PROBLEMS AND SOLUTIONS

Edited by:
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Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before February 28, 1993 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgement is desired.

*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10238. *Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.*

(a) Show that there exist infinitely many positive integers a such that both $a+1$ and $3a+1$ are perfect squares.

(b) Let $a_1 < a_2 < \dots$ be the sequence of all solutions of (a). Show that $a_n a_{n+1} + 1$ is also a perfect square.

10239. *Proposed by Ismor Fischer, Naval Postgraduate School, Monterey, CA.*

A continuous vector field $\vec{F}$ (in $\mathbb{R}^2$ or $\mathbb{R}^3$) and a simple closed curve Γ are given. Show that, for every point $x \in \Gamma$, there exists a point $y \in \Gamma$ and a path γ from x to y (nontrivial if $x = y$) such that the work $W = \int_\gamma \vec{F} \cdot d\vec{r}$ is zero.

10240. *Proposed by Michael Golomb, Purdue University, West Lafayette, IN.*

Fix an integer n. For each integer m with $0 \le m \le n$, let p_m be a polynomial of degree n for which $\int_0^1 p_m(x)x^l\,dx = 0$ for $0 \le l \le n$ with $l \ne m$, while $\int_0^1 p_m(x)x^m\,dx = 1$.

(a) Determine the value of $\int_0^1 p_m^2(x)\,dx$.

(b) Find an explicit expression for p_m and prove that the coefficient of x^l in p_m is the same as the coefficient of x^m in p_l for $0 \le l < m \le n$.

10241. *Proposed by Roger W. Johnson, Carleton College, Northfield, MN.*

Let m and n be positive integers with $m \ge n$. Show that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \left(\frac{\sin(mx)}{x}\right) dx = \frac{\pi}{2}.$$

10242. *Proposed by S. Brocco, Brandeis University, Waltham, MA, and F. Mignosi, Institut Blaise Pascal, Paris, France and Università di Palermo, Palermo, Italy.*

Let α be a fixed irrational number.

(a) For fixed integer n with $n > 1$, show that it is possible to find a constant $c(n)$ such that there are infinitely many rationals p/q with q relatively prime to n and $|\alpha - p/q| < c(n)/q^2$.

(b) If the continued fraction of α has unbounded partial quotients and $\varepsilon > 0$ is given, can one find $c(n) < \varepsilon$ satisfying the above condition?

10243. *Proposed by Michel Balazard, Université Bordeaux I, Talence, France.*

Define a sequence of functions $f_k(t)$ for $t > k$ recursively by

$$f_1(t) = 1$$

$$f_{k+1}(t) = \int_k^{t-1} f_k(u)\frac{du}{u}.$$

Prove that, for every real number $t > 1$, the sequence $\langle f_k(t) : 1 \le k < t\rangle$ is unimodal.

10244. *Proposed by Ken Bromberg (student), Brown University and Stan Wagon, The Geometry Center, Minneapolis, MN and Macalester College, St. Paul, MN.*

A classical construction of Miquel starts with an n-vertex polygon and a point P in the plane (not a vertex of the n-gon), and forms another n-gon as follows:

1. draw the perpendiculars from P to the (extended) sides of the polygon;
2. connect the feet to obtain another n-gon.

These steps are then repeated n times (provided that none of the polygons has P as a vertex). The resulting polygon, denoted $M(P)$ is similar to the initial n-gon.

(a) Given a triangle, construct the point P for which $M(P)$ is largest.

(b)* Given a quadrilateral, is there a Euclidean construction of the point P for which $M(P)$ is largest?

10245. *Proposed by M. A. Bezem, Utrecht University, Utrecht, The Netherlands and A. J. C. Hurkens, Catholic University, Nijmegen, The Netherlands.*

Let $\mathscr{S}$ be a set of finite, non-empty sets. A *transversal* of $\mathscr{S}$ is a set which has a non-empty intersection with every element of $\mathscr{S}$. The Principle of Minimal Transversal states that every such $\mathscr{S}$ has a transversal which is minimal with respect to set inclusion. Prove that the Axiom of Choice is equivalent to the Principle of Minimal Transversal.

10246. *Proposed by B. C. Carlson, Iowa State University, Ames, IA.*

For integers m and n with $n \geq 0$ and $-n \leq m \leq n$, find values of A, N, and λ such that

$$\frac{1}{2\pi(1-\rho^2)^{1/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{n+m}y^{n-m}\exp\left(\frac{-x^2+2\rho xy-y^2}{2(1-\rho^2)}\right)dx\,dy = AC_N^{\lambda}(-\rho)$$

for $-1<\rho<1$, where C_N^{λ} is a Gegenbauer polynomial.

NOTES

(10242) It is known that, without the condition on relative primality, there are infinitely many p/q with $|\alpha - p/q| < 1/q^2$. Furthermore, if the continued fraction of α has unbounded partial quotients, then $|\alpha - p/q| < \varepsilon/q^2$ has infinitely many solutions. **(10243)** A sequence $\langle s_k \rangle$ is called "unimodal" if there is an index k_0 such that $s_k < s_{k+1}$ for $k < k_0$ and $s_k > s_{k+1}$ for $k > k_0$. **(10244)** Further details of this construction can be found in B. M. Stewart, "Cyclic properties of Miquel polygons", this MONTHLY, 47 (1940), 462–466, and in H. S. M. Coxeter, *Introduction to Geometry*, p. 16. **(10245)** The book, H. Rubin & J. E. Rubin, *Equivalents of the Axiom of Choice, II*, North-Holland, 1985 gives a description of some recent work on the Axiom of Choice and its relatives. **(10246)** The Gegenbauer (or ultraspherical) polynomials are defined by the generating function $(1-2tz+t^2)^{-\lambda} = \Sigma_{N=0}^{\infty} t^N C_N^{\lambda}(z)$. Details may be found in A. Erdélyi, et al., *Higher Transcendental Functions*, vol. 2, Sect. 10.9.

SOLUTIONS

The Product of Two Sides of a Triangle

E 3417 [1991, 54]. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, WI.*

Suppose ABC is a triangle with $AB \neq AC$, and let D, E, F, G be points on the line through B and C defined as follows: D is the midpoint of BC, AE is the bisector of the angle BAC, F is the foot of the perpendicular from A to BC, and AG is perpendicular to AE (i.e., AG bisects one of the exterior angles at A).

Prove that $AB \cdot AC = DF \cdot EG$.

Solution I by S. Belbas, University of Alabama, Tuscaloosa, AL. All symbols refer to non-oriented line segment lengths. We may assume $AB < AC$. We set $a = BC$, $b = AC$, $c = AB$, $h = AF$, $x = BE$, $y = BG$, $z = DF$.