Graphs and Combinatorics
DOI 10.1007/s00373-015-1625-3

ORIGINAL PAPER

Free Choosability of Outerplanar Graphs

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Received: 31 October 2014 / Revised: 17 July 2015

Abstract A graph G is free (a, b)-choosable if for any vertex v with b colors assigned and for any list of colors of size a associated with each vertex $u \neq v$, the coloring can be completed by choosing for u a subset of b colors such that adjacent vertices are colored with disjoint color sets. In this note, a necessary and sufficient condition for a cycle to be free (a, b)-choosable is given. As a corollary, we obtain almost optimal results about the free (a, b)-choosability of outerplanar graphs.

Keywords Coloring · Choosability · Free choosability · Cycle · Outerplanar graph

Mathematics Subject Classification 05C15 · 05C38 · 05C10

1 Introduction

We consider only simple graphs, i.e. graphs without loops or parallel edges. For a graph G, we denote its vertex set by $V(G)$ and edge set by $E(G)$. A list assignment L of a graph G is an assignment of lists of integers (colors) to the vertices of G. For

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Published online: 23 September 2015

an integer a, a a-assignment L of G is a list assignment such that $|L(v)| = a$ for any $v \in V(G)$.

In 1996, Voigt considered the following problem: let G be a graph and L a list assignment and assume that an arbitrary vertex $v \in V(G)$ is precolored by a color $f \in L(v)$. Under which hypothesis is it always possible to complete this precoloring to a proper list coloring ? This question leads to the concept of free choosability introduced by Voigt [9].

Formally, for a graph G, integers a, b and a a-assignment L of G, an (L, b)-*coloring* of G is a mapping c that associates to each vertex u a subset $c(u)$ of $L(u)$ such that $|c(u)| = b$ and $c(u) \cap c(v) = \emptyset$ for any edge $uv \in E(G)$. The graph G is (a, b)-*choosable* if for any a-assignment L of G, there exists an (L, b)-coloring . Moreover, G is *free* (a, b)-*choosable* if for any a-assignment L, any vertex v_0 and any set $c_0 \subset L(v_0)$ of b colors, there exists an (L, b)-coloring c such that $c(v_0) = c_0$. Remark that another equivalent way to view the problem is to consider that the list assignment $L(v)$ is of size a for any vertex except for v_0 for which $|L(v_0)| = b$. We will use interchangeably both views.

As shown by Voigt [9], there are examples of graphs G that are (a, b)-choosable but not free (a, b)-choosable. Some related recent results concern defective free choosability of outerplanar graphs [7]. We investigate, in the next section, the free-choosability of the first interesting case, namely the cycle. We derive a necessary and sufficient condition for a cycle to be free (a, b)-choosable (Theorem 1). Then we use it to obtain free (a, b)-choosability results of outerplanar graphs in Sect. 3. We end the paper with some algorithmic issues in Sect. 4.

In order to get concise statements, we introduce the free-choice ratio of a graph, in the same way that Alon, Tuza and Voigt [1] introduced the choice ratio (which is equal to the so-called fractional chromatic number).

For any real x, let $\mathrm{FCH}(x)$ be the set of graphs G which are free (a, b)-choosable for all a, b such that $\frac{a}{b} \geq x$:

$$\mathrm{FCH}(x) = \left\{G \mid \forall\ \frac{a}{b} \geq x,\ \ G \text{ is free } (a, b)\text{-choosable}\right\}.$$

Moreover, we can define the free-choice ratio $\mathrm{fchr}(G)$ of a graph G by:

$$\mathrm{fchr}(G) := \inf\left\{\frac{a}{b} \mid G \text{ is free } (a, b)\text{-choosable}\right\}.$$

Remark 1 Erdős, Rubin and Taylor have asked [4] the following question: If G is (a, b)-choosable, and $\frac{c}{d} > \frac{a}{b}$, does it imply that G is (c, d)-choosable ? Gutner and Tarsi have shown [6] that the answer is negative in general. If we consider the analogue question for the free choosability, then Theorem 1 implies that it is true for the cycle.

