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PROPRIÉTÉS ASYMPTOTIQUES DES CORPS GLOBAUX

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Introduction

Deux parties principales constituent le sujet de cette thèse. La première partie est consacrée à l'étude des propriétés asymptotiques des fonctions zêta, des fonctions L, des corps globaux et des variétés sur ces corps. Le but de la deuxième partie est l'étude des jacobiennes parmi les variétés abéliennes de dimension 3. La thématique étant large nous allons donner une description détaillée du contenu de chaque partie et de chaque chapitre.

Première partie.

La théorie asymptotique des corps globaux a été développée dans les années 1990 par M. Tsfasman et S. Vlăduţ, d'abord pour les corps de fonctions puis pour les corps de nombres. La théorie avait pour origine le problème suivant : étant donné un nombre entier positif g et une puissance d'un nombre premier q, trouver le nombre maximal de points sur une courbe de genre g sur le corps fini $\mathbb{F}_q$. Le problème s'avère très difficile et la réponse complète n'est connue que pour $g = 1$ et $g = 2$. Il y a aussi des résultats partiels pour $g = 3$, qui sont obtenus en étudiant les jacobiennes parmi les variétés abéliennes de dimension 3, ce qui fait l'objet de la deuxième partie de cette thèse.

V. Drinfeld, S. Vlăduţ puis M. Tsfasman ont réussi à obtenir des résultats intéressants en considérant ce problème sous un angle différent. Ils ont, notamment, obtenu des bornes asymptotiques pour le nombre maximal de points quand $g \to \infty$, qui sont optimales quand q est un carré. Tout cela a eu de nombreuses applications en théorie des codes correcteurs, en théorie des empilements de sphères, etc.

Cette théorie asymptotique a été développée bien au delà de ces bornes pour le nombre de points et elle réunit maintenant des résultats très divers. Citons par exemple : le théorème de Brauer–Siegel généralisé pour les corps de fonctions et pour les corps de nombres, les bornes pour les régulateurs et pour les discriminants, la théorie asymptotique des fonctions zêta des corps globaux, les bornes pour le nombre de points sur les variétés sur les corps finis...

Le but de la première partie de la thèse est, d'abord d'étudier plus profondément la théorie asymptotique des corps globaux et surtout celle des corps de nombres où les résultats étaient bien moins précis à cause des difficultés analytiques inhérentes au sujet. Puis, nous entreprenons la quête d'autres cas où la théorie asymptotique pourrait être applicable. Plus précisément, nous étudions les trois cas suivants avec des points de vue différents : les fonctions zêta des variétés de dimension supérieure sur les corps finis, les fonctions L des surfaces elliptiques sur les corps finis et les fonctions L des formes modulaires. Nous avons le sentiment que ces trois cas ne sont que le début d'une longue histoire qui est encore à écrire.

Maintenant nous allons expliquer le contenu de chaque chapitre.

Chapitre 1.

Dans ce chapitre nous étudions le théorème de Brauer–Siegel pour les corps de nombres. On a l'énoncé suivant du théorème de Brauer–Siegel classique : si k parcourt une suite d'extensions normales de $\mathbb{Q}$ telle que $n_k/\log|D_k| \to 0$, alors $\log h_k R_k/\log\sqrt{|D_k|} \to 1$[1].

Tout d'abord, nous démontrons une généralisation de ce théorème au cas des suites de corps presque normaux (où un corps presque normal est un corps qui admet une tour de sous-extensions, dont chaque étage est une extension normale). Le cas asymptotiquement bon (c'est à dire celui où $\lim n_k/\log|D_k| > 0$) était déjà connu avec cette généralité grâce aux travaux de M. Tsfasman et S. Vlăduţ. Cependant, leurs méthodes n'étaient pas applicables au cas asymptotiquement mauvais. Nous nous servons de la technique de H. Stark ainsi que de certaines inégalités de S. Louboutin pour démontrer ce résultat.

Ensuite, en utilisant une approche de F. Hajir et C. Maire, nous construisons quelques tours asymptotiquement bonnes avec $\lim \log h_k R_k/\log\sqrt{|D_k|}$ plus petit que dans les exemples connus auparavant.

Les résultats de ce chapitre sont parus dans *Moscow Mathematical Journal, Vol. 5, Num 4, pp. 961–968.*

Chapitre 2.

Ce chapitre présente un travail effectué en collaboration avec Philippe Lebacque.

Ici nous étudions le comportement asymptotique des dérivées logarithmiques des fonctions zêta dans des familles de corps globaux. Ce problème est important car il est lié d'une part à l'inégalité fondamentale de M. Tsfasman et S. Vlăduţ (dans le cas de corps de fonctions il s'agit d'une estimation pour le nombre de points sur les courbes sur les corps finis) et d'autre part au théorème de Brauer–Siegel explicite. Nous démontrons une formule asymptotique pour la dérivée logarithmique d'une fonction zêta dans la domaine $\mathrm{Re}\, s > \frac{1}{2}$ avec un terme d'erreur explicite. En particulier, sous GRH cela implique une amélioration du théorème de Brauer–Siegel explicite démontré par P. Lebacque.

L'article correspondant à ce chapitre est soumis pour publication.

Chapitre 3.

Dans ce chapitre nous avons deux buts principaux. En premier lieu, nous donnons un panorama des généralisations du théorème de Brauer–Siegel classique. Cela comprend les développements récents qui concernent le cas des variétés de dimension supérieure, en particulier nous donnons une explication détaillée des approches de M. Hindry et A. Pachecko, et de B. Kunyavskii et M. Tsfasman.

Le deuxième but dans ce chapitre est de démontrer une version du théorème de Brauer–Siegel pour les variétés de dimension $d \geq 1$ sur les corps finis dans lequel il s'agit du comportement asymptotique du résidu en $s = d$ de leurs fonctions zêta.

Les résultats sont à paraître dans *Proceedings of the Conference AGCT 11 (2007), Contemp. Math. series, AMS, 2009.*

Chapitre 4.

Ce chapitre est consacré à l'étude de la distribution des zéros des fonctions L des formes modulaires. Pour les fonctions zêta des corps globaux de tels résultats ont été établis par M. Tsfasman et S. Vlăduţ. En utilisant leurs méthodes, nous démontrons que sous GRH les zéros

[1] n_k est le degré, D_k est le discriminant, R_k est le régulateur et h_k est le nombre de classes d'idéaux de k.

des fonctions L des formes modulaires deviennent équirépartis sur la droite critique quand le niveau N ou le poids k (ou les deux) tend vers l'infini.

L'article correspondant à ce chapitre est soumis pour publication.

Chapitre 5.

Dans ce chapitre nous étudions les propriétés asymptotiques des familles de fonctions zêta et de fonctions L sur les corps finis. Nous le faisons dans le contexte des trois problèmes suivants : l'inégalité fondamentale, les résultats de type Brauer–Siegel et la distribution des zéros. Nous donnons une définition algébrique des fonctions L et des fonctions zêta auxquelles s'appliquent nos méthodes, de façon que la plupart des résultats précédents de G. Lachaud, M. Tsfasman, S. Vlăduţ sur les mêmes problèmes pour les fonctions zêta des courbes et des variétés sur les corps finis soient inclus dans notre schéma. Nous analysons dans quelle mesure les résultats classiques pour les courbes restent vrais dans ce contexte général.

Ensuite, nous passons à des applications concrètes. Le cas le plus intéressant est celui des fonctions L des familles de surfaces elliptiques, qui a été récemment étudié par B. Kunyavskii, M. Tsfasman, M. Hindry et A. Pacheko. Les résultats que nous avons obtenus permettent d'approcher quelques-unes de leurs conjectures. En outre, nos méthodes donnent une généralisation d'un résultat de P. Michel sur l'équirépartition des zéros des fonctions L des courbes elliptiques sur $\mathbb{F}_q(t)$.

Dans le cas classique des courbes sur les corps finis, nous arrivons à démontrer un théorème pour les fonctions zêta limites dont un corollaire est une généralisation d'un résultat de Y. Ihara sur la constante d'Euler–Kronecker.

Deuxième partie.

Cette partie présente un travail effectué en collaboration avec Gilles Lachaud et Christophe Ritzenthaler.

Le point de départ historique est le même problème que dans la première partie : trouver le nombre maximal de points sur une courbe sur un corps fini. Ici on s'intéresse à ce problème quand le genre g est petit, tandis que dans la première partie on supposait, au contraire, que $g \to \infty$. La différence des méthodes employées est très sensible.

Une approche de ce problème proposée par J.-P. Serre, consiste à résoudre la même question pour les variétés abéliennes (ce qui est facile grâce au théorème de Honda–Tate) et puis de choisir entre toutes les variétés abéliennes celles qui correspondent aux jacobiennes des courbes. C'est ce dernier problème que nous étudions dans cette partie de la thèse.

En utilisant les formes modulaires de Siegel nous donnons une réponse complète à ce problème quand le corps de définition k est contenu dans $\mathbb{C}$. Plus précisément, nous réalisons la stratégie suivante. Pour un corps k et une forme modulaire de Siegel f sur k de poids $h \geq 0$ et de genre $g > 1$ nous définissons un invariant des k-classes d'isomorphisme des variétés abéliennes principalement polarisées (A, a). De plus, si (A, a) est une jacobienne d'une courbe plane projective et lisse nous montrons comment associer à f un invariant plan classique. Comme premier corollaire de ces constructions, pour $g = 3$ et $k \subset \mathbb{C}$ nous obtenons une nouvelle démonstration de la formule de Klein qui relie la forme modulaire de Siegel χ_{18} au discriminant des quartiques planes. Le deuxième corollaire est une démonstration du fait qu'on peut décider si (A, a) est une jacobienne sur k en regardant si la valeur de χ_{18} au point correspondant à (A, a) est un carré dans k.

Cela fournit une réponse à la question de J.-P. Serre sur la caractérisation des jacobiennes.

L'article correspondant à cette partie est soumis pour publication.

Première partie

Propriétés asymptotiques des fonctions zêta et L

Chapitre 1

The Brauer-Siegel and Tsfasman-Vlăduţ Theorems for almost normal extensions of number fields

1.1 Introduction

Let K be an algebraic number field of degree $n_K = [K : \mathbb{Q}]$ and discriminant D_K. We define the genus of K as $g_K = \log \sqrt{D_K}$. By h_K we denote the class-number of K, R_K denotes its regulator. We call a sequence $\{K_i\}$ of number fields a family if K_i is non-isomorphic to K_j for $i \neq j$. A family is called a tower if also $K_i \subset K_{i+1}$ for any i. For a family of number fields we consider the limit

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log h_{K_i} R_{K_i}}{g_{K_i}}.$$

The classical Brauer-Siegel theorem, proved by Brauer (see [4]), states that for a family $\mathcal{K} = \{K_i\}$ we have $\mathrm{BS}(\mathcal{K}) = 1$ if the family satisfies two conditions :

(i) $\lim\limits_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0$;

(ii) either the generalized Riemann hypothesis (GRH) holds, or all the fields K_i are normal over $\mathbb{Q}$.

We call a number field almost normal if there exists a finite tower of number fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K$ such that all the extensions K_i/K_{i-1} are normal. Weakening the condition (ii), we prove the following generalization of the classical Brauer-Siegel theorem to the case of almost normal number fields :

Theorem 1.1.1. *Let $\mathcal{K} = \{K_i\}$ be a family of almost normal number fields for which $n_{K_i}/g_{K_i} \to 0$ as $i \to \infty$. Then we have* $\mathrm{BS}(\mathcal{K}) = 1$.

It was shown by M. A. Tsfasman and S. G. Vlăduţ that, taking in account non-archimedian places, one may generalize the Brauer-Siegel theorem to the case of extensions where the condition (i) does not hold.

For a prime power q we set

$$\Phi_q(K_i) := |\{v \in P(K_i) : \mathrm{Norm}(v) = q\}|,$$

where $P(K_i)$ is the set of non-archimedian places of K_i. We also put $\Phi_{\mathbb{R}}(K_i) = r_1(K_i)$ and $\Phi_{\mathbb{C}}(K_i) = r_2(K_i)$, where r_1 and r_2 stand for the number of real and (pairs of) complex embeddings.

