Differentially 4-uniform functions

Yves Aubry and François Rodier

Abstract. We give a geometric characterization of vectorial Boolean functions with differential uniformity $\leq 4$. This enables us to give a necessary condition on the degree of the base field for a function of degree $2^r - 1$ to be differentially 4-uniform.

1. Introduction

We are interested in vectorial Boolean functions from the $F_2$-vectorial space $F_2^m$ to itself in $m$ variables, viewed as polynomial functions $f : F_2^m \rightarrow F_2^m$ over the field $F_2^m$ in one variable of degree at most $2^m - 1$. For a function $f : F_2^m \rightarrow F_2^m$, we consider, after K. Nyberg (see [16]), its differential uniformity

$$\delta(f) = \max_{\alpha \neq 0, \beta} \sharp \{x \in F_2^m \mid f(x + \alpha) + f(x) = \beta\}.$$ 

This is clearly a strictly positive even integer.

Functions $f$ with small $\delta(f)$ have applications in cryptography (see [16]). Such functions with $\delta(f) = 2$ are called almost perfect nonlinear (APN) and have been extensively studied: see [16] and [9] for the genesis of the topic and more recently [3] and [6] for a synthesis of open problems; see also [7] for new constructions and [20] for a geometric point of view of differential uniformity.

Functions with $\delta(f) = 4$ are also useful; for example the function $x \rightarrow x^{-1}$, which is used in the AES algorithm over the field $F_2^8$, has differential uniformity 4 on $F_2^m$ for any even $m$. Some results on these functions have been collected by C. Bracken and G. Leander [4, 5].

We consider here the class of functions $f$ such that $\delta(f) \leq 4$, called differentially 4-uniform functions. We will show that for polynomial functions $f$ of degree $d = 2^r - 1$ such that $\delta(f) \leq 4$ on the field $F_2^m$, the number $m$ is bounded by an expression depending on $d$. The second author demonstrated the same bound in the case of APN functions [17, 18]. The principle of the method we apply here was already used by H. Janwa et al. [13] to study cyclic codes and by A. Canteaut [8] to show that certain power functions could not be APN when the exponent is too large.

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Henceforth we fix $q = 2^m$. In order to simplify our study of such functions, let us recall the following elementary results on differential uniformity; the proofs are straightforward:

**Proposition 1.** (i) Adding a polynomial whose monomials are of degree 0 or a power of 2 to a function $f$ does not change $\delta(f)$.
(ii) For all $a$, $b$ and $c$ in $\mathbb{F}_q$, such that $a \neq 0$ and $c \neq 0$ we have
$$\delta(cf(ax + b)) = \delta(f).$$
(iii) One has $\delta(f^2) = \delta(f)$.

Hence, without loss of generality, from now on we can assume that $f$ is a polynomial mapping from $\mathbb{F}_q$ to itself which has neither terms of degree a power of 2 nor a constant term, and which has at least one term of odd degree.

To any function $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$, we associate the polynomial
$$f(x) + f(y) + f(z) + f(x + y + z).$$
Since this polynomial is clearly divisible by
$$(x + y)(x + z)(y + z),$$
we can consider the polynomial
$$P_f(x, y, z) := \frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)}$$
which has degree $\deg(f) - 3$ if $\deg(f)$ is not a power of 2.

2. A characterization of functions with $\delta \leq 4$

We will give, as in [17], a geometric criterion for a function to have $\delta \leq 4$. We consider in this section the algebraic set $X$ defined by the elements $(x, y, z, t)$ in the affine space $\mathbb{A}^4(\mathbb{F}_q)$ such that
$$P_f(x, y, z) = P_f(x, y, t) = 0.$$

We set also $V$ the hypersurface of the affine space $\mathbb{A}^4(\mathbb{F}_q)$ defined by
$$\text{(1)} \quad (x + y)(x + z)(y + t)(y + z)(x + y + z + t) = 0.$$

The hypersurface $V$ is the union of the seven hyperplanes $H_1, \ldots, H_7$ defined respectively by the equations $x + y = 0, \ldots, x + y + z + t = 0$.