The *path* P_{n+1} of length n is the graph with vertex set $V = \{v_0, v_1, \ldots, v_n\}$ and edge set $E = \bigcup_{i=0}^{n-1}\{v_i v_{i+1}\}$. The cycle C_n of length n is the graph with vertex set $V = \{v_0, \ldots, v_{n-1}\}$ and edge set $E = \bigcup_{i=0}^{n-1}\{v_i v_{i+1 (mod\ n)}\}$. To simplify the notation, for a list assignment L of P_n or C_n, we let $L(i)$ denote $L(v_i)$ and $c(i)$ denote $c(v_i)$.

The notion of waterfall list assignment was introduced in [2] to obtain choosability results on the weighted path and then used to prove the $(5m, 2m)$-choosability of triangle-free induced subgraphs of the triangular lattice. We recall one of the results from [2] that will be used in this note, with the function Even being defined for any real x by: $\text{Even}(x)$ is the smallest even integer p such that $p \geq x$.

Proposition 1 ([2]) *Let L be a list assignment of P_{n+1} such that $|L(0)| = |L(n)| = b$, and $|L(i)| = a = 2b + e$ for all $i \in \{1, \ldots, n-1\}$ (with $e > 0$).*

$$\text{If } n \geq \text{Even}\left(\frac{2b}{e}\right) \text{ then } P_{n+1} \text{ is } (L, b)\text{-colorable}.$$

As an example, let P_{n+1} be the path of length n with a list assignment L such that $|L(0)| = |L(n)| = 4$, and $|L(i)| = 9$ for all $i \in \{1, \ldots, n-1\}$. Then the previous proposition asserts us that we can find an $(L, 4)$-coloring of P_{n+1} whenever $n \geq 8$. In other words, if $n \geq 8$, we can choose 4 colors on each vertex such that adjacent vertices receive disjoint colors sets. On the other side, there are list assignments L for which P_{n+1} is not $(L, 4)$-colorable for $n < 8$. For instance, there is no $(L, 4)$-coloring of the path P_8 with list assignment L such that $L(0) = L(7) = \{1, 2, 3, 4\}$ and $L(i) = \{1, \ldots, 9\}$ for $1 \leq i \leq 6$. But if $|L(i)| = 11$ for all $i \in \{1, \ldots, n-1\}$, then P_{n+1} is $(L, 4)$-colorable whenever $n \geq 4$.

2 Free Choosability of the Cycle

We begin with a negative result for the even-length cycle:

Lemma 1 *For any integers a, b, p such that $p \geq 2$, and $\frac{a}{b} < 2 + \frac{1}{p}$, the cycle C_{2p} is not free (a, b)-choosable.*

Proof We construct a counterexample for the free-choosability of C_{2p}: let L be the a-assignment of C_{2p} such that

$$L(i) = \begin{cases} \{1, \ldots, a\}, & \text{if } i \in \{0, 1\}; \\ \{\frac{i-1}{2}a + 1, \ldots, \frac{i-1}{2}a + a\}, & \text{if } i \neq 2p-1 \text{ is odd}; \\ \{b + \frac{i-2}{2}a + 1, \ldots, b + (\frac{i-2}{2} + 1)a\}, & \text{if } i \text{ is even and } i \neq 0; \\ \{1, \ldots, b, 1 + (p-1)a, \ldots, 1 + pa - b - 1\}, & \text{if } i = 2p-1. \end{cases}$$