We consider the set $A = \{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \ldots\}$ of all prime powers plus two auxiliary symbols $\mathbb{R}$ and $\mathbb{C}$ as the set of indices. A family $\mathcal{K} = \{K_i\}$ is called asymptotically exact if and only if for any $\alpha \in A$ the following limit exists :

$$\phi_\alpha = \phi_\alpha(\mathcal{K}) := \lim_{i\to\infty} \frac{\Phi_\alpha(K_i)}{g_{K_i}}.$$

We call an asymptotically exact family $\mathcal{K}$ asymptotically good (respectively, bad) if there exists $\alpha \in A$ with $\phi_\alpha > 0$ (respectively, $\phi_\alpha = 0$ for any $\alpha \in A$). The condition on a family to be asymptotically bad is, in the number field case, obviously equivalent to the condition (i) in the classical Brauer-Siegel theorem. For an asymptotically good tower of number fields the following generalization of the Brauer-Siegel theorem was proved in [87] :

Theorem 1.1.2 (Tsfasman-Vlăduţ Theorem, see [87], Theorem 7.3)**.** *Assume that for an asymptotically good tower $\mathcal{K}$ any of the following conditions is satisfied :*

– *GRH holds*
– *All the fields K_i are almost normal over $\mathbb{Q}$.*

Then the limit $\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log h_{K_i} R_{K_i}}{g_{K_i}}$ *exists and we have :*

$$\mathrm{BS}(\mathcal{K}) = 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi, \tag{1.1}$$

the sum beeing taken over all prime powers q.

For an asymptotically bad tower of number fields we have $\phi_{\mathbb{R}} = 0$ and $\phi_{\mathbb{C}} = 0$ as well as $\phi_q = 0$ for all prime powers q, so the right hand side of the formula (1.1) equals to one. We also notice that the condition on a family to be asymptotically bad is equivalent to $\lim_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0$. So, combining our theorem 1.1.1 with the theorem 1.1.2 we get the following corollary :

Corollary 1.1.3. *For any tower $\mathcal{K} = \{K_i\}$, $K_1 \subset K_2 \subset \ldots$ of almost normal number fields the limit* $\mathrm{BS}(\mathcal{K})$ *exists and we have :*

$$\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log(h_i R_i)}{g_i} = 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

the sum beeing taken over all prime powers q.

In [87] bounds on the ratio $\mathrm{BS}(\mathcal{K})$ were given, together with examples showing that the value of $\mathrm{BS}(\mathcal{K})$ may be different from 1. We corrected some of these erroneous bounds and managed to precise a few of the estimates in the examples. Also, using the infinite tamely ramified towers, found by Hajir and Maire (see [27]), we get (under GRH) new examples, both totaly complex and totally real, with the values of $\mathrm{BS}(\mathcal{K})$ smaller than those of totally real and totally complex examples of [87]. The result is as follows :

Theorem 1.1.4. 1. *Let $k = \mathbb{Q}(\xi)$, where ξ is a root of $f(x) = x^6 + x^4 - 4x^3 - 7x^2 - x + 1$, $K = k(\sqrt{\xi^5 - 467\xi^4 + 994\xi^3 - 3360\xi^2 - 2314\xi + 961})$. Then K is totally complex and has an infinite tamely ramified 2-tower $\mathcal{K}$, for which, under GRH, we have :*

$$\mathrm{BS}_{\text{lower}} \leq \mathrm{BS}(\mathcal{K}) \leq \mathrm{BS}_{\text{upper}},$$

where $\mathrm{BS}_{\text{lower}} \approx 0.56498\ldots$, $\mathrm{BS}_{\text{upper}} \approx 0.59748\ldots$.

2. *Let* $k = \mathbb{Q}(\xi)$*, where* ξ *is a root of* $f(x) = x^6 - x^5 - 10x^4 + 4x^3 + 29x^2 + 3x - 13$*,* $K = k(\sqrt{-2993\xi^5 + 7230\xi^4 + 18937\xi^3 - 38788\xi^2 - 32096\xi + 44590})$*. Then* K *is totally real and has an infinite tamely ramified 2-tower* $\mathcal{K}$*, for which, under GRH, we have :*

$$\mathrm{BS}_{\mathrm{lower}} \leq \mathrm{BS}(\mathcal{K}) \leq \mathrm{BS}_{\mathrm{upper}},$$

where $\mathrm{BS}_{\mathrm{lower}} \approx 0.79144\ldots$, $\mathrm{BS}_{\mathrm{upper}} \approx 0.81209\ldots$.

However, unconditionally (without GRH), the estimates for totally complex fields that may be obtained using the methods developed by Tsfasman and Vlăduţ lead to slightly worse results, than those already known from [87]. This is due to a rather large number of prime ideals of small norm in the field K. For the same reasons the upper bounds for the Brauer-Siegel ratio for other fields constructed in [27] are too high, though the lower bounds are still good enough.

Finally we present the table (the ameliorated version of the table of [87]), where all the bounds and estimates are given together :

		lower bound	lower example	upper example	upper bound
GRH	all fields	0.5165	0.5649-0.5975	1.0602-1.0798	1.0938
	totally real	0.7419	0.7914-0.8121	1.0602-1.0798	1.0938
	totally complex	0.5165	0.5649-0.5975	1.0482-1.0653	1.0764
Unconditional	all fields	0.4087	0.5939-0.6208	1.0602-1.1133	1.1588
	totally real	0.6625	0.8009-0.9081	1.0602-1.1133	1.1588
	totally complex	0.4087	0.5939-0.6208	1.0482-1.1026	1.1310

1.2 Proof of Theorem 1.1.1

Let $\zeta_K(s)$ be the Dedekind zeta function of the number field K and $\varkappa_K$ its residue at $s = 1$. By w_K we denote the number of roots of unity in K, and by r_1, r_2 the number of real and complex places of K respectively. We have the following residue formula (see [54], Chapter VIII, Section 3) :

$$\varkappa = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{D_K}}.$$

Since

$$\sqrt{w_K/2} \leq \varphi(w_K) = [\mathbb{Q}(\zeta_{w_K}) : \mathbb{Q}] \leq [K : \mathbb{Q}] = n_K,$$

we note that $w_K \leq 2n_K^2$ so $\log w_{K_j}/g_{K_j} \to 0$. Thus, it is enough to prove that $\log \varkappa_{K_j}/\log D_{K_j} \to 0$.

As for the upper bound we have

Theorem 1.2.1 (See [59], Theorem 1)**.** *Let* K *be a number field of degree* $n \geq 2$*. Then,*

$$\varkappa_K \leq \left(\frac{e\log D_K}{2(n-1)}\right)^{n-1}. \tag{1.2}$$

Moreover, $1/2 \leq \rho < 1$ *and* $\zeta_K(\rho) = 0$ *imply*

$$\varkappa_K \leq (1-\rho)\left(\frac{e\log D_K}{2n}\right)^{n}. \tag{1.3}$$

Using the estimate (1.2) we get (even without the assumption of almost normality) the "easy inequality" :

$$\frac{\log \varkappa_{K_j}}{\log D_{K_j}} \leq \frac{n_j - 1}{\log D_{K_j}}\left(\log\frac{e}{2} + \log\frac{\log D_{K_j}}{n_j - 1}\right) \to 0.$$

As for the lower bound the business is much more tricky and we will proceed to the proof after giving a few preliminary statements.

Let K be a number field other than $\mathbb{Q}$. A real number ρ is called an *exceptional zero* of $\zeta_K(s)$ if $\zeta_K(\rho) = 0$ and

$$1 - (4 \log D_K)^{-1} \leq \rho < 1;$$

an exceptional zero ρ of $\zeta_K(s)$ is called its *Siegel zero* if

$$1 - (16 \log D_K)^{-1} \leq \rho < 1.$$

Our proof will be based on the following fundamental property of Siegel zeroes proved by Stark :

Theorem 1.2.2 (See [80], Lemma 10)**.** *Let K be an almost normal number field, and let ρ be a Siegel zero of $\zeta_K(s)$. Then there exists a quadratic subfield k of K such that $\zeta_k(\rho) = 0$.*

The next estimate is also due to Stark :

Theorem 1.2.3 (See [80], Lemma 4 or [60], Theorem 1)**.** *Let K be a number field and let ρ be the exceptional zero of $\zeta_K(s)$ if it exists and $\rho = 1 - (4 \log D_K)^{-1}$ otherwise. Then there is an absolute constant $c < 1$ (effectively computable) such that*

$$\varkappa_K > c(1 - \rho) \tag{1.4}$$

Our proof of Theorem 1.1.1 will be similar to the proof of the classical Brauer-Siegel theorem given in [61]. We will use the Brauer-Siegel result for quadratic fields, a simple proof of which is given in [23]. There are two cases to consider.

1. First, assume that $\zeta_{K_j}(s)$ has no Siegel zero. From (1.4) we deduce that

$$\varkappa_{K_j} > c(1 - \rho) \geq c \left(1 - \left(1 - \frac{1}{16 \log D_{K_j}}\right)\right) = \frac{c}{16 \log D_{K_j}}. \tag{1.5}$$

2. Second, assume that there exists a Siegel zero ρ of $\zeta_{K_j}(s)$. From Theorem 1.2.2 we see that there exists a quadratic subfield k_j of K_j such that $\zeta_{k_j}(\rho) = 0$. Applying (1.3) and (1.4) we obtain :

$$\varkappa_{K_j} = \frac{\varkappa_{K_j}}{\varkappa_{k_j}} \varkappa_{k_j} \geq \frac{c(1 - \rho)}{(1 - \rho)\left(\frac{e \log D_{k_j}}{4}\right)^2} \varkappa_{k_j} = \frac{16c}{e^2 \log^2 D_{k_j}} \varkappa_{k_j}. \tag{1.6}$$

If the number of fields K_j for which the second case holds is finite, then, using the fact that $\log D_{K_j} \to \infty$, we get the desired lower estimate from (1.5).

Otherwise, we note that for a number field there exists at most one exceptional zero (See [80], Lemma 3), so, applying this statement to the fields k_j, we get that only finitely many of them may be isomorphic to each other and so $D_{k_j} \to \infty$ as $j \to \infty$. Thus we may use the Brauer-Siegel result for quadratic fields :

$$\frac{\log \varkappa_{k_j}}{\log D_{K_j}} \leq \frac{\log \varkappa_{k_j}}{\log D_{k_j}} \to 0.$$

Finally from (1.6), we get :

$$\frac{\log \varkappa_{K_j}}{\log D_{K_j}} \geq \frac{16c}{e^2 \log D_{K_j}} - 2 \frac{\log \log D_{k_j}}{\log D_{K_j}} + \frac{\log \varkappa_{k_j}}{\log D_{K_j}} \to 0.$$

This concludes the proof. □

Remark 1.2.1. Our proof of Theorem 1.1.1 is explicit and effective if all the fields in the family $\mathcal{K}$ contain no quadratic subfield and thus the corresponding zeta function does not have Siegel zeroes.

1.3 Proof of Theorem 1.1.4

First we recall briefly some constructions related to class field towers. Let us fix a prime number ℓ. For a finitely generated pro-ℓ group G, we let $d(G) = \dim_{\mathbb{F}_\ell} H^1(G, \mathbb{F}_\ell)$ be its generator rank. Let T be a finite set of ideals of a number field K such that no prime in T is a divisor of ℓ. We denote by K_T the maximal ℓ-extension of K unramified outside $T, G_T = \mathrm{Gal}(K_T/K)$. We let

$$\theta_{K,T} = \begin{cases} 1, & \text{if } T \neq \varnothing \text{ and } K \text{ contains a primitive } \ell\text{th root of unity}\,; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have (see [76], theorems 1 and 5) :

Theorem 1.3.1. *If $d(G_T) \geq 2 + 2\sqrt{r_1(K) + r_2(K) + \theta_{K,T}}$, then K_T is infinite.*

To estimate $d(G_T)$ we use the following theorem

Theorem 1.3.2 (See [62], section 2)**.** *Let K/k be a finite Galois extension, $r_1 = r_1(k)$, $r_2 = r_2(k)$, ρ be the number of real places of k, ramified in K, t be the number of primes in k, ramified in K. We set $\delta_\ell = 1$, if k contains a primitive root of degree ℓ of unity and $\delta_\ell = 0$ otherwise. Then we have :*

$$d(G_T) \geq d(G_\varnothing) \geq t - r_1 - r_2 + \rho - \delta_\ell$$

The number field arithmetic behind the construction of our theorem 1.1.4 was mainly carried out with the help of the computer package PARI. However, we would like to present our examples in the way suitable for non-computer check. We give here the proof of the first part of our theorem, as the proof of the second part is very much similar and may be carried out simply by repeating all the steps of the proof given here.