We begin with a simple lemma:

**Lemma 2.** The following two properties are equivalent:
(i) there exist 6 distinct elements $x_0, x_1, x_2, x_3, x_4, x_5$ in $\mathbb{F}_q$ such that
$$\begin{align*}
x_0 + x_1 &= \alpha, \quad f(x_0) + f(x_1) = \beta \\
x_2 + x_3 &= \alpha, \quad f(x_2) + f(x_3) = \beta \\
x_4 + x_5 &= \alpha, \quad f(x_4) + f(x_5) = \beta
\end{align*}$$
(ii) there exist 4 distinct elements $x_0, x_1, x_2, x_4$ in $\mathbb{F}_q$ such that $x_0 + x_1 + x_2 + x_4 \neq 0$ and such that
$$\begin{align*}
f(x_0) + f(x_1) + f(x_2) + f(x_0 + x_1 + x_2) &= 0 \\
f(x_0) + f(x_1) + f(x_4) + f(x_0 + x_1 + x_4) &= 0.
\end{align*}$$
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Proof. Suppose that (i) is true. Then we have \( x_0 + x_1 + x_2 = \alpha + x_2 = x_3 \) and so \( f(x_0) + f(x_1) + f(x_2) + f(x_0 + x_1 + x_2) = f(x_0) + f(x_1) + f(x_2) + f(x_3) = 0 \). The second equation holds true in the same way. Finally, we have \( x_0 + x_1 + x_2 + x_4 = x_3 + x_4 \neq 0 \).

Conversely, let us set \( \alpha = x_0 + x_1, \beta = f(x_0) + f(x_1) \) and \( x_3 = \alpha + x_2 = x_0 + x_1 + x_2 \). Then \( f(x_2) + f(x_3) = f(x_2) + f(x_0 + x_1 + x_2) = f(x_0) + f(x_1) = \beta \). Furthermore, we have \( x_3 \neq x_0 \) because \( x_1 \neq x_2 \) and we have \( x_3 \neq x_4 \) since otherwise we would have \( x_2 = \alpha + x_3 = \alpha + x_1 = x_0 \).

Setting \( x_5 = \alpha + x_4 = x_0 + x_1 + x_4 \) we have \( f(x_4) + f(x_5) = f(x_4) + f(x_0 + x_1 + x_4) = f(x_0) + f(x_1) = \beta \). We have \( x_3 \neq x_4 \) since otherwise we would have \( 0 = x_3 + x_4 = x_0 + x_1 + x_2 + x_4 \) which is not the case by hypothesis.

Finally \( x_3 \neq x_5 \) since otherwise we would have \( x_2 = x_4 \), and so all the six elements \( x_0, x_1, x_2, x_3, x_4, x_5 \) are different. \( \square \)

We can now state a geometric characterization of differentially 4-uniform functions:

**Theorem 3.** The differential uniformity of a function \( f : \mathbb{F}_q \rightarrow \mathbb{F}_q \) is not larger than 4 if and only if:

\[
X(\mathbb{F}_q) \subset V
\]

where \( X(\mathbb{F}_q) \) denotes the set of rational points over \( \mathbb{F}_q \) of \( X \).