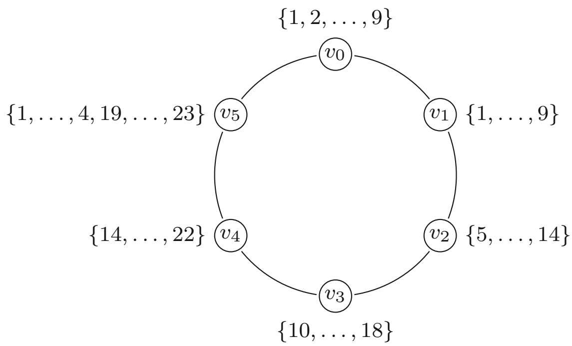
If we choose $c_0 = \{1, \ldots, b\} \subset L(0)$, we can check that there does not exist an (L, b)-coloring of C_{2p} such that $c(0) = c_0$, so C_{2p} is not free (a, b)-choosable. See Fig. 1 for an illustration when $p = 3$, $a = 9$ and $b = 4$. □

Now, if $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real x, we can state:

Theorem 1 *For the cycle C_n of length n,*

$$C_n \in \text{FCH}\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Fig. 1 The cycle C_6, along with a nine-assignment L for which there is no $(L, 4)$-coloring c such that $c(v_0) = \{1, 2, 3, 4\}$



Moreover, we have:

$$\mathrm{fchr}(C_n) = 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}.$$

Proof Let a, b be two integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Let L be a a-assignment of C_n. Without loss of generality, we can suppose that v_0 is the vertex chosen for the free-choosability and $c_0 \subset L(v_0)$ has b elements. It remains to construct an (L, b)-coloring c of C_n such that $c(v_0) = c_0$. Hence we have to construct an (L', b)-coloring c of P_{n+1} such that $L'(0) = L'(n) = L_0$ and for all $i \in \{1, ..., n-1\}$, $L'(i) = L(v_i)$. We have $|L'(0)| = |L'(n)| = b$ and for all $i \in \{1, ..., n-1\}$, $|L'(i)| = a$. Since $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ and $e = a - 2b$, we get $e/b \geq \lfloor \frac{n}{2} \rfloor^{-1}$ hence $n \geq \mathrm{Even}(2b/e)$. Using Proposition 1, we get:

$$C_n \in \mathrm{FCH}\left(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}\right).$$

Hence, we have that $\mathrm{fchr}(C_n) \leq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Moreover, let us prove that $M = 2 + \lfloor \frac{n}{2} \rfloor^{-1}$ is reached. For n odd, Voigt has proved [10] that the choice ratio $\mathrm{chr}(C_n)$ of a cycle of odd length n is exactly M. Hence $\mathrm{fchr}(C_n) \geq \mathrm{chr}(C_n) = M$, and the result is proved. For n even, Lemma 1 asserts that C_n is not free (a, b)-choosable for $\frac{a}{b} < 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. □

Remark 2 In particular, the previous theorem implies that if b, e, n are integers such that $n \geq \mathrm{Even}(\frac{2b}{e})$, then the cycle C_n of length n is free $(2b + e, b)$-choosable.

Remark 3 As a comparison with the ordinary choice ratio, we can note that the free choice ratio of an odd-length cycle is the same than its choice ratio whereas for the even-length cycle C_{2p}, we have $\mathrm{chr}(C_{2p}) = 2 < \mathrm{fchr}(C_{2p}) = 2 + 1/p$.

3 Free Choosability of Outerplanar Graphs

An *outerplanar graph* is a graph that has a crossing-free embedding in the plane such that all vertices are on the same face (without loss of generality, it may be assumed that it is the unbounded one). Biconnected graphs have interesting decomposition properties (see [3, p.124–125]): for a subgraph F of a graph G, an *ear* of F in G is a nontrivial path P in G whose endpoints lie in F but whose internal vertices do not. An *ear decomposition* is a sequence $(G_0, G_1, \ldots, G_r)$ of subgraphs of G such that G_0 is a cycle, $G_{i+1} = G_i \cup Q_i$, where Q_i is an ear of G_i in G, $0 \leq i < r$, and $G_r = G$. As noticed in [3], the ear decomposition can start from any cycle of the graph.