The following construction is taken from [27]. We let $k = \mathbb{Q}(\xi)$, where ξ is a root of $f(x) = x^6 + x^4 - 4x^3 - 7x^2 - x + 1$. Then k is a field of signature $(4, 1)$ and discriminant $d_f = d_k = -23{\cdot}35509$. Its ring of integers is $\mathcal{O}_k = \mathbb{Z}[\xi]$ and its class number is equal to 1. The principal ideal of norm $7{\cdot}13{\cdot}19^2{\cdot}23^2{\cdot}29{\cdot}31$ generated by $\eta = -671\xi^5 + 467\xi^4 - 994\xi^3 + 3360\xi^2 + 2314\xi - 961$ factors into eight different prime ideals of $\mathcal{O}_k$. In fact, one may see that $\eta = \pi_7\pi_{13}\pi_{19}\pi'_{19}\pi_{23}\pi'_{23}\pi_{29}\pi_{31}$, where

$$\begin{aligned}
\pi_7 &= -9\xi^5 + 6\xi^4 - 13\xi^3 + 44\xi^2 + 31\xi - 12, \\
\pi_{13} &= -7\xi^5 + 5\xi^4 - 11\xi^3 + 36\xi^2 + 23\xi - 9, \\
\pi_{19} &= 5\xi^5 - 4\xi^4 + 8\xi^3 - 26\xi^2 - 15\xi + 6, \\
\pi'_{19} &= 5\xi^5 - 3\xi^4 + 7\xi^3 - 24\xi^2 - 20\xi + 6, \\
\pi_{23} &= -5\xi^5 + 4\xi^4 - 8\xi^3 + 26\xi^2 + 15\xi - 9, \\
\pi'_{23} &= 6\xi^5 - 4\xi^4 + 9\xi^3 - 30\xi^2 - 22\xi + 6, \\
\pi_{29} &= 11\xi^5 - 8\xi^4 + 17\xi^3 - 56\xi^2 - 35\xi + 16, \\
\pi_{31} &= 7\xi^5 - 5\xi^4 + 11\xi^3 - 36\xi^2 - 22\xi + 7.
\end{aligned}$$

$K = k(\sqrt{\eta})$ is a totally complex field of degree 12 over $\mathbb{Q}$ with the relative discriminant $\mathcal{D}_{K/k}$ equal to (η) as $\eta = \beta^2 + 4\gamma$, where $\beta = \xi^5 + \xi^4 + \xi^3 + 1$, $\gamma = -173\xi^5 + 112\xi^4 - 270\xi^3 + 815\xi^2 + 576\xi - 237$. From this we see that $d_K = 7 \cdot 13 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 23^2 \cdot 35509^2$. From Theorem 1.3.2 we deduce that

$$d(G_\varnothing) \geq t - r_1(k) - r_2(k) + \rho - 1 = 8 - 4 - 1 + 4 - 1 = 6.$$

The right hand side of the inequality from Theorem 1.3.1 is equal to $2 + 2\sqrt{6} \approx 6.8989 < 7$, so it is enough to show that $d(G_T) > d(G_\varnothing)$, and to do this it is enough to construct a set of prime ideals T and an extension of K, ramified exactly at T.

Let $\pi_3 = -6\xi^5 + 4\xi^4 - 9\xi^3 + 30\xi^2 + 21\xi - 7$ be the generator of a prime ideal of norm 3 in $\mathcal{O}_k$ and T be the set consisting of one prime ideal of $\mathcal{O}_K$ over $\pi_3\mathcal{O}_k$. We see that $\pi_3\pi_{19} = 11\xi^5 - 8\xi^4 + 17\xi^3 - 56\xi^2 - 35\xi + 14 = \rho^2 + 4\sigma$, where $\rho = \xi^5 + \xi^3 + \xi^2 + 1$, $\sigma = 2\xi^5 - 8\xi^4 - 14\xi^3 - 28\xi^2 - 9\xi + 5$, so $k(\sqrt{\pi_3\pi_{19}})/k$ is ramified exactly at π_3 and π_{19}. But π_{19} already ramifies in K that is why $K(\sqrt{\pi_3\pi_{19}})/K$ is ramified exactly at T. Thus we have showed that $d(G_T) \geq 7$ and K_T/K is indeed infinite.

To complete our proof we need a few more results.

Theorem 1.3.3 (GRH Basic Inequality, see [87], Theorem 3.1)**.** *For an asymptotitically exact family of number fields under GRH one has :*

$$\sum_q \frac{\phi_q \log q}{\sqrt{q} - 1} + \phi_{\mathbb{R}} \left(\log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\gamma}{2} \right) + \phi_{\mathbb{C}} (\log 8\pi + \gamma) \leq 1, \tag{1.7}$$

the sum beeing taken over all prime powers q.

Theorem 1.3.4 (See [27], Theorem 1)**.** *Let K be a number field of degree n over $\mathbb{Q}$, such that K_T is infinite and assume that $K_T = \bigcup_{i=1}^{\infty} K_i$. Then*

$$\lim_{i \to \infty} \frac{g_i}{n_i} \leq \frac{g_K}{n_K} + \frac{\sum_{\mathfrak{p} \in T} \log(N_{K/\mathbb{Q}}\mathfrak{p})}{2n_K}.$$

For our previously constructed field K the genus is equal to $g_K \approx 25.3490\ldots$. From Theorem 1.3.4 we easily see that $\phi_{\mathbb{R}} = 0$ and $\frac{12}{2g_K + 2\log 3} \leq \phi_{\mathbb{C}} \leq \frac{12}{2g_K}$, i. e., $0.23669 < \phi_{\mathbb{C}} < 0.22687$. The lower bound for $\mathrm{BS}(K_T)$ is clearly equal to

$$\mathrm{BS}_{\text{lower}} = 1 - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log(2\pi) \leq 0.56498\ldots.$$

Knowing the decomposition in K of small primes of $\mathbb{Q}$, we may now apply the linear programming approach to get the upper bound for $\mathrm{BS}(K_T)$. This is done using the explicit formula (1.1) for the Brauer-Siegel ratio along with the basic inequality (1.7) and the inequality

$$\sum_{m=1}^{\infty} m\phi_{p^m} \leq \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}},$$

taken as the restrictions. This was done using the PARI package. As the calculations are rather cumbersome we will give here only the final result : $\mathrm{BS}_{\text{upper}} \approx 0.59748\ldots$, and the bound is attained for $\phi_7 = \phi_9 = \phi_{13} = 0.03944\ldots, \phi_{19} = 0.01002\ldots$. $\square$

Chapitre 2

On logarithmic derivatives of zeta functions in families of global fields (with P. Lebacque)

2.1 Introduction

The goal of this chapter is to prove a formula for the limit of logarithmic derivatives of zeta functions in families of global fields (assuming GRH in the number field case) with an explicit error term. This result is close in spirit both to the explicit Brauer–Siegel and Mertens theorems from [56] as well as to the generalized Brauer–Siegel type theorems from chapter 5. We also improve the error term in the explicit Brauer–Siegel theorem from [56], allowing its dependence on the family of global fields under consideration.

Throughout the chapter the constants involved in O and $\ll$ are absolute and effective (and, in fact, not very large). Let K be a global field that is a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_r(t)$, in the latter case $K = \mathbb{F}_r(X)$ for a smooth absolutely irreducible projective curve over $\mathbb{F}_r$, where $\mathbb{F}_r$ is the finite field with r elements. We will often use the acronyms NF or FF for the statements proven in the number field and the function field cases respectively. We shall often omit the index K in our notation in cases when it creates no confusion.

For a number field K let n_K and D_K denote its degree and its discriminant respectively. Let g_K be the genus of a function field, that is the genus of the corresponding smooth projective curve and let $g_K = \log\sqrt{D_K}$ in the number field case. Let $\mathcal{P}(K)$ be the set of (finite) places of K and let $\Phi_q = \Phi_q(K)$ be the number of places of norm q in K, i. e. $\Phi_q = |\{\mathfrak{p} \in \mathcal{P}(K) | \mathrm{N}\mathfrak{p} = q\}|$. In the number field case we denote by $\Phi_{\mathbb{R}} = r_1$ and $\Phi_{\mathbb{C}} = r_2$ the number of real and complex places of K respectively.

Recall that the zeta function of a global field K is defined as

$$\zeta_K(s) = \prod_q (1 - q^{-s})^{-\Phi_q},$$

where the product runs over all prime powers q. Let us denote by $Z_K(s) = -\sum\limits_q \frac{\Phi_q \log q}{q^s - 1}$ its logarithmic derivative. One knows that $\zeta_K(s)$ can be analytically continued to the whole complex plane and satisfies a functional equation relating $\zeta_K(s)$ and $\zeta_K(1-s)$. Furthermore, in the function field case $\zeta_K(s)$ is a rational function of $t = r^{-s}$. Moreover,

$$\zeta_K(s) = \frac{\prod\limits_{j=1}^{g} (\pi_j t - 1)(\bar{\pi}_j t - 1)}{(1-t)(1-rt)}, \tag{2.1}$$

and $|\pi_j| = \sqrt{r}$ (the Riemann hypothesis). For the rest of the chapter we will assume that the Generalized Riemann Hypothesis is true for zeta functions of number fields, that is all the non-trivial zeroes of $\zeta_K(s)$ are on the line $\operatorname{Re} s = \frac{1}{2}$.

Here are our first main results :

Theorem 2.1.1 (FF)**.** *For any function field K, any integer $N \geq 10$ and any $\epsilon = \epsilon_0 + i\epsilon_1$ such that $\epsilon_0 = \operatorname{Re}\epsilon > 0$ we have :*

$$\sum_{f=1}^{N} \frac{f\Phi_{r^f}}{r^{(\frac{1}{2}+\epsilon)f} - 1} + \frac{1}{\log r} \cdot Z_K\left(\frac{1}{2} + \epsilon\right) + \frac{1}{r^{-\frac{1}{2}+\epsilon} - 1} = O\left(\frac{g_K}{r^{\epsilon_0 N}}\left(1 + \frac{1}{\epsilon_0}\right)\right) + O\left(r^{\frac{N}{2}}\right).$$

Theorem 2.1.2 (NF, GRH)**.** *For a number field K, an integer $N \geq 10$ and any $\epsilon = \epsilon_0 + i\epsilon_1$ such that $\epsilon_0 = \operatorname{Re}\epsilon > 0$ we have :*

$$\sum_{q \leq N} \frac{\Phi_q \log q}{q^{\frac{1}{2}+\epsilon} - 1} + Z_K\left(\frac{1}{2} + \epsilon\right) + \frac{1}{\epsilon - \frac{1}{2}} =$$

$$= O\left(\frac{|\epsilon|^4 + |\epsilon|}{\epsilon_0^2}(g + n\log N)\frac{\log^2 N}{N^{\epsilon_0}}\right) + O\left(\sqrt{N}\right).$$

Let us explain a little bit the meaning of these theorems. It was known before (see chapter 5 and also below) that the identities (without the error terms) of the theorems are true in the asymptotic sense (when $N = \infty$ and $g = \infty$ for families of global fields). Our theorems give the "finite level" versions of these results. They allow to estimate how well the cutoffs of the series for $Z_K(s)$ approximate it away from the domain of convergence of this series (which is $\operatorname{Re} s > 1$) when we vary K.

We give the proof of these theorems in sections 2.2 and 2.3 respectively. Both proofs are based on the Weil explicit formula. However, in the number field case the analytical difficulties are rather considerable, so the explicit formula has to be applied three times with different choices of test functions. We note that, as indicated in the remarks in the corresponding sections, in both cases we obtain the new proofs of the basic inequalities from [85] and [87].