**Proof.** The differential uniformity is not larger than 4 if and only if for any \( \alpha \in \mathbb{F}_q \) and any \( \beta \in \mathbb{F}_q \), the equation

\[
f(x + \alpha) + f(x) = \beta
\]

has at most 4 solutions, that is to say

\[
\sharp\{x \in \mathbb{F}_q | f(x) + f(y) = \beta, \ x + y = \alpha\} \leq 4.
\]

But this is equivalent to saying that we cannot find 6 distinct elements \( x_0, x_1, x_2, x_3, x_4, x_5 \) in \( \mathbb{F}_q \) such that

\[
\begin{align*}
x_0 + x_1 &= \alpha, & f(x_0) + f(x_1) &= \beta \\
x_2 + x_3 &= \alpha, & f(x_2) + f(x_3) &= \beta \\
x_4 + x_5 &= \alpha, & f(x_4) + f(x_5) &= \beta.
\end{align*}
\]

By the previous lemma, this is equivalent to saying that we cannot find 4 distinct elements \( x_0, x_1, x_2, x_4 \) in \( \mathbb{F}_q \) such that \( x_0 + x_1 + x_2 + x_4 \neq 0 \) and such that

\[
\begin{align*}
f(x_0) + f(x_1) + f(x_2) + f(x_0 + x_1 + x_2) &= 0 \\
f(x_0) + f(x_1) + f(x_4) + f(x_0 + x_1 + x_4) &= 0.
\end{align*}
\]

But this can be reformulated by saying that the rational points over \( \mathbb{F}_q \) of the variety \( X \) are contained in the variety \( V \), that is to say \( X(\mathbb{F}_q) \subset V \). \( \square \)

3. Monomial functions with \( \delta \leq 4 \)

If the function \( f \) is a monomial of degree \( d > 3 \):

\[
f(x) = x^d
\]
then the polynomials $P_f(x, y, z)$ and $P_f(x, y, t)$ are homogeneous polynomials and we can consider the intersection $X$ of the projective cones $S_1$ and $S_2$ of dimension 2 defined respectively by $P_f(x, y, z) = 0$ and $P_f(x, y, t) = 0$ with projective coordinates $(x : y : z : t)$ in the projective space $\mathbb{P}^3(\mathbb{F}_q)$.

Even if $X$ is now a projective algebraic subset of the projective space $\mathbb{P}^3(\mathbb{F}_q)$, Theorem 3 tells us also that:

$$\delta(f) \leq 4 \text{ if and only if } X(\mathbb{F}_q) \subset V,$$

where $V$ is the hypersurface of $\mathbb{P}^3(\mathbb{F}_q)$ defined by Equation (1).

Indeed, the algebraic sets $X$ and $V$ in this section are closely related to but not equal to the sets $X$ and $V$ of the previous section. The set $X$ of this section (resp. $V$) is the set of lines through the origin of the set $X$ (resp. $V$) of the previous section which is invariant under homotheties with center the origin. For convenience, we keep the same notations.

**Lemma 4.** The projective algebraic set $X$ has dimension 1, i.e. it is a projective curve.

**Proof.** We have to show that the projective surfaces $S_1$ and $S_2$ do not have common irreducible components. Since $S_1$ and $S_2$ are two cones, it is enough to prove that the vertex of one of the cones doesn’t lie in the other cone. The coordinates of the vertex of the cone $S_2$ is $(0 : 0 : 1 : 0)$. To show that $S_1$ doesn’t lie in $S_2$, we will prove that $P_f(0 : 0 : 1 : 0) \neq 0$. Indeed, $S_1$ is defined by the polynomial

$$P_f(x, y, z) = \frac{x^d + y^d + z^d + (x + y + z)^d}{(x + y)(x + z)(y + z)}.$$

Setting $x + y = u$, we obtain:

$$P_f(x, y, z) = \frac{x^d + (x + u)^d + z^d + (u + z)^d}{u(x + z)(x + u + z)},$$

which gives

$$P_f(x, y, z) = \frac{x^{d-1} + z^{d-1} + uQ(x, z)}{(x + z)(x + u + z)},$$

where $Q$ is some polynomial in $x$ and $z$. This expression takes the value 1 at the point $(0 : 0 : 1 : 0)$. $\square$

Now we know that $X$ is a projective curve in $\mathbb{P}^3(\mathbb{F}_q)$, and in order to estimate its number of rational points over $\mathbb{F}_q$, we must determine its irreducibility. We will prove that the curve $C_7$, defined as the intersection of $S_2$ with the projective plane $H_7$ of equation $x + y + z + t = 0$, is an absolutely irreducible component of $X$, and hence that $X$ is reducible.