Bi-connected outerplanar graphs have special ear decompositions:

Observation 2 *Any 2-connected outerplanar graph G embedded on the plane admits an ear decomposition $(G_0, G_1, \ldots, G_r)$ such that G_0 is any face of G and the endpoints of each ear are adjacent vertices in G.*

Proof Since G is 2-connected, any crossing-free embedding of G on the plane has an ear decomposition $(G_0, G_1, \ldots, G_r)$ with G_0 being any of its faces. We now show that the ends x_i, y_i of each ear $Q_i = G_{i+1} - G_i$, $0 \leq i < r$ are adjacent vertices in G. By contradiction, assume that there exists j such that $x_j y_j \notin E(G)$. Then, since G is 2-connected, there is another path Q' of length at least 2 between x_j and y_j that share no edge with Q_j. But any internal vertex of Q' will not remain on the unbounded face if the ear Q_j is added to G_j, contradicting the hypothesis that G is embedded on the plane. □

An example of an ear decomposition of a 2-connected outerplanar graph is given in Fig. 2.

In order to restrict our argument to biconnected graphs, we first show the following:

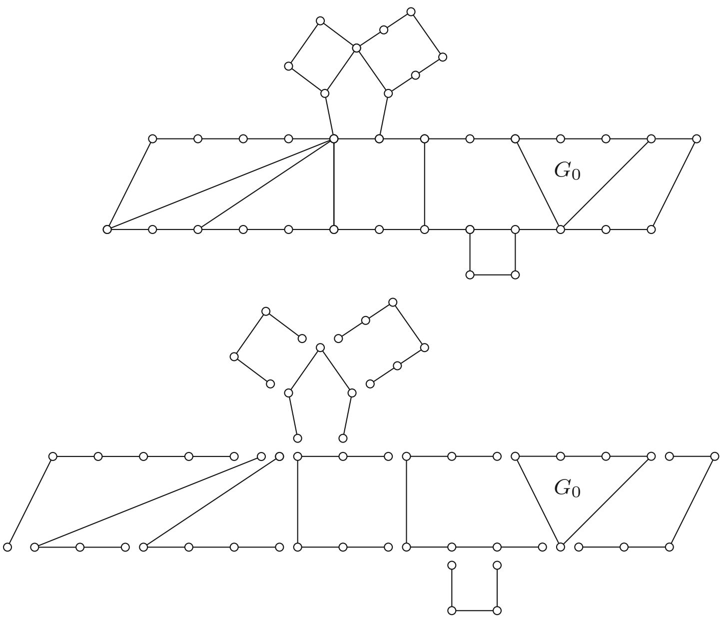


Fig. 2 A 2-connected outerplanar graph G (on the *top*) and an ear decomposition of G from G_0 (on the *bottom*)

Lemma 2 *Let a, b be integers and let G_1, G_2 be two free (a, b)-choosable graphs. Then the graph obtained from G_1 and G_2 by identifying any vertex of G_1 with any vertex of G_2 is free (a, b)-choosable.*

Proof Let G be the graph obtained by identifying vertex x_1 of G_1 with vertex x_2 of G_2, resulting in a vertex named x. Let $x_0 \in V(G)$ and let L be a list assignment of G with $|L(v)| = a$ for $v \in V(G) \setminus \{x_0\}$ and $|L(x_0)| = b$ (i.e. x_0 is the precolored vertex). Assume without loss of generality, that $x_0 \in V(G_1)$. Let L_i, $i = 1, 2$, be the sublist assignment of L restricted to vertices of G_i. As G_1, G_2 are both free (a, b)-choosable, there exists an (L_1, b)-coloring c_1 of G_1 and an (L_2, b)-coloring c_2 of G_2 such that $c_2(x) = c_1(x)$ (i.e. x is the precolored vertex of G_2). The union of colorings c_1 and c_2 is an (L, b)-coloring of G. □

We now show a free choosability result for graphs having restricted ear decompositions. A graph G is *k-ear decomposable* if it has an ear decomposition starting from any cycle of length at least k and such that the length of each ear is at least k.

