$$\int_T^{T+H} \left|G_1(t)\right|^2 dt \geqslant \int_{T+101U}^{T+H} \left|\widetilde{G}_1(t)\right|^2 dt$$
$$\geqslant \int_{T+101U}^{T+H} \left(\left|\widetilde{G}_2(t)\right|^2 + 2\Re\overline{F(t)\widetilde{G}_2(t)} + 2\Re\overline{F(t)G_3(t)} + 2\Re G_2(t)\overline{G_3(t)} \right) dt\,.$$

Since $\int_T^{T+H} |G_3(t)|^2 dt \ll HSM^{-2}$, the terms involving $G_3(t)$ give a small error term by the Cauchy-Schwarz inequality.
Since $G_2(t)$ is $\sum_{n\leqslant H} a_n n^{-s}$ at $s = 1/2 + it$, and $F(t)$ is the analytic continuation of $\sum_{n>H+H^{1/4}} a_n n^{-s}$ at $s = 1/2 + it$, the integral $\int_{T+U}^{T+H-101U} G_2(t)\overline{F(t)}$ is of the form discussed in Lemma 4 and hence small. Again by Lemma 5,

$$\int\limits_{T+101U}^{T+H} |\widetilde{G}_2(t)|^2\, dt \geqslant \int\limits_{T+101U}^{T+H-101U} |G_2(t)|^2 dt = \sum_{n\leqslant H} \big(H - 204U + O(n)\big)\frac{|a_n|^2}{n}\,.$$

This completes the proof.

Applications to omega results for the error term in the summatory functions of arithmetic functions were also given.

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A continuity property connected with Nyman's criterion for the Riemann hypothesis

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(joint work with Nicolas Jousse)

Let $\{u\}$ denote the fractional part of the real number u, and, for $u > 0$, $\alpha \geqslant 0$, define $g_\alpha(u) := \{\alpha/u\}$. Let $\mathcal{B}$ be the set of linear combinations of the g_α's, and $\mathcal{N}$ be the subset of those elements of $\mathcal{B}$ which vanish on $(1, +\infty)$. We observe that $\mathcal{B} \subset L^p(0, +\infty)$ for $1 < p \leqslant +\infty$, and that $\mathcal{N} \subset L^\infty(0, 1)$.

A striking result from Nyman's thesis (1950) is the following.

THEOREM 1. (Nyman [2]) *The Riemann hypothesis is equivalent to* $\mathcal{N}$ *being dense in* $L^2(0, 1)$.

It is easy to see that $\mathcal{N}$ is dense in $L^2(0,1)$ if, and only if the characteristic function χ of $(0,1)$ lies in the closure of $\mathcal{B}$ in $L^2(0,+\infty)$. Thus Nyman's theorem gives a reformulation of the Riemann hypothesis (RH) as an approximation problem in a Hilbert space. A further rephrasing of (RH) involves the quantity

$$\delta_n := \inf\Big\{\Big\|\chi - \sum_{k=1}^{n} c_k g_{\alpha_k}\Big\| : \, c_k \in \mathbb{C},\ 0 \leqslant \alpha_k \leqslant 1,\ k = 1, \ldots, n\Big\}, \tag{1}$$

the distance in $L^2(0,+\infty)$ between χ and the set of all n-terms linear combinations of the g_α's ; (RH) is plainly equivalent to $\delta_n = o(1)$, $n \to +\infty$.

The inequality $\delta_n \leqslant \delta_{n+1}$ is obvious. Is the sequence (δ_n) strictly decreasing? The answer is positive and was given by Nicolas Jousse in his thesis (2004).

Theorem 2. (Jousse [1]) *or every* $n \geqslant 1$, *one has* $\delta_{n+1} < \delta_n$.