**Proposition 5.** The intersection of the curve $X$ with the plane $H_7$ with the equation $x + y + z + t = 0$ is equal to the curve $C_7 := S_2 \cap H_7$.

**Proof.** Since $X = S_1 \cap S_2$, it is enough to prove that $C_7 \subset S_1$. Since $t = x + y + z$ the points of intersection of the cone $S_2$ with the plane $x + y + z + t = 0$
satisfy:
\[
0 = P_f(x, y, t) = \frac{x^d + y^d + t^d + (x + y + t)^d}{(x + y)(y + t)(y + t)} = \frac{x^d + y^d + (x + y + z)^d + z^d}{(x + y)(y + z)(x + z)} = P_f(x, y, z),
\]
so they belong to \( S_1 \).

**Proposition 6.** The projective plane curve \( C_7 \) is isomorphic to the projective plane curve \( C \) with equation
\[
P_f(x, y, z) = \frac{x^d + y^d + z^d + (x + y + z)^d}{(x + y)(x + z)(y + z)} = 0.
\]

**Proof.** The projection from the vertex of the cone \( S_1 \) defines an isomorphism of the projective plane \( H_7 \) with equation \( x + y + z + t = 0 \) onto the plane with equation \( t = 0 \), and it maps \( C_7 \) onto the curve \( C \) with equation \( P_f(x, y, z) = 0 \).

**Proposition 7.** Let \( C \) be a plane curve of degree \( \deg(C) \) and which is not contained in \( V \). Then:
\[
\sharp(C \cap V)(\mathbb{F}_q) \leq 7 \deg(C).
\]

**Proof.** The variety \( V \) is the union of seven projective planes. Each plane cannot contain more than \( \deg(C) \) points, therefore \( V \) contains at most \( 7 \deg(C) \) rational points in \( C \).

In order to get a lower bound for the number of rational points over \( \mathbb{F}_q \) on the curve \( C \), hence on the curve \( X \), we need to know if \( C \) is absolutely irreducible or not. This question has been discussed by H. Janwa, G. McGuire and R. M. Wilson in [14] and very recently by F. Hernando and G. McGuire in [10].

**Proposition 8.** If \( d = 2^r - 1 \) with \( r \geq 3 \), then the projective curve \( X \) has an absolutely irreducible component \( C' \) defined over \( \mathbb{F}_2 \) in the plane \( x + z + t = 0 \) and this component \( C' \) is isomorphic to the curve \( C \).

**Proof.** One checks that the intersection of the cone \( S_1 \) with the plane \( x + z + t = 0 \) is the same as the intersection of the cone \( S_2 \) with that plane. Hence one can show, as in Proposition 6, that the intersection of the curve \( X \) with the plane \( x + z + t = 0 \) is isomorphic to the curve \( C \). Furthermore, it is proved in [14] that the curve \( C \) is absolutely irreducible since, \( \deg(C) = 2^r - 1 \equiv 3 \pmod{4} \).

Hence we can state

**Theorem 9.** Consider the function \( f : \mathbb{F}_q \rightarrow \mathbb{F}_q \) defined by \( f(x) = x^d \) with \( d = 2^r - 1 \) and \( r \geq 3 \). If \( 5 \leq d < q^{1/4} + 4.6 \), then \( f \) has differential uniformity strictly greater than 4.