Theorem 3 *If a graph G is k-ear decomposable, then*

$$G \in \mathrm{FCH}\left(2 + \left\lfloor \frac{k}{2} \right\rfloor^{-1}\right).$$

Proof Let a, b be integers such that $\frac{a}{b} \geq 2 + \lfloor \frac{k}{2} \rfloor^{-1}$. We show that G is free (a, b)-choosable. Let L be a list assignment of G such that $|L(v)| = a$ for $v \neq x_0$ and $|L(x_0)| = b$ (x_0 is the precolored vertex) and let C_0 be any cycle of G on which x_0 lies. As G is k-ear decomposable, it possesses an ear decomposition $(G_0, \ldots, G_r)$, with $G_0 = C_0$ and with ears of length at least k. First color the vertices of G_0 thanks to Theorem 1. Now, we show that the ears $Q_i = G_{i+1} - G_i$, $0 \leq i < r$, can be colored in sequence: by hypothesis, we have

$$\begin{aligned} \frac{a}{b} &\geq 2 + \left\lfloor \frac{k}{2} \right\rfloor^{-1} \\ &\Leftrightarrow \left\lfloor \frac{k}{2} \right\rfloor \geq \frac{b}{a - 2b}. \end{aligned} \tag{1}$$

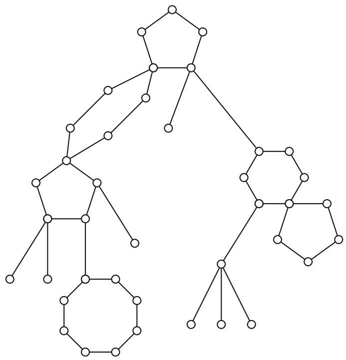
If k is even, then Inequality 1 becomes $\frac{k}{2} \geq \frac{b}{a-2b}$, hence $k \geq \mathrm{Even}(\frac{2b}{a-2b})$. If k is odd, then Inequality 1 becomes $\frac{k-1}{2} \geq \frac{b}{a-2b}$, hence $k \geq \mathrm{Even}(\frac{2b}{a-2b})$. Therefore, in both cases, the conditions of Proposition 1 are satisfied, hence the coloring of G_i can be extended to G_{i+1}. □

Corollary 1 *If G is an outerplanar graph of girth g, then*

$$G \in \mathrm{FCH}\left(2 + \left\lfloor \frac{g-1}{2} \right\rfloor^{-1}\right).$$

Proof Let a, b be integers such that $\frac{a}{b} \geq 2 + \lfloor \frac{g-1}{2} \rfloor^{-1}$. We show that G is free (a, b)-choosable. First, if G is not 2-connected, then it can be decomposed into blocks (that

Fig. 3 A cactus of girth 5



are maximally 2-connected subgraphs or single edges). Trivially, a single edge K_2 is free $(2b, b)$-choosable and thus free (a, b)-choosable. By Lemma 2, if each block is free (a, b)-choosable, then the whole graph G is also free (a, b)-choosable. Therefore, we can suppose that G is 2-connected, and, since it is outerplanar, it possesses an ear decomposition. Moreover, since G is of girth g and the endvertices of the ears are adjacent vertices, the length of each ear is at least $g-1$. Therefore, G is $(g-1)$-ear decomposable and, by Theorem 3, G is free (a, b)-choosable. □

Notice that for odd g, the above result is tight since $\lfloor \frac{g-1}{2} \rfloor = \lfloor \frac{g}{2} \rfloor$. For even g, however, we were not able to find an outerplanar graph of girth g that is not free (a, b)-choosable for $\frac{a}{b} = 2 + \lfloor \frac{g}{2} \rfloor^{-1}$. We propose the following conjecture:

Conjecture 1 If G is an outerplanar graph of girth g, then

$$G \in \mathrm{FCH}\Big(2 + \Big\lfloor \frac{g}{2} \Big\rfloor^{-1}\Big).$$

As a first step, we now show that Conjecture 1 is true for cactus graphs, a subclass of outerplanar graphs. A *cactus* is a graph in which every edge is part of at most one cycle (see Fig. 3 for an example).