Our next results concern families of global fields $\{K_i\}$ with growing genus $g_i = g(K_i)$. Recall ([86],[87]) that a family of global fields is called asymptotically exact if the limits

$$\phi_\alpha = \phi_\alpha(\{K_i\}) = \lim_{i\to\infty} \frac{\Phi_\alpha(K_i)}{g_i}$$

exist for each α which is a power of r in the function field case and each prime power and $\alpha = \mathbb{R}$ and $\alpha = \mathbb{C}$ in the number field case. The numbers ϕ_α are called the Tsfasman–Vlăduţ invariants of the family $\{K_i\}$. From now on we assume that all our families are asymptotically exact.

We introduce the limit zeta function of a family $\{K_i\}$ as

$$\zeta_{\{K_i\}}(s) = \prod_q (1 - q^{-s})^{-\phi_q}.$$

We will also denote by $Z_{\{K_i\}}(s) = -\sum_q \frac{\phi_q \log q}{q^s - 1}$ its logarithmic derivative. It follows from the basic inequality (cf. [85] and [87] or sections 2.2 and 2.3 of this chapter) that both the product and the sum converge absolutely for $\operatorname{Re} s \geq \frac{1}{2}$ and thus define analytic functions for $\operatorname{Re} s > \frac{1}{2}$.

Let us first formulate a corollary of theorems 2.1.1 and 2.1.2.

Corollary 2.1.3. *For an asymptotically exact family of global fields $\{K_i\}$, an integer $N \geq 10$ and any $\epsilon = \epsilon_0 + i\epsilon_1$ such that $\epsilon_0 = \operatorname{Re}\epsilon > 0$ the following holds :*

1. *in the function field case :*

$$\sum_{f=1}^{N} \frac{f\phi_{r^f}}{r^{(\frac{1}{2}+\epsilon)f}-1} + \frac{1}{\log r}\cdot Z_{\{K_i\}}\left(\frac{1}{2}+\epsilon\right) = O\left(\frac{1}{r^{\epsilon_0 N}}\left(1+\frac{1}{\epsilon_0}\right)\right);$$

2. *in the number field case with the assumption of GRH :*

$$\sum_{q\leq N} \frac{\phi_q \log q}{q^{\frac{1}{2}+\epsilon}-1} + Z_{\{K_i\}}\left(\frac{1}{2}+\epsilon\right) = O\left(\frac{(|\epsilon|^4+|\epsilon|)\log^3 N}{\epsilon_0^2 N^{\epsilon_0}}\right).$$

This corollary, in particular, implies the convergence of the logarithmic derivatives of zeta functions of global fields to the logarithmic derivative of the limit zeta function for $\operatorname{Re} s > \frac{1}{2}$. This result (without an explicit error term but with a much easier proof) is also obtained in chapter 5.

Our next result concerns the behaviour of $Z_{\{K_i\}}(s)$ at $s=\frac{1}{2}$.

Theorem 2.1.4. *For an asymptotically exact family of global fields $\{K_i\}$ there exists a number $\delta > 0$ depending on $\{K_i\}$ such that :*

1. *in the function field case :*

$$\sum_{f=1}^{N} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1} + \frac{1}{\log r}\cdot Z_{\{K_i\}}\left(\frac{1}{2}\right) = O(r^{-\delta N});$$

2. *in the number field case, assuming GRH, we have :*

$$\sum_{q\leq N} \frac{\phi_q \log q}{\sqrt{q}-1} + Z_{\{K_i\}}\left(\frac{1}{2}\right) = O(N^{-\delta}).$$

Let us formulate a corollary of this result which, in a sense, improves the explicit Brauer–Siegel theorem from [56]. We denote by $\varkappa_{K_i} = \operatorname{Res}_{s=1}\zeta_{K_i}(s)$ the residue of $\zeta_{K_i}(s)$ at $s=1$. We let $\kappa = \kappa_{\{K_i\}} = \lim\limits_{i\to\infty} \frac{\log \varkappa_{K_i}}{g_i}$. One knows ([86] and [87]) that for an asymptotically exact family this limit exists and equals $\log \zeta_{\{K_i\}}(1)$ (we assume GRH in the number field case). In fact, in the number field case it can be seen as a generalization of the classical Brauer–Siegel theorem (cf. [54]).

Corollary 2.1.5. *For an asymptotically exact family of global fields $\{K_i\}$ there exists a number $\delta > 0$ depending on $\{K_i\}$ such that :*

1. *in the function field case :*

$$\sum_{f=1}^{N} \phi_{r^f} \log\frac{r^f}{r^f-1} = \kappa + O\left(\frac{1}{r^{(\frac{1}{2}+\delta)N}}\right);$$

2. *assuming GRH, in the number field case :*

$$\sum_{q\leq N} \phi_q \log\frac{q}{q-1} = \kappa + O\left(\frac{1}{N^{\frac{1}{2}+\delta}}\right).$$

We prove theorem 2.1.4 and both of the corollaries 2.1.3 and 2.1.5 in the section 2.4.

2.2 Proof of theorem 2.1.1

We will use the following analogue of Weil explicit formula for zeta functions of function fields, see [74] or [51] (in the case of varieties over finite fields) for a proof.

Theorem 2.2.1. *For a sequence $v = (v_n)$ such that $\sum\limits_{n=1}^{\infty} v_n r^{\frac{n}{2}}$ is convergent, the series $\sum\limits_{n=1}^{\infty} v_n r^{-\frac{n}{2}} \sum\limits_{m|n} m\Phi_{r^m}$ is also convergent and one has the following equality :*

$$\sum_{n=1}^{\infty} v_n r^{-\frac{n}{2}} \sum_{f|n} f\Phi_{r^f} = \psi_v(r^{1/2}) + \psi_v(r^{-1/2}) - \sum_{j=1}^{g} \left(\psi_v \left(\frac{\pi_j}{\sqrt{r}} \right) + \psi_v \left(\frac{\bar{\pi}_j}{\sqrt{r}} \right) \right),$$

where the $\pi_j, \bar{\pi}_j$ are the inverse roots of the numerator of the zeta function of K, $g = g_K$ and $\psi_v(t) = \sum\limits_{n=1}^{\infty} v_n t^n$.

Let us take the test sequence $v_n = v_n(N) = \frac{1}{r^{n\epsilon}}$ if $n \leq N$ and 0 otherwise. Introducing it in the explicit formulae, we get $S_0(N,\epsilon) = S_1(N,\epsilon) + S_2(N,\epsilon) - S_3(N,\epsilon)$, where

$$\begin{aligned}
S_0(N,\epsilon) &= \sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\epsilon\right)} \sum_{f|n} f\Phi_{r^f}, \\
S_1(N,\epsilon) &= \sum_{n=1}^{N} r^{n\left(\frac{1}{2}-\epsilon\right)}, \\
S_2(N,\epsilon) &= \sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\epsilon\right)}, \\
S_3(N,\epsilon) &= \sum_{j=1}^{g} \sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\epsilon\right)} (\pi_j^n + \bar{\pi}_j^n).
\end{aligned}$$

Let us estimate each of the S_i.

Calculation of S_0 :

Let us first change the summation order in S_0 :

$$S_0(N,\epsilon) = \sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\epsilon\right)} \sum_{f|n} f\Phi_{r^f} = \sum_{f=1}^{N} f\Phi_{r^f} \sum_{m=1}^{[N/f]} \frac{1}{r^{fm\left(\frac{1}{2}+\epsilon\right)}}.$$

Now

$$\begin{aligned}
R_0(N,\epsilon) &= \sum_{f=1}^{N} f\Phi_{r^f} \frac{1}{r^{\left(\frac{1}{2}+\epsilon\right)f} - 1} - S_0(N,\epsilon) \\
&= \sum_{f=1}^{N} f\Phi_{r^f} \left(\frac{1}{r^{\left(\frac{1}{2}+\epsilon\right)f} - 1} - \sum_{m=1}^{[N/f]} r^{-fm\left(\frac{1}{2}+\epsilon\right)} \right) \\
&= \sum_{f=1}^{N} f\Phi_{r^f} \sum_{m=[N/f]+1}^{\infty} r^{-fm\left(\frac{1}{2}+\epsilon\right)}.
\end{aligned}$$

Taking the absolute values, we can assume that ϵ is real. Summing the geometric series, we obtain

$$0 \leq \sum_{f=1}^{N} f\Phi_{r^f} \frac{1}{r^{\left(\frac{1}{2}+\epsilon\right)f}-1} - S_0(N,\epsilon) \leq \sum_{f=1}^{N} f\Phi_{r^f} r^{-\left(\frac{1}{2}+\epsilon\right)[N/f]f} \frac{1}{r^{\left(\frac{1}{2}+\epsilon\right)f}-1}.$$

We now use the Weil inequality $f\Phi_{r^f} \leq r^f + 1 + 2g\sqrt{r^f}$, and split the above sum into two parts in the following way. For $f > [N/2]$ we have $[N/f] = 1$ and for $f \leq [N/2]$ we use the inequality $f[N/f] \geq N - f$.

$$\begin{aligned}
|R_0(N,\epsilon)| &\leq \sum_{f=1}^{N} \frac{\left(1 + r^f + 2g\sqrt{r^f}\right)}{r^{f\left(\frac{1}{2}+\epsilon\right)[N/f]}\left(r^{\left(\frac{1}{2}+\epsilon\right)f}-1\right)} \\
&\leq 8\sum_{f=1}^{[N/2]} \frac{r^{\left(\frac{1}{2}-\epsilon\right)f} + 2g\,r^{-f\epsilon}}{r^{(N-f)\left(\frac{1}{2}+\epsilon\right)}} + 6\sum_{f>[N/2]}^{N} \frac{r^{\left(\frac{1}{2}-\epsilon\right)f} + 2g\,r^{-f\epsilon}}{r^{f\left(\frac{1}{2}+\epsilon\right)}} \\
&\leq \frac{6}{r^{N\left(\frac{1}{2}+\epsilon\right)}} \sum_{f=1}^{[N/2]} (r^f + 2\,g\,r^{\frac{f}{2}}) + 6\sum_{f>[N/2]} (r^{-2\epsilon f} + 2\,g\,r^{-\left(\frac{1}{2}+2\epsilon\right)f}) \\
&\leq \frac{6}{r^{N\left(\frac{1}{2}+\epsilon\right)}} \left(\frac{r^{\frac{N}{2}+1}-r}{r-1} + 2g\frac{r^{\frac{N}{4}+\frac{1}{2}} - r^{\frac{1}{2}}}{r^{\frac{1}{2}}-1}\right) + \frac{6r^{-\epsilon N}}{1-r^{-2\epsilon}} + \frac{12gr^{-\frac{N}{4}-\epsilon N}}{1-r^{-\frac{1}{2}-2\epsilon}} \\
&\leq \frac{48}{r^{\epsilon N}}\left(2gr^{-\frac{N}{4}} + \frac{1}{r^{\epsilon}-1} + 1\right) \leq \frac{96}{r^{\epsilon N}}\left(g\,r^{-\frac{N}{4}} + \frac{1+\epsilon}{\epsilon}\right).
\end{aligned}$$

Calculation of S_1 :

$$0 \leq |S_1(N,\epsilon)| \leq r^{\frac{1}{2}-\epsilon_0} \cdot \frac{r^{\left(\frac{1}{2}-\epsilon_0\right)N}-1}{r^{\frac{1}{2}-\epsilon_0}-1} \leq 4r^{N/2},$$

since the function $t \mapsto t \cdot \frac{t^N-1}{t-1}$ is a continuous and increasing.

