an integer. In fact, let $F(x)$ be the number of positive integers n not exceeding x that have exactly k prime divisors and such that each of these exceeds $x^{1/t}$. Then

$$f_k(t) = \lim_{x\to\infty} F(x) \Big/ (x/\log x).$$

For $k = 1$ and $t > 1$ this is the prime number theorem.

He also remarks that if $k_0(t)$ is any function such that $r_k(t) > 1$ for $k < k_0(t)$ and $r_k(t) < 1$ for $k > k_0(t)$, then there is an elementary proof that $k_0(t) \le \log t + 1$. The theory outlined here leads to a lower bound on $k_0(t)$ that is also of the form $\log t + O(1)$ but he claims that, "the proof is no longer elementary".

Solved also by by the proposer.

An Odd Square

10263 [1992, 873]. *Proposed by J. G. Mauldon, Amherst College, Amherst, MA.*

Let m and n be odd integers, and suppose that $n^2 - 1$ is a multiple of $|m^2 + 1 - n^2|$. Prove or disprove that this requires that $|m^2 + 1 - n^2|$ be the square of an integer.

Solution by the proposer. The implication holds, and it also holds if m, n are nonzero even integers. Furthermore, $m^2 \ge n^2$, so that $m^2 + 1 - n^2$ itself is a square.

By hypothesis, we have $n^2 - 1 = t(m^2 + 1 - n^2)$ for some integer t, which we rewrite as $m^2 = (t+1)(m^2 + 1 - n^2)$. Let $k = t + 1$; this integer is nonzero. It suffices to show that k is a square, since that will show that the integer $m^2 + 1 - n^2$ is a rational square. However, the only integers that are squares of rationals are squares of integers.

Let S_k be the set of integer ordered pairs (x, y) such that $(x+y)^2 = k(1+4xy)$. Because $((m+n)/2, (m-n)/2) \in S_k$, the set is nonempty. Let $a = \min\{|x| : (x, y) \in S_k\}$. It suffices to show that $a = 0$.

The set S_k is invariant under negation and under interchange of coordinates. Hence a belongs to a pair in S_k. The other member of such a pair must solve the quadratic equation $(x+a)^2 = k(1+4ax)$. Hence the two roots b_1, b_2 must both be integers and have absolute value at least a. From the quadratic equation, we have $b_1 + b_2 = 4ak - 2a$ and $b_1 b_2 = a^2 - k$. By direct computation, these yield $(a + b_1)(a + b_2) = (4a^2 - 1)k$.

If $k < 0$, then the equations above yield $b_1 b_2 > 0$ and $b_1 + b_2 \le 0$. Hence each of b_1, b_2 is negative. Since $a < |b_i|$, this implies $a + b_i < 0$, and hence $(4a^2 - 1)k > 0$. With $k < 0$, this requires $a = 0$, but we have noted that $a = 0$ implies k is a square. Hence we may assume $k > 0$.

If $k > 0$ and $a > 0$, the equations yield $b_1 + b_2 > 0$ and $(a + b_1)(a + b_2) > 0$, so each of b_1, b_2 is positive. By the choice of a, we have $a^2 \le b_1 b_2$, but also $k > 0$ implies $b_1 b_2 = a^2 - k < a^2$. The contradiction implies $a = 0$, and hence k is a square.

Editorial comment. The other correct solutions reduced the problem to solving a Pell equation (i. e., a Diophantine equation of the form $u^2 - Dv^2 = 1$ for some non-square positive integer D). In this case, D is the *squarefree part* of $k(k-1)$, and the special form of D is exploited in the solution. The incorrect solution claimed that the conditions hold only when $m = \pm n$ or $n = \pm 1$, which omits the solution $(m, n) = (105, 99)$.

Solved also by J. C. Binz (Switzerland), R. J. Chapman (U. K.), I. Kastanas, J. P. Robertson, and the GCHQ Problem Solving Group (U. K.). One incorrect solution was received.