Proposition 2 *For any cactus G of girth g, we have $G \in \mathrm{FCH}(2 + \lfloor \frac{g}{2} \rfloor^{-1})$ and* $\mathrm{fchr}(G) = (2 + \lfloor \frac{g}{2} \rfloor^{-1})$.

Proof Let G be a cactus of girth g and let a, b be integers such that $\frac{a}{b} \geq 2 + \lfloor \frac{g}{2} \rfloor^{-1}$. We show that G is free (a, b)-choosable. Since G is a cactus, each of its blocks $B_0, \ldots, B_r$ is either a cycle (of length at least g) or a single edge, and they are connected in a treelike structure. By virtue of Theorem 1, any cycle of length at least g is free (a, b)-choosable and trivially, any edge is free (a, b)-choosable. Therefore, by virtue of Lemma 2, G is free (a, b)-choosable. Moreover, since G contains a cycle of length g, then by Theorem 1, $\mathrm{fchr}(G) \geq (2 + \lfloor \frac{g}{2} \rfloor^{-1})$. □

Notice that Conjecture 1 is no longer true if we consider planar graphs instead of outerplanar graphs, as the next result shows:

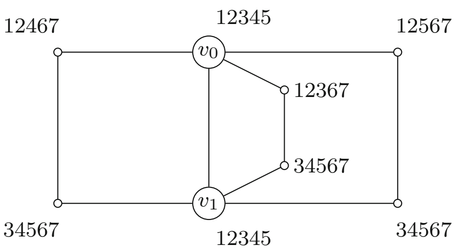


Fig. 4 A planar graph G along with a five-assignment L (in condensed notation) for which G is not free $(L, 2)$-choosable

Proposition 3 *There are planar graphs of girth* 4 *that are not in* $\mathrm{FCH}(\frac{5}{2})$.

Proof The graph depicted on Fig. 4 is not free (5, 2)-choosable: a 5-assignment L for which the graph is not free $(L, 2)$-colorable is given on the figure. If the vertex chosen for the free choosability is v_0 (the one of degree 4 on the top) and if $c(v_0) = \{1, 2\}$, then it can be seen that anyway we color the other vertex v_1 of degree 4, it will not be possible to complete the coloring. Effectively, each of the three antipodal vertices of v_1 needs to have at least a color in common with those of v_1 in order the last vertex of each C_4 be colored, which is impossible. □

We believe that there is no counterexample with smaller order than the one of Fig. 4. Finding the smallest counterexamples for other values of the girth g could also have some interest.

4 Algorithmic considerations

Let $n \geq 3$ be an integer and let a, b be two integers such that $a/b \geq 2 + \lfloor \frac{n}{2} \rfloor^{-1}$. Let L be a a-assignment of C_n.

As defined in [2], a *waterfall* list L of a path P_{n+1} of length n is a list L such that for all $i, j \in \{0, \ldots, n\}$ with $|i - j| \geq 2$, we have $L(i) \cap L(j) = \emptyset$. Let $m = |\cup_{i=0}^{n} L(i)|$ be the total number of colors of the color-list L.

The algorithm behind the proof of Proposition 1 consists in three steps: first, the transformation of the list L into a waterfall list L' by renaming some colors; second, the construction of the (L', b)-coloring by coloring vertices from 0 to $n - 1$, giving to vertex i the first b-colors that are not used by the previous vertex; third, the backward transformation to obtain an (L, b)-coloring from the (L', b)-coloring by coming back to original colors and resolving color conflicts if any. It can be seen that the time complexity of the first step is $O(mn)$; that of the second one is $O(a^2 n)$ and that of the third one is $O(\max(a, b^3)n)$. Therefore, the total running time for computing a free (L, b)-coloring of the cycle C_n is $O(\max(m, a^2, b^3)n)$.

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