Calculation of S_2 :

$$0 \leq |S_2(N,\epsilon)| \leq \frac{1 - r^{-\left(\frac{1}{2}+\epsilon_0\right)N}}{r^{\frac{1}{2}+\epsilon_0}-1} \leq 4.$$

Calculation of S_3 :

$$R_3(N,\epsilon) = S_3(N,\epsilon) - \sum_{j=1}^{g}\left(\frac{\pi_j}{r^{\frac{1}{2}+\epsilon}-\pi_j} + \frac{\bar{\pi}_j}{r^{\frac{1}{2}+\epsilon}-\bar{\pi}_j}\right) = -\sum_{j=1}^{g}\sum_{n=N+1}^{\infty}\left(\frac{\pi_j}{r^{\frac{1}{2}+\epsilon}}\right)^n + \left(\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\epsilon}}\right)^n.$$

The absolute value of the right hand side can be bounded using the fact that $|\pi_j| \leq r^{\frac{1}{2}}$:

$$|R_3(N,\epsilon)| = \left|\sum_{j=1}^{g}\sum_{n=N+1}^{\infty}\left(\frac{\pi_j}{r^{\frac{1}{2}+\epsilon}}\right)^n + \left(\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\epsilon}}\right)^n\right| \leq 2g\,\frac{r^{-N\epsilon_0}}{r^{\epsilon_0}-1} \leq 4g\,\frac{r^{-N\epsilon_0}}{\epsilon_0}.$$

From the expression (2.1) of $\zeta_K(s)$ as rational function in $t = r^{-s}$ we can easily deduce the following formula for its logarithmic derivative :

$$\frac{1}{\log r} \cdot Z_K\left(\frac{1}{2}+\epsilon\right) = -\frac{1}{r^{\frac{1}{2}+\epsilon}-1} - \frac{1}{r^{-\frac{1}{2}+\epsilon}-1} + \sum_{j=1}^{g}\left(\frac{\pi_j}{r^{\frac{1}{2}+\epsilon}-\pi_j} + \frac{\bar{\pi}_j}{r^{\frac{1}{2}+\epsilon}-\bar{\pi}_j}\right).$$

Putting it all together we get the statement of the theorem. □

Remark 2.2.1. Using our theorem we can easily reprove the basic inequality from [86]. We take a real $\epsilon < \frac{1}{4}$, and remark that

$$\frac{1}{\log r} \cdot Z_K\left(\frac{1}{2}+\epsilon\right) + \frac{1}{r^{\frac{1}{2}+\epsilon}-1} + \frac{1}{r^{-\frac{1}{2}+\epsilon}-1} + g = \sum_{j=1}^{g}\left(\frac{\pi_j}{r^{\frac{1}{2}+\epsilon}-\pi_j} + \frac{\bar{\pi}_j}{r^{\frac{1}{2}+\epsilon}-\bar{\pi}_j} + 1\right) \geq 0,$$

as

$$\frac{\pi_j}{r^{\frac{1}{2}+\epsilon}-\pi_j} + \frac{\bar{\pi}_j}{r^{\frac{1}{2}+\epsilon}-\bar{\pi}_j} + 1 = \frac{r^{1+2\epsilon}-|\pi_j|^2}{(r^{\frac{1}{2}+\epsilon}-\pi_j)(r^{\frac{1}{2}+\epsilon}-\bar{\pi}_j)} \geq 0.$$

Now, from the theorem we get that

$$\sum_{f=1}^{N} \frac{f\Phi_{rf}}{r^{(\frac{1}{2}+\epsilon)f}-1} \leq g + O\left(\frac{g}{\epsilon r^{\epsilon N}}\right) + O(r^{\frac{N}{2}}).$$

We divide by g and first let $g \to \infty$ (varying K), after that we let $N \to \infty$ and finally we take the limit when $\epsilon \to 0$. In doing so we obtain the basic inequality from [85] :

$$\sum_{f=1}^{\infty} \frac{f\phi_{rf}}{r^{\frac{f}{2}}-1} \leq 1.$$

2.3 Proof of theorem 2.1.2

Our starting point will be the Weil explicit formula, the proof of which can be found in [70] or in [54, chap. XVII] (with slightly more general conditions on the test functions).

Consider the class (W) of even real valued functions, satisfying the following conditions :

1. there exists $\epsilon > 0$ such that $\int_0^\infty F(x)e^{(\frac{1}{2}+\epsilon)x}\,dx$ is convergent in the sense of Cauchy ;
2. there exists $\epsilon > 0$ such that $F(x)e^{(\frac{1}{2}+\epsilon)x}$ has bounded variation ;
3. $\frac{F(0)-F(x)}{x}$ has bounded variation ;
4. for any x we have $F(x) = \frac{F(x-0)+F(x+0)}{2}$.

For such a function F we define

$$\phi(s) = \int_{-\infty}^{+\infty} F(x)e^{(s-\frac{1}{2})x}\,dx. \tag{2.2}$$

The Weil explicit formula for Dedekind zeta functions of number fields can be stated as follows :

Theorem 2.3.1 (Weil). *Let K be a number field. Let F belong to the class (W) and let $\phi(s)$ be defined by (2.2). Then the sum $\sum\limits_{|\operatorname{Im}\rho|<T} \phi(\rho)$, where ρ runs through the non-trivial zeroes of the Dedekind zeta-function of K, is convergent when $T \to \infty$ and the limit $\sum\limits_{\rho}\phi(\rho)$ is given by :*

$$\sum_{\rho}\phi(\rho) = F(0)\left(2g - n(\gamma+\log 8\pi) - r_1\frac{\pi}{2}\right) + 4\int_0^{\infty} F(x)\,\mathrm{ch}\left(\frac{x}{2}\right)$$
$$+ r_1\int_0^{\infty}\frac{F(0)-F(x)}{2\,\mathrm{ch}(\frac{x}{2})}\,dx + n\int_0^{\infty}\frac{F(0)-F(x)}{2\,\mathrm{sh}(\frac{x}{2})}\,dx - 2\sum_{\mathfrak{p},m}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\frac{m}{2}}}F(m\log \mathrm{N}\mathfrak{p}), \quad (2.3)$$

where the last sum is taken over all prime ideals $\mathfrak{p}$ in K and all integers $m \geq 1$.

First of all, we remark that, if we have a complex valued function $F(x)$ with both real and imaginary parts $F_0(x)$ and $F_1(x)$ being even and lying in (W), we can apply (2.3) separately to $F_0(x)$ and $F_1(x)$. Thus the explicit formula, being linear in the test function, is also applicable to the initial complex valued function $F(x)$.

We apply the explicit formula to the function defined by

$$F_{N,\epsilon}(x) = \begin{cases} e^{-\epsilon|x|} & \text{if } |x| < \log(N+\frac{1}{2}), \\ 0 & \text{if } |x| > \log(N+\frac{1}{2}) \end{cases}$$

(here $N+\frac{1}{2}$ is take to avoid counting some of the terms with the factor $\frac{1}{2}$).

Next, we estimate each of the terms in (2.3).

2.3.1 The sum over the primes

$$\sum_{\mathfrak{p},m}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\frac{m}{2}}}F_{N,\epsilon}(m\log \mathrm{N}\mathfrak{p}) = \sum_{\mathrm{N}\mathfrak{p}^m\leq N}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{(\frac{1}{2}+\epsilon)m}}$$
$$= \sum_{\mathrm{N}\mathfrak{p}\leq N}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\frac{1}{2}+\epsilon}-1} - \sum_{\mathrm{N}\mathfrak{p}\leq N}\log \mathrm{N}\mathfrak{p}\sum_{m>\frac{\log N}{\log \mathrm{N}\mathfrak{p}}}\frac{1}{\mathrm{N}\mathfrak{p}^{(\frac{1}{2}+\epsilon)m}}.$$

We have to estimate the sum :

$$\Delta(N,\epsilon) = \sum_{\mathrm{N}\mathfrak{p}\leq N}\log \mathrm{N}\mathfrak{p}\sum_{m>\frac{\log N}{\log \mathrm{N}\mathfrak{p}}}\frac{1}{\mathrm{N}\mathfrak{p}^{(\frac{1}{2}+\epsilon)m}}.$$

Taking the absolute values, we can assume that ϵ is real. Calculating the remainder term of the geometric series, we get :

$$\Delta(N,\epsilon) \leq (2+\sqrt{2})\sum_{\mathrm{N}\mathfrak{p}\leq N}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{(\frac{1}{2}+\epsilon)([\frac{\log N}{\log \mathrm{N}\mathfrak{p}}]+1)}}$$

(for $(1-\mathrm{N}\mathfrak{p}^{-1/2-\epsilon})^{-1} \leq (1-2^{-1/2})^{-1} \leq \sqrt{2}(1+\sqrt{2})$).

Let us split the sum into two parts according as whether $\mathrm{N}\mathfrak{p} > \sqrt{N}$ or not. Taking into account that $\log \mathrm{N}\mathfrak{p}[\log N/\log \mathrm{N}\mathfrak{p}] \geq \log N - \log \mathrm{N}\mathfrak{p}$ for $\log \mathrm{N}\mathfrak{p} \leq [\log\sqrt{\mathrm{N}\mathfrak{p}}]$, we obtain :

$$\Delta(N,\epsilon) \leq (2+\sqrt{2})\left(\sum_{\mathrm{N}\mathfrak{p}\leq\sqrt{N}}\frac{\log \mathrm{N}\mathfrak{p}}{e^{\log \mathrm{N}(\frac{1}{2}+\epsilon)}} + \sum_{\sqrt{N}<\mathrm{N}\mathfrak{p}\leq N}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{(1+2\epsilon)}}\right).$$

Write

$$\Delta_1(N,\epsilon) = \sum_{\mathrm{N}\mathfrak{p}\leq\sqrt{N}}\frac{\log \mathrm{N}\mathfrak{p}}{e^{\log \mathrm{N}(\frac{1}{2}+\epsilon)}},$$

$$\Delta_2(N,\epsilon) = \sum_{\sqrt{N}<\mathrm{N}\mathfrak{p}\le N} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{(1+2\epsilon)}}.$$

For $\Delta_1(N,\epsilon)$ we have :

$$\Delta_1(N,\epsilon) \le \frac{1}{N^{\frac{1}{2}+\epsilon}} \sum_{\mathrm{N}\mathfrak{p}\le\sqrt{N}} \log \mathrm{N}\mathfrak{p}.$$

The last sum can be estimated with the help of Lagarias and Odlyzko results (which use GRH, cf. [52, Theorem 9.1]) :

$$\sum_{\mathrm{N}\mathfrak{p}\le\sqrt{N}} \log \mathrm{N}\mathfrak{p} \le \sum_{\mathrm{N}\mathfrak{p}^k\le\sqrt{N}} \log \mathrm{N}\mathfrak{p} = \sqrt{N} + O(N^{\frac{1}{4}} \log N(g + n\log N))$$

with an effectively computable absolute constant in O. Thus we get :

$$\Delta_1(N,\epsilon) \le \frac{2+\sqrt{2}}{N^\epsilon} + a_0 \frac{g\log N + n\log^2 N}{N^{\frac{1}{4}+\epsilon}}.$$

We can estimate the sum $\Delta_2(N,\epsilon)$ as follows :

$$\Delta_2(N,\epsilon) \le \int_{\sqrt{N}}^{\infty} \frac{\log t}{t^{1+2\epsilon}}\, d\pi(t),$$

where $\pi(t)$ is the prime counting function $\pi(t) = \sum_{\mathrm{N}\mathfrak{p}\le t} 1$. As before, according to Lagarias and Odlyzko, $\pi(t) = \int_2^t \frac{dx}{\log x} + \delta(t)$, with $|\delta(t)| \le a_1\sqrt{t}(g + n\log t)$. Thus, substituting, we get :

$$\Delta_2(N,\epsilon) \le \int_{\sqrt{N}}^{\infty} t^{-1-2\epsilon}\, dt + 2|\delta(\sqrt{N})| \frac{\log N}{N^{\frac{1}{2}+\epsilon}} + \left| \int_{\sqrt{N}}^{\infty} \delta(t) \frac{1-(1+2\epsilon)\log t}{t^{2+2\epsilon}}\, dt \right|.$$

We deduce that

$$\Delta_2(N,\epsilon) \le \frac{1}{2\epsilon N^\epsilon} + 2a_1(g + n\log N)\frac{\log N}{N^{\frac{1}{4}+\epsilon}} + \int_{\sqrt{N}}^{\infty} a_1(g + n\log t)\frac{|1-(1+2\epsilon)\log t|}{t^{\frac{3}{2}+2\epsilon}}\, dt.$$