**Proof.** The curve \( C' \) is an absolutely irreducible plane curve of arithmetic genus \( \pi_{C'} = (d - 4)(d - 5)/2 \). According to [1] (see also [2] for a more general statement), the number of rational points of the (possibly singular) absolutely irreducible curve \( C' \) satisfies
\[
|\#C'({\mathbb{F}_q}) - (q + 1)| \leq 2 \pi_{C'} q^{1/2}.
\]
Hence
\[ \#C'(\mathbb{F}_q) \geq q + 1 - 2\pi c q^{1/2}. \]

The maximum number of rational points on the curve \( C' \) on the surface \( V \) is \( 7(d-3) \) by Proposition 7. If \( q + 1 - 2\pi c q^{1/2} > 7(d-3) \), then \( C'(\mathbb{F}_q) \not\subset V \), therefore \( X(\mathbb{F}_q) \not\subset V \), and \( \delta(f) > 4 \) by Theorem 3. But this condition is equivalent to
\[ q - 2\pi c q^{1/2} - 7(d-3) + 1 > 0. \]
The condition is satisfied when
\[ q^{1/2} > \pi c + \sqrt{7(d-3) - 1 + \pi^2 c}, \]
and hence when
\[ q > d^4 - 18d^3 + 121d^2 - 348d + 362 \]
or
\[ 5 \leq d < q^{1/4} + 4.6. \]

\[ \square \]

4. Polynomials functions with \( \delta \leq 4 \)

If the function \( f \) is a polynomial of one variable with coefficients in \( \mathbb{F}_q \) of degree \( d > 3 \), we consider again as in section 3 the intersection \( X \) of \( S_1 \) and \( S_2 \), which are now cylinders in the affine space \( \mathbb{A}^3(\mathbb{F}_q) \) with equations respectively \( P_f(x, y, z) = 0 \) and \( P_f(x, y, t) = 0 \) and which are of dimension 3 as affine varieties.

**Lemma 10.** The algebraic set \( X \) has dimension 2, i.e. it is an affine surface. Moreover, it has degree \( (d-3)^2 \).

**Proof.** We have to show that the hypersurfaces \( S_1 \) and \( S_2 \) do not have a common irreducible component. Since these hypersurfaces are two cylinders, it is enough to prove that the polynomial defining \( S_1 \) does not vanish on the whole of a straight line \((x_0, y_0, z, t_0)\) where \( x_0, y_0, t_0 \) are fixed and satisfy \( P_f(x_0, y_0, t_0) = 0 \). Indeed, \( S_1 \) is defined by the polynomial \( P_f(x, y, z) \), which takes the value
\[ P_f(x_0, y_0, z) = \frac{f(x_0) + f(y_0) + f(z) + f(x_0 + y_0 + z)}{(x_0 + y_0)(x_0 + z)(y_0 + z)} \]
at the point \((x_0, y_0, z, t_0)\). If we set \( x_0 + y_0 = s_0 \), the homogeneous term of degree \( d_i \) in \( P_f(x, y, z) \) becomes
\[ d_i(x_0^{d_i-1} + z^{d_i-1}) + s_0 Q_i(x_0, z) \]
where \( Q_i \) is a polynomial in \( x_0 \) and \( z \) of degree \( d_i - 2 \). If \( d_i \) is odd, the numerator of this term is of degree \( d_i - 2 \), and hence does not vanish, so it is the same for the polynomial \( P_f(x_0, y_0, z) \). Hence, \( X \) has dimension 2. Moreover, \( X \) is the intersection of two hypersurfaces of degree \( d - 3 \), thus it has degree \( (d - 3)^2 \). \[ \square \]

The surface \( X \) is reducible. Let \( X = \bigcup X_i \) be its decomposition in absolutely irreducible components.