It turns out that the main step in the proof of Theorem 2 consists in showing that the infimum in (1) is in fact a minimum. With this goal in mind, we denote by P_V the orthogonal projection on the closed subspace V of the (implicit) Hilbert space H. In the case $H = L^2(0,+\infty)$, one has

$$\delta_n = \inf\Big\{\big\|\chi - P_{\mathrm{Vect}(g_{\alpha_1},\ldots,g_{\alpha_n})}(\chi)\big\| : \, 0 \leqslant \alpha_1, \ldots, \alpha_n \leqslant 1\Big\},$$

so that a sufficient condition for this infimum to be a minimum is the continuity of the map

$$\begin{aligned} [0,+\infty[^n &\to H \\ (\alpha_1, \ldots, \alpha_n) &\mapsto P_{\mathrm{Vect}(g_{\alpha_1},\ldots,g_{\alpha_n})}(\chi)\,. \end{aligned}$$

This last question is an instance of a general problem studied by Jousse. We discuss now the simplest form of this problem, whereas the application to (δ_n) is handled by means of a slightly modified variant.

Let G be a locally compact abelian group, noted multiplicatively, with Haar measure μ. For $G \in H := L^2(G)$, and $\alpha \in G$, define $g_\alpha(x) := g(x\alpha^{-1})$, $x \in G$.

Definition. *The function* $g \in L^2(G)$ *is admissible if, for every* $y_0 \in G$, *and every positive integer* n, *the map*

$$\begin{aligned} G^n &\to H \\ (\alpha_1, \ldots, \alpha_n) &\mapsto P_{\mathrm{Vect}(g_{\alpha_1},\ldots,g_{\alpha_n})}(y_0)\,. \end{aligned}$$

is continuous.

The general problem is to obtain a characterization of admissible functions. It may be difficult to get a complete answer. Let us note that some very regular functions, such as non-zero continuously differentiable functions with compact support (in the case where G is the real line), are *not* admissible.

Jousse described a class of admissible functions. We begin with two definitions and then state his result.

Definition. *A measurable function* $f\colon g \to \mathbb{C}$ *is an exponential polynomial if the vector space* $\mathrm{Vect}(f_\alpha,\, \alpha \in \mathbb{C})$ *has finite dimension.*

In the case where $G = (]0,+\infty[,\times)$, the exponential polynomials are linear combinations of functions $x \mapsto x^\rho \log^k x$, $\rho \in \mathbb{C}$, $k \in \mathbb{N}$.

Definition. *A measurable function* $f : g \to \mathbb{C}$ *is countably simple if* $f(G)$ *is countable.*

Theorem 3. (Jousse [1]) *Assume* G *is* σ*-compact, metrizable and not compact. Let* $f \in L^2(G)$ *be such that* $f = f' + f''$, *where* f' *is an exponential polynomial, and* f'' *is countably simple. Then* f *is admissible.*

Observe that the error terms of analytic number theory are often sums (or differences) of an exponential polynomial on $]0,+\infty[$ and a countably simple function. In particular, this is the case for $t \mapsto \{1/t\}$.

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Non-vanishing of class group L-functions at the central point

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The question as to whether an L-function vanishes at a special point on the critical line has arisen in various contexts and is apparently a fundamental one. By now there are numerous results that many members – in some cases even a positive proportion – of a certain family of L-functions do not vanish at the central point. This is of interest in various aspects such as the Birch–Swinnerton–Dyer conjecture, the Siegel zero (see [4]) and the theory of modular forms of half-integral weight (see [6]). A large number of old and new results around this theme can be found in [5].

Here we consider L-functions attached to class group characters of an imaginary quadratic field. Let $K = \mathbb{Q}(\sqrt{-D})$ be the imaginary quadratic field of discriminant $-D$. We denote its class group by $\mathcal{C}$ and write $h = \#\mathcal{C}$. For each character $\chi \in \hat{\mathcal{C}}$ we have an L-function

$$L_K(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N\,\mathfrak{a})^{-s},$$

the summation being taken over all nonzero integral ideals $\mathfrak{a}$. For real characters $L_K(s,\chi)$ is the product of two Dirichlet L-functions, while for complex characters $L_K(s,\chi)$ comes from the cusp form $\sum_{\mathfrak{a}} \chi(\mathfrak{a}) e(zN\,\mathfrak{a})$ of weight 1 for $\Gamma_0(D)$ and character χ_D. We shall obtain the following result [1].

Theorem. *There is an absolute constant* $c > 0$ *such that*

$$\frac{1}{h}\#\{\chi \in \hat{\mathcal{C}} \mid L(1/2,\chi) \neq 0\} \geqslant c \prod_{p \mid D}\Big(1 - \frac{1}{p}\Big) \tag{1}$$