For $N \ge 8$ we have :

$$\int_{\sqrt{N}}^{\infty} a_1(g + n\log t)\frac{|1-(1+2\epsilon)\log t|}{t^{\frac{3}{2}+2\epsilon}}\, dt \le \int_{\sqrt{N}}^{\infty} a_1(g + n\log t)\frac{(1+2\epsilon)\log t}{t^{\frac{3}{2}+2\epsilon}}\, dt.$$

Integrating by parts, we can find that

$$\int_{\sqrt{N}}^{\infty} \frac{\log t}{t^{\frac{3}{2}+2\epsilon}}\, dt = \frac{\log N}{2(\frac{1}{2}+2\epsilon)N^{\frac{1}{4}+\epsilon}} + \frac{1}{(\frac{1}{2}+2\epsilon)^2 N^{\frac{1}{4}+\epsilon}},$$

and

$$\int_{\sqrt{N}}^{\infty} \frac{\log^2 t}{t^{\frac{3}{2}+2\epsilon}}\, dt = \frac{\log^2 N}{4(\frac{1}{2}+2\epsilon)N^{\frac{1}{4}+\epsilon}} + \frac{\log N}{2(\frac{1}{2}+2\epsilon)^2 N^{\frac{1}{4}+\epsilon}} + \frac{1}{(\frac{1}{2}+2\epsilon)^3 N^{\frac{1}{4}+\epsilon}}.$$

We conclude that the following estimate holds :

$$\Delta_2(N,\epsilon) \le \frac{1}{2\epsilon N^\epsilon} + a_2\left(\frac{n\log^2 N}{N^{\frac{1}{4}+\epsilon}} + \frac{g\log N}{N^{\frac{1}{4}+\epsilon}}\right).$$

Putting everything together, we see that :

$$|\Delta(N,\epsilon)| \ll \frac{1}{\epsilon_0 N^{\epsilon_0}} + \frac{\log N}{N^{\frac{1}{4}+\epsilon_0}}(n\log N + g). \tag{2.4}$$

2.3.2 Archimedean terms

First of all,

$$\left|\int_0^\infty F_{N,\epsilon}(x)\operatorname{ch}\left(\frac{x}{2}\right)\,dx\right| \leq \int_0^{\log(N+\frac{1}{2})} e^{(\frac{1}{2}-\epsilon_0)x}\,dx = \frac{(N+\frac{1}{2})^{\frac{1}{2}-\epsilon_0}-1}{\frac{1}{2}-\epsilon_0} \ll \sqrt{N}. \tag{2.5}$$

Let

$$I_{N,\epsilon} = \int_0^\infty \frac{1-F_{N,\epsilon}(x)}{2\operatorname{sh}(\frac{x}{2})}\,dx$$

and

$$I_{\infty,\epsilon} = \int_0^\infty \frac{1-e^{-\epsilon x}}{2\operatorname{sh}(\frac{x}{2})}\,dx.$$

We have for $N \geq 4$:

$$|I_{\infty,\epsilon} - I_{N,\epsilon}| \leq \int_{\log N}^\infty \frac{2}{e^{\frac{x}{2}}}\,dx \leq \frac{4}{\sqrt{N}}.$$

Now,

$$\begin{aligned}
I_{\infty,\epsilon} &= \int_0^\infty \left(\frac{e^{-\frac{x}{2}}}{1-e^{-x}} - \frac{e^{-(\frac{1}{2}+\epsilon)x}}{1-e^{-x}}\right)dx \\
&= \int_0^\infty \left(\left(\frac{e^{-\frac{x}{2}}}{1-e^{-x}} - \frac{e^{-x}}{x}\right) + \left(\frac{e^{-x}}{x} - \frac{e^{-(\frac{1}{2}+\epsilon)x}}{1-e^{-x}}\right)\right)dx \\
&= \psi\left(\frac{1}{2}+\epsilon\right) - \psi\left(\frac{1}{2}\right),
\end{aligned}$$

as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \int_0^\infty \left(\frac{e^{-t}}{x} - \frac{e^{-xt}}{1-e^{-t}}\right)dt.$$

The second integral

$$J_{N,\epsilon} = \int_0^\infty \frac{1-F_{N,\epsilon}(x)}{2\operatorname{ch}(\frac{x}{2})}\,dx$$

can be estimated along the same lines using an integral from [25, 3.541] :

$$\int_0^\infty \frac{e^{-\epsilon x}}{\operatorname{ch}(\frac{x}{2})}\,dx = \psi\left(\frac{1}{4}+\frac{\epsilon}{2}\right) - \psi\left(\frac{3}{4}+\frac{\epsilon}{2}\right).$$

Taking into account that $\psi(2x) = \frac{1}{2}\left(\psi(x) + \psi\left(x+\frac{1}{2}\right)\right) + \log 2$, we finally obtain :

$$\begin{aligned}
J_{N,\epsilon} &= \frac{\pi}{2} + \log 2 + \psi\left(\frac{1}{4}+\frac{\epsilon}{2}\right) - \psi\left(\frac{1}{2}+\epsilon\right) + O\left(\frac{1}{\sqrt{N}}\right), \\
I_{N,\epsilon} &= \gamma + \log 4 + \psi\left(\frac{1}{2}+\epsilon\right) + O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned} \tag{2.6}$$

2.3.3 The sum over the zeroes : the main term

Let us estimate now the sum $\sum_\rho \phi(\rho)$ over zeroes of $\zeta_K(s)$. Let $\rho = \frac{1}{2} + it$ be a zero of the zeta function of K on the critical line. Put $y = \log(N + \frac{1}{2})$. We have

$$\phi(\rho) = \int_{-y}^{y} e^{-\epsilon|x|+itx}\, dx = \int_0^y e^{(-\epsilon+it)x}\, dx + \int_0^y e^{(-\epsilon-it)x}\, dx,$$

so

$$\phi(\rho) = \frac{2}{\epsilon^2 + t^2}(\epsilon + e^{-\epsilon y}(-\epsilon \cos(ty) + t \sin(ty))).$$

We divide the sum over ρ into three parts :

$$S_1(\epsilon) = \sum_{\rho=\frac{1}{2}+it} \frac{\epsilon}{\epsilon^2 + t^2};$$

$$S_2(y,\epsilon) = \sum_{\rho=\frac{1}{2}+it} \frac{\cos(ty)}{\epsilon^2 + t^2};$$

$$S_3(y,\epsilon) = \sum_{\rho=\frac{1}{2}+it} \frac{t\sin(ty)}{\epsilon^2 + t^2};$$

so that

$$\sum_\rho \phi(\rho) = 2S_1(\epsilon) - 2\epsilon e^{-\epsilon y} S_2(y,\epsilon) + 2e^{-\epsilon y} S_3(y,\epsilon).$$

Let us relate the sum $S_1(\epsilon)$ to $Z_K(s)$, the logarithmic derivative of $\zeta_K(s)$. Stark's formula (cf. [80, (9)]) gives us the following :

$$\sum_\rho \frac{1}{s-\rho} = \frac{1}{s-1} + \frac{1}{s} + g - \frac{n}{2}\log \pi + \frac{r_1}{2}\psi\left(\frac{s}{2}\right) + r_2(\psi(s) - \log 2) + Z_K(s), \tag{2.7}$$

where as before $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$. Specializing at $s = \frac{1}{2} + \epsilon$, we obtain :

$$\sum_{\rho=\frac{1}{2}+it} \frac{\epsilon}{\epsilon^2 + t^2} = \frac{1}{\epsilon - \frac{1}{2}} + \frac{1}{\epsilon + \frac{1}{2}} + g - \frac{n}{2}\log\pi - r_2 \log 2 + \frac{r_1}{2}\psi\left(\frac{1}{4} + \frac{\epsilon}{2}\right) + r_2\psi\left(\frac{1}{2} + \epsilon\right) + Z_K\left(\frac{1}{2} + \epsilon\right). \tag{2.8}$$

We note that the archimedean factors from the Stark formula and from the initial Weil explicit formula cancel each other. We are left to prove that $S_2(y,\epsilon)$ and $S_3(y,\epsilon)$ are sufficiently small.

2.3.4 The sum over the zeroes : the remainder term.

To estimate

$$S_2(y,\epsilon) = \sum_{\rho=\frac{1}{2}+it} \frac{\cos(ty)}{\epsilon^2 + t^2}$$

we take the absolute values of all the terms in the sum so that

$$|S_2(y,\epsilon)| \le \sum_{\rho=\frac{1}{2}+it} \frac{1}{|\epsilon^2 + t^2|} \le \sum_{\rho=\frac{1}{2}+it} \frac{n(j)}{\epsilon_0^2 + (t - |\epsilon_1|)^2}, \tag{2.9}$$

where $n(j)$ is the number of zeroes with $|t-j|<1$. A standard estimate from [52, Lemma 5.4] yields $n(j) \ll g + n\log(j+2)$, thus

$$\begin{aligned}|S_2(y,\epsilon)| &\ll \frac{g+n\log(|\epsilon_1|+2)}{\epsilon_0^2} + g + n\sum_{j=1}^{|\epsilon_1|+1}\frac{\log j}{|\epsilon_1|+2-j} + g + n\log(|\epsilon_1|+2) \\ &\ll (g+n\log^2(|\epsilon_1|+2))\left(1+\frac{1}{\epsilon_0^2}\right).\end{aligned}$$

Let us finally estimate

$$S_3(y,\epsilon) = \sum_{\rho=\frac{1}{2}+it}\frac{t\sin(ty)}{\epsilon^2+t^2}.$$

We have

$$S_3(y,\epsilon) = \sum_{\rho=\frac{1}{2}+it}\frac{\sin ty}{t} - \sum_{\rho=\frac{1}{2}+it}\frac{\epsilon^2\sin(ty)}{t(\epsilon^2+t^2)} = A(y) - B(y,\epsilon).$$

The series for the formal derivative of $B(y,\epsilon)$ with respect to y is given by

$$\sum_{\rho=\frac{1}{2}+it}\frac{\epsilon^2\cos(ty)}{\epsilon^2+t^2}.$$

Using the estimates for $S_2(y,\epsilon)$ we deduce that on any compact subset of $[0,+\infty)$ this series is absolutely and uniformly convergent to $B'(y)$, and we have $|B'(y,\epsilon)| \ll |\epsilon|^2(g+n\log^2(|\epsilon_1|+2))\left(1+\frac{1}{\epsilon_0^2}\right)$. Thus we see that $|B(y)| \ll y|\epsilon|^2(g+n\log^2(|\epsilon_1|+2))\left(1+\frac{1}{\epsilon_0^2}\right)$, since $B(0,\epsilon)=0$.

2.3.5 The sum over the zeroes : the difficult part.

We are left to estimate the term $A(y)$.

Let us recall a particular case of Weil explicit formula which is due to Landau (cf. [53]) :

$$\sum_\rho \frac{x^\rho}{\rho} = x - \Psi(x) - r\log x - b - \frac{r_1}{2}\log(1-x^{-2}) - r_2\log(1-x^{-1}), \tag{2.10}$$

where $\Psi(x) = \sum_{\mathrm{N}\mathfrak{p}^k\le x}\log \mathrm{N}\mathfrak{p}$, b is the constant term of the expansion of $Z_K(s)$ at 0, $r = r_1+r_2-1$ and x is not a prime power. This formula is stated in [53] for $x \ge \frac{3}{2}$, however, applying theorem 2.3.1 to the function

$$F_x(y) = \begin{cases} e^{|y|/2} & \text{if } |y| < \log x, \\ 0 & \text{if } |y| > \log x,\end{cases}$$

one can see that it is valid for any $x>1$. We also note that by an effective version of the prime ideals theorem ([52, Theorem 9.1]) we have the following estimate :

$$\Psi(x) - x = O\left(x^{\frac{1}{2}}\log x(g+n\log x)\right). \tag{2.11}$$

Now, we introduce $C(x) = \sum_\rho \frac{x^\rho}{\rho}$, $D(x) = \sum_{\rho\ne\frac{1}{2}}\frac{x^\rho}{\rho-\frac{1}{2}}$ and $E(x) = D(x) - C(x)$. From (2.10) and (2.11) we see that $C(x)$ is an integrable function on compact subsets of $(1,+\infty)$. Using the arguments similar to those from the previous subsection we can deduce that the series for $E(x)$ is absolutely and uniformly convergent on compact subsets of $[1,+\infty)$ and thus $E(x)$ is a continuous function on this interval. From this we conclude that the series for $D(x)$ is also convergent to a locally integrable function.