We embed the affine surface \( X \) into a projective space \( \mathbb{P}^4(\mathbb{F}_q) \) with homogeneous coordinates \((x : y : z : t : u)\). Consider the hyperplane at infinity \( H_\infty \) defined by the equation \( u = 0 \) and let \( X_\infty \) be the intersection of the projective closure \( X \) of \( X \) with \( H_\infty \). Then \( X_\infty \) is the intersection of two surfaces in this hyperplane, which
are respectively the intersections $S_{1,\infty}$ and $S_{2,\infty}$ of the cylinders $S_1$ and $S_2$ with that hyperplane. The homogeneous equations of $S_{1,\infty}$ and $S_{2,\infty}$ are

$$P_{x^d}(x, y, z) = \frac{x^d + y^d + z^d + (x + y + z)^d}{(x + y)(x + z)(y + z)}$$

and

$$P_{x^d}(x, y, t) = \frac{x^d + y^d + t^d + (x + y + t)^d}{(x + y)(x + t)(y + t)}.$$

By Proposition 8, the intersection of the curve $X_\infty$ with the plane $x + z + t = 0$ (inside the hyperplane at infinity) is an absolutely irreducible component $C'$ of the curve $X_\infty$ of multiplicity 1, defined over $\mathbb{F}_2$. So the only absolutely irreducible component of $X$, say $X_1$, which contains $C'$ is defined over $\mathbb{F}_q$.

**Proposition 11.** Let $X$ be an absolutely irreducible projective surface of degree $d > 1$. Then the maximum number of rational points on $X$ which are contained in the hypersurface $V \cup H_\infty$ is

$$\sharp (X \cap (V \cup H_\infty)) \leq 8(\deg(X)q + 1).$$

**Proof.** As $\deg(X) > 1$, the surface $X$ is not contained in any hyperplane. Thus, a hyperplane section of $X$ is a curve of degree $\deg(X)$. Using the bound on the maximum number of rational points on a general hypersurface of given degree proved by Serre in [19], we get the result. □

**Theorem 12.** Let $q = 2^m$. Consider a function $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ of degree $d = 2^r - 1$ with $r \geq 3$. If $31 \leq d < q^{1/8} + 2$, then $\delta(f) > 4$. For $d < 31$, we get $\delta(f) > 4$ for $d = 7$ and $m \geq 22$ and also if $d = 15$ and $m \geq 30$.

**Proof.** From an improvement of a result of S. Lang and A. Weil [15] proved by S. Ghorpade and G. Lachaud [11, section 11], we deduce

$$|\#X_1(\mathbb{F}_q) - q^2 - q - 1| \leq ((d - 3)^2 - 1)((d - 3)^2 - 2)q^{3/2} + 36(2d - 3)^5q$$

$$\leq (d - 3)^4q^{3/2} + 36(2d - 3)^5q.$$ 

Hence

$$\#X_1(\mathbb{F}_q) \geq q^2 + q + 1 - (d - 3)^4q^{3/2} - 36(2d - 3)^5q.$$

Therefore, if

$$q^2 + q + 1 - (d - 3)^4q^{3/2} - 36(2d - 3)^5q > 8((d - 3)q + 1),$$

then $\#X(\mathbb{F}_q) \geq \#X_1(\mathbb{F}_q) > 8((d - 3)q + 1)$, and hence $X(\mathbb{F}_q) \nsubseteq V \cup H_\infty$ by Proposition 11. As $X$ is the set of affine points of the projective surface $X$, we deduce that $X(\mathbb{F}_q) \not\subseteq V$ and so the differential uniformity of $f$ is at least 6 from Theorem 3. This condition can be written

$$q - (d - 3)^4q^{1/2} - 36(2d - 3)^5 - 8(d - 3) > 0.$$ 

This condition is satisfied when

$$q^{1/2} > d^4 - 12d^3 + 54d^2 + 1044d + 5265 + 25920/d$$

if $d \geq 2$, or $d < q^{1/8} + 2$ if $d \geq 31$. □
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Institut de Mathématiques de Toulon, Université du Sud Toulon-Var, France, and, Institut de Mathématiques de Luminy, Marseille, France
E-mail address: yves.aubry@univ-tln.fr and rodier@iml.univ-mrs.fr