If we put $x = e^y$, we get

$$\operatorname{Re} D(e^y) = e^{\frac{y}{2}} \sum_{\rho \neq \frac{1}{2}} \frac{\sin(ty)}{t},$$

which is equal to $e^{\frac{y}{2}} A(y)$ up to a term corresponding to a possible zero of $\zeta_K(s)$ at $\rho = \frac{1}{2}$.

Since the series for $C(x)$ is not uniformly convergent, we will have to work with distributions defined by $C(x), D(x)$ and $E(x)$. See [72] for the basic notions and results used here. From the fact that a convergent series of distributions can be differentiated term by term we deduce that the following equality holds :

$$\frac{d}{dx} \frac{E(x)}{\sqrt{x}} = \frac{C(x)}{2\sqrt{x^3}}.$$

We apply (2.10) to the right hand side of this formula and integrate from $1+\delta$ to x (here $\delta > 0$). The obtained equality will be valid in the sense of distributions, thus almost everywhere for the corresponding locally integrable functions defining these distributions. Since $E(x)$ is continuous, we see that the resulting identity

$$\begin{aligned} \frac{E(x)}{\sqrt{x}} = E(1+\delta) + \int_{1+\delta}^{x} \frac{t - \Psi(t)}{2t^{\frac{3}{2}}}\, dt - r \int_{1+\delta}^{x} \frac{\log t}{2t^{\frac{3}{2}}}\, dt \\ - \int_{1+\delta}^{x} \frac{b}{2t^{\frac{3}{2}}}\, dt - \frac{r_1}{2} \int_{1+\delta}^{x} \frac{\log(1-t^{-2})}{2t^{\frac{3}{2}}}\, dt - r_2 \int_{1+\delta}^{x} \frac{\log(1-t^{-1})}{2t^{\frac{3}{2}}}\, dt \end{aligned}$$

actually holds pointwise on $[1+\delta, +\infty)$. We use (2.11) to estimate $t - \Psi(t)$. It is easily seen that all the integrals converge when $\delta \to 0$. From [53, 10.RH] it follows that $b \ll g + n$.

$$E(1) = \sum_{\rho \neq \frac{1}{2}} \frac{1}{\rho - \frac{1}{2}} - \sum_{\rho} \frac{1}{\rho} = -\frac{1}{2} \sum_{\rho = \frac{1}{2} + it} \frac{1}{\frac{1}{4} + t^2},$$

the first sum being zero as the term in ρ and $1-\rho$ cancel each other. An estimate for the last sum can be made using (2.9). This gives $|E(1)| \ll g + n$. Putting it all together we see that $|E(x)| \ll \sqrt{x} \log^2 x (g + n \log x)$. The estimate $|C(x)| \ll \sqrt{x} \log^2 x (n+g)$ can be obtained directly using (2.11). Thus, we conclude that $|A(y)| \ll y^2 (g + ny)$.

Finally, combining all together we get :

$$\sum_{\rho} \phi(\rho) = 2S_1(\epsilon) + O\left(\frac{|\epsilon|^4 + |\epsilon|}{\epsilon_0^2} (g + n \log N) \frac{\log^2 N}{N^{\epsilon_0}} \right).$$

This estimate together with (2.4), (2.5), (2.6) and (2.8) completes the proof of the theorem. □

Remark 2.3.1. Using our theorem we can derive the basic inequality from [87]. Indeed, we apply the formula (2.8) to express $Z_K\left(\frac{1}{2} + \epsilon\right)$ via $\sum\limits_{\rho = \frac{1}{2} + it} \frac{\epsilon}{\epsilon^2 + t^2}$ plus some archimedean terms. For a real positive $\epsilon < \frac{1}{4}$ the latter sum is non-negative, thus we see that

$$\begin{aligned} \sum_{q \leq N} \frac{\Phi_q \log q}{q^{\frac{1}{2}+\epsilon} - 1} + \frac{n}{2} \log \pi + r_2 \log 2 - \frac{r_1}{2} \psi\left(\frac{1}{4} + \frac{\epsilon}{2}\right) - r_2 \psi\left(\frac{1}{2} + \epsilon\right) \\ \leq g + O\left((g + n \log N) \frac{\log^2 N}{\epsilon N^{\epsilon}} \right) + O\left(\sqrt{N}\right). \end{aligned}$$

Now, we divide by g and first let $g \to \infty$ (varying K), after that we let $N \to \infty$ and finally we take the limit when $\epsilon \to 0$. Taking into account that $\psi(\frac{1}{2}) = -\gamma - 2\log 2$ and $\psi(\frac{1}{4}) = -\frac{\pi}{2} - \gamma - 3\log 2$, we obtain the basic inequality from [85] :

$$\sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + \phi_{\mathbb{R}} \left(\log(2\sqrt{2\pi}) + \frac{\pi}{4} + \frac{\gamma}{2}\right) + \phi_{\mathbb{C}} \left(\log(8\pi) + \gamma\right) \leq 1.$$

Remark 2.3.2. The choice of the test function $F_{N,\epsilon}(x)$ in the explicit formula is not accidental. Indeed, the resulting formulas "approximate" the Stark formula (2.7) when $N \to \infty$.

2.4 Proof of theorem 2.1.4 and of the corollaries

We will carry out the proofs in the function field case, the calculations in the number field case being exactly the same.

Proof of the corollary 2.1.3 : Assume first that $\epsilon \neq \frac{1}{2} + \frac{2\pi i k}{\log r}$, $k \in \mathbb{Z}$. We note that

$$\left|\sum_{f=1}^{\infty} \frac{f\phi_{rf}}{r^{(\frac{1}{2}+\epsilon)f}-1} + \frac{1}{g_j \log r} Z_{K_j}\left(\frac{1}{2}+\epsilon\right)\right| \leq$$

$$\leq \left|\sum_{f=N+1}^{\infty} \frac{f\phi_{rf}}{r^{(\frac{1}{2}+\epsilon)f}-1}\right| + \sum_{f=1}^{N} \frac{f\left|\frac{\Phi_{rf}}{g_j} - \phi_{rf}\right|}{r^{(\frac{1}{2}+\epsilon)f}-1} + \frac{1}{g_j}\left|\sum_{f=1}^{N} \frac{f\Phi_{rf}}{r^{(\frac{1}{2}+\epsilon)f}-1} + \frac{1}{\log r} Z_{K_j}\left(\frac{1}{2}+\epsilon\right)\right|.$$

Given $\delta > 0$ we choose an integer N such that the first sum is less than δ (this is possible due to the basic inequality) and such that $\frac{1}{r^{\epsilon_0 N}}\left(1 + \frac{1}{\epsilon_0}\right) \leq \delta$. Now, taking g sufficiently large, and using theorem 2.1.1 as well as the convergence of $\frac{\Phi_{rf}}{g_j}$ to ϕ_{rf}, we conclude that the whole sum is $\ll \delta$. Thus, we deduce that

$$\lim_{j\to\infty} \frac{Z_{K_j}\left(\frac{1}{2}+\epsilon\right)}{g_j} = Z_{\{K_j\}}\left(\frac{1}{2}+\epsilon\right). \tag{2.12}$$

Now, the corollary immediately follows from theorem 2.1.1 and (2.12). Though we initially assumed that $\epsilon \neq \frac{1}{2} + \frac{2\pi i k}{\log r}$, the statement still holds for $\epsilon = \frac{1}{2} + \frac{2\pi i k}{\log r}$ as all the function are continuous (and even analytic) for $\operatorname{Re}\epsilon > 0$. □

Remark 2.4.1. The formula (2.12) no longer holds when $\epsilon = 0$ as can be seen from the fact that $Z_K\left(\frac{1}{2}\right) = g_K - 1$. In fact, the identity holds if and only if our family is asymptotically optimal. Whether it holds or not for the logarithm of $\zeta_K(s)$ and not for its derivative seems to be very difficult to say at the moment. Even for quadratic fields this question is far from being obvious. It is known that in the number field case there exists a sequence (d_i) in $\mathbb{N}$ of density at least $\frac{1}{2}$ such that

$$\lim_{i\to\infty} \frac{\log \zeta_{\mathbb{Q}(\sqrt{d_i})}(\frac{1}{2})}{\log d_i} = 0$$

(cf. [43]). The techniques of the evaluation of mollified moments of Dirichlet L- functions used in that paper is rather involved. In general one can prove an upper bound for the limit (cf. chapter 5). This is analogous to the "easy" inequality in the classical Brauer–Siegel theorem.

The interest of the question about the behaviour of $\log Z_K\left(\frac{1}{2}\right)$ can be in particular explained by its connection to the behaviour of the order of the Shafarevich–Tate group and the regulator of constant supersingular elliptic curves over function fields, the connection being provided by the Birch and Swinnerton–Dyer conjecture. In general, a similar question can be asked about the behaviour of these invariants in arbitrary families of elliptic curves. Some discussion on the problem is given in [49] (beware, however, that the proof of the main result there can not be seen as a correct one as the change of limits, which is a key point, is not justified).

Proof of theorem 2.1.4 : It follows from the basic inequality that the series defining $\log \zeta_{\{K_i\}}(s)$ converges absolutely for $\operatorname{Re} s \geq \frac{1}{2}$. The function $\log \zeta_{\{K_i\}}(s)$ has a Dirichlet series expansion with positive coefficients, converging for $\operatorname{Re} s \geq \frac{1}{2}$. Thus, from a standard theorem on Dirichlet series (cf. [42, Lemma 5.56]), it must converge in some open domain $\operatorname{Re} s > \frac{1}{2} - \delta_0$ for $\delta_0 > 0$, defining an analytic function there. It follows that in the same domain the series for $Z_{\{K_i\}}(s)$ converges. Taking any δ with $0 < \delta < \delta_0$ we obtain :

$$\begin{aligned}\left|\sum_{f=1}^{N} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1} - \frac{1}{\log r} Z_{\{K_i\}}\left(\frac{1}{2}\right)\right| &= \left|\sum_{f=N+1}^{\infty} \frac{f\phi_{r^f}}{r^{\left(\frac{1}{2}-\delta\right)f}-1} \cdot \frac{r^{\left(\frac{1}{2}-\delta\right)f}-1}{r^{\frac{f}{2}}-1}\right| \\ &\leq \left|\sum_{f=1}^{\infty} \frac{f\phi_{r^f}}{r^{\left(\frac{1}{2}-\delta\right)f}-1}\right| \cdot \frac{r^{\left(\frac{1}{2}-\delta\right)N}-1}{r^{\frac{N}{2}}-1} = O(r^{-\delta N}).\end{aligned}$$

This gives the necessary result. □

Proof of the corollary 2.1.5 *:* We use theorem 2.1.4 to obtain the necessary estimate much in the same spirit as in the proof of theorem 2.1.4 itself. Using the function field Brauer–Siegel theorem to find the value for κ, we get :

$$\begin{aligned}\left|\sum_{f=1}^{N} \phi_{r^f} \log \frac{r^f}{r^f-1} - \kappa\right| &= \left|\sum_{f=N+1}^{\infty} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1} \cdot (r^{\frac{f}{2}}-1) \cdot \log \frac{r^f}{r^f-1}\right| \\ &\leq \left|\sum_{f=N+1}^{\infty} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1}\right| \cdot (r^{\frac{N}{2}}-1) \cdot \log \frac{r^N}{r^N-1} \\ &= O(r^{-\delta N}) \cdot O\left(r^{-\frac{N}{2}}\right).\end{aligned}$$

Indeed, $N \mapsto (r^{\frac{N}{2}}-1)\log \frac{r^N}{r^N-1}$ is decreasing for $N \geq 2$. The required estimate follows. □

Remark 2.4.2. Actually, our method gives an easy and conceptual proof of the explicit version of the Brauer–Siegel theorem from [56] (which is roughly speaking the statement of corollary 2.1.5 with $\delta = 0$). It shows that the rate of convergence in the Brauer–Siegel theorem essentially depends on how far to the left the limit zeta function $\zeta_{\{K_i\}}(s)$ is analytic. In the number field case we even save $\log^2 N$ in the estimate of the error term compared to what is proven in [56].

Chapitre 3

On the generalizations of the Brauer–Siegel theorem

3.1 Introduction

Let K be an algebraic number field of degree $n_K = [K : \mathbb{Q}]$ and discriminant D_K. We define the genus of K as $g_K = \log \sqrt{D_K}$. By h_K we denote the class-number of K, R_K denotes its regulator. We call a sequence $\{K_i\}$ of number fields a family if K_i is non-isomorphic to K_j for $i \neq j$. A family is called a tower if also $K_i \subset K_{i+1}$ for any i. For a family of number fields we consider the limit

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_{K_i} R_{K_i})}{g_{K_i}}.$$

The classical Brauer–Siegel theorem, proved by Brauer (see [4]) can be stated as follows :

Theorem 3.1.1 (Brauer–Siegel)**.** *For a family $\mathcal{K} = \{K_i\}$ we have*

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_{K_i} R_{K_i})}{g_{K_i}} = 1$$

if the family satisfies two conditions :

(i) $\lim\limits_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0$;

(ii) either the generalized Riemann hypothesis (GRH) holds, or all the fields K_i are normal over $\mathbb{Q}$.

The initial motivation for the Brauer–Siegel theorem can be traced back to a conjecture of Gauss :

Conjecture 3.1.2 (Gauss)**.** *There are only 9 imaginary quadratic fields with class number equal to one, namely those having their discriminants equal to* -3, -4, -7, -8, -11, -19, -43, -67, -163.

The first result towards this conjecture was proven by Heilbronn in [29]. He proved that $h_K \to \infty$ as $D_K \to -\infty$. Moreover, together with Linfoot [30] he was able to verify that Gauss' list was complete with the exception of at most one discriminant. However, this "at most one" part was completely ineffective. The initial question of Gauss was settled independently by Heegner [28], Stark [79] and Baker [2] (initially the paper by Heegner was not acknowledged as giving the complete proof). We refer to [90] for a more thorough discussion of the history of the Gauss class number problem.

A natural question was to find out what happens with the class number in the case of arbitrary number fields. Here the situation is more complicated. In particular a new invariant comes into play : the regulator of number fields, which is very difficult to separate from the class number in asymtotic considerations (in particular, for this reason the other conjecture of Gauss on the infinitude of real quadratic fields having class number one is still unproven). A major step in this direction was made by Siegel [77] who was able to prove Theorem 3.1.1 in the case of quadratic fields. He was followed by Brauer [4] who actually proved what we call the classical Brauer–Siegel theorem.

Ever since a lot of different aspects of the problem have been studied. For example, the major difficulty in applying the Brauer–Siegel theorem to the class number problem is its ineffectiveness. Thus many attempts to obtain good explicit bounds on $h_K R_K$ were undertaken. In particular we should mention the important paper of Stark [80] giving an explicit version of the Brauer–Siegel theorem in the case when the field contains no quadratic subfields. See also some more recent papers by Louboutin [59], [60] where better explicit bounds are proven in certain cases. Even stronger effective results were needed to solve (at least in the normal case) the class-number-one problem for CM fields, see [33], [68], [3].

In another direction, assuming the generalized Riemann hypothesis (GRH) one can obtain more precise bounds on the class number then those given by the Brauer–Siegel theorem. For example in the case of quadratic fields we have $h_K << D_K^{1/2}(\log\log D_K/\log D_K)$. In particular they are known to be optimal in many cases (see [14], [15], [9]).

A full survey of the problems stemming from the study of the Brauer–Siegel type questions definitely lies beyond the scope of this chapter. Our goal is more modest. Here we survey the results that generalize the classical Brauer–Siegel theorem. In §3.2 the case of families of number fields violating one (or both) of the conditions (i) and (ii) of theorem 3.1.1 is discussed. In particular we introduce the notion of Tsfasman–Vlăduţ invariants of global fields that allow to express the Brauer–Siegel limit in general. In §3.3 we survey the known results and conjectures about the Brauer–Siegel type statements in the higher dimensional situation. Finally, in the last §3.4 we prove a Brauer–Siegel type result (theorem 3.3.1) for families of varieties over finite fields. This theorem expresses the asymptotic properties of the residue at $s = d$ of the zeta function of smooth projective varieties over finite fields via the asymptotics of the number of $\mathbb{F}_{q^m}$-points on them.

3.2 The case of global fields : Tsfasman–Vlăduţ approach

A natural question is whether one can weaken the conditions (i) and (ii) of theorem 3.1.1. The first condition seems to be the most restrictive one. Tsfasman and Vlăduţ were able to deal with it first in the function field case [85], [86] and then in the number field case [87] (which was as usual more difficult, especially from the analytical point of view). It turned out that one has to take in account non-archimedian place to be able to treat the general situation. Let us introduce the necessary notation in the number field case (for the function field case see §3.3).

For a prime power q we set

$$\Phi_q(K_i) := |\{v \in P(K_i) : \mathrm{Norm}(v) = q\}|,$$

where $P(K_i)$ is the set of non-archimedian places of K_i. Taking in account the archimedian places we also put $\Phi_{\mathbb{R}}(K_i) = r_1(K_i)$ and $\Phi_{\mathbb{C}}(K_i) = r_2(K_i)$, where r_1 and r_2 stand for the number of real and (pairs of) complex embeddings.

We consider the set $A = \{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \ldots\}$ of all prime powers plus two auxiliary symbols $\mathbb{R}$ and $\mathbb{C}$ as the set of indices.

Definition 3.2.1. *A family $\mathcal{K} = \{K_i\}$ is called asymptotically exact if and only if for any $\alpha \in A$ the following limit exists :*

$$\phi_\alpha = \phi_\alpha(\mathcal{K}) := \lim_{i\to\infty} \frac{\Phi_\alpha(K_i)}{g_{K_i}}.$$

We call an asymptotically exact family $\mathcal{K}$ asymptotically good (respectively, bad) if there exists $\alpha \in A$ with $\phi_\alpha > 0$ (respectively, $\phi_\alpha = 0$ for any $\alpha \in A$). The ϕ_α are called the Tsfasman–Vlăduţ invariants of the family $\{K_i\}$.

One knows that any family of number fields contains an asymptotically exact subfamily so the condition on a family to be asymptotically exact is not very restrictive. On the other hand, the condition of asymptotical goodness is indeed quite restrictive. It is easy to see that a family is asymptotically bad if and only if it satisfies the condition (i) of the classical Brauer–Siegel theorem. In fact, before the work of Golod and Shafarevich [24] even the existence of asymptotically good families of number fields was unclear. Up to now the only method to construct asymptotically good families in the number field case is essentially based on the ideas of Golod and Shafarevich and consists of the usage of classfield towers (quite often in a rather elaborate way). This method has the disadvantage of beeing very inexplicit and the resulting families are hard to controll (ex. splitting of the ideals, ramification, etc.). In the function field case we dispose of a much wider range of constructions such as the towers coming from supersingular points on modular curves or Drinfeld modular curves ([40], [89]), the explicit iterated towers proposed by Garcia and Stichtenoth [18], [19] and of course the classfield towers as in the number field case (see [74] for the treatement of the function field case).

This partly explains why so little is known about the above set of invariants ϕ_α. Very few general results about the structure of the set of possible values of (ϕ_α) are available. For instance, we do not know whether the set $\{\alpha \mid \phi_\alpha \neq 0\}$ can be infinite for some family $\mathcal{K}$. We refer to [57] for an exposition of most of the known results on the invariants ϕ_α.

Before formulating the generalization of the Brauer–Siegel theorem proven by Tsfasman and Vlăduţ in [87] we have to give one more definition. We call a number field almost normal if there exists a finite tower of number fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K$ such that all the extensions K_i/K_{i-1} are normal.

Theorem 3.2.1 (Tsfasman–Vlăduţ)**.** *Assume that for an asymptotically good tower $\mathcal{K}$ any of the following conditions is satisfied :*

– *GRH holds*
– *All the fields K_i are almost normal over $\mathbb{Q}$.*

Then the limit $\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log(h_{K_i}R_{K_i})}{g_{K_i}}$ exists and we have :

$$\mathrm{BS}(\mathcal{K}) = 1 + \sum_q \phi_q \log\frac{q}{q-1} - \phi_\mathbb{R}\log 2 - \phi_\mathbb{C}\log 2\pi,$$

the sum beeing taken over all prime powers q.

We see that in the above theorem both the conditions (i) and (ii) of the classical Brauer–Siegel theorem are weakend. A natural supplement to the above theorem is the following result obtained in [91] (see also chapter 1) :

Theorem 3.2.2 (Zykin)**.** *Let $\mathcal{K} = \{K_i\}$ be an asymptotically bad family of almost normal number fields (i. e. a family for which $n_{K_i}/g_{K_i} \to 0$ as $i \to \infty$). Then we have $\mathrm{BS}(\mathcal{K}) = 1$.*

One may ask if the values of the Brauer–Siegel ratio $\mathrm{BS}(\mathcal{K})$ can really be different from one. The answer is “yes”. However, due to our lack of understanding of the set of possible (ϕ_α) there are only partial results. Under GRH one can prove (see [87]) the following bounds on $\mathrm{BS}(\mathcal{K})$: $0.5165 \le \mathrm{BS}(\mathcal{K}) \le 1.0938$. The existence bounds are weaker. There is an example of a (class field) tower with $0.5649 \le \mathrm{BS}(\mathcal{K}) \le 0.5975$ and another one with $1.0602 \le \mathrm{BS}(\mathcal{K}) \le 1.0938$ (see [87] and [91]). Our inability to get the exact value of $\mathrm{BS}(\mathcal{K})$ lies in the inexplicitness of the construction : as it was said before, class field towers are hard to control. A natural question is whether all the values of $\mathrm{BS}(\mathcal{K})$ between the bounds in the examples are attained. This seems difficult to prove at the moment though one may hope that some density results (i. e. the density of the values of $\mathrm{BS}(\mathcal{K})$ in a certain interval) are within reach of the current techniques.

Let us formulate yet another version of the generalized Brauer–Siegel theorem proven by Lebacque in [56]. It assumes GRH but has the advantage of beeing explicit in a certain (unfortunately rather weak) sense :

Theorem 3.2.3 (Lebacque). *Let $\mathcal{K} = \{K_i\}$ be an asymptotically exact family of number fields. Assume that* GRH *in true. Then the limit* $\mathrm{BS}(\mathcal{K})$ *exists, and we have :*

$$\sum_{q \le x} \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi = \mathrm{BS}(\mathcal{K}) + \mathcal{O}\left(\frac{\log x}{\sqrt{x}}\right).$$

This theorem is an easy corollary of the generalised Mertens theorem proven in [56]. We should also note that Lebacque's apporoach leads to a unified proof of theorems 3.2.1 and 3.2.2 with or without the assumption of GRH.

3.3 Varieties over global fields

Once we are in the realm of higher dimensional varieties over global fields the question of finding a proper analogue of the Brauer–Siegel theorem becomes more complicated and the answers which are currently available are far from being complete. Here we have essentially three approaches : the one by the author (which leads to a fairly simple result), another one by Kunyavskii and Tsfasman and the last one by Hindry and Pacheko (which for the moment gives only plausible conjectures). We will present all of them one by one.

The proof of the cassical Brauer–Siegel theorem as well as those of its generalisations discussed in the previous section passes through the residue formula. Let $\zeta_K(s)$ be the Dedekind zeta function of a number field K and $\varkappa_K$ its residue at $s = 1$. By w_K we denote the number of roots of unity in K. Then we have the following classical residue formula :

$$\varkappa_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{D_K}}.$$

This formula immediately reduces the proof of the Brauer–Siegel theorem to an appropriate asymptotical estimate for $\varkappa_K$ as K varies in a family (by the way, this makes clear the connection with GRH which appears in the statement of the Brauer–Siegel theorem). So, in the higher dimensional situation we face two completely different problems :

(i) Study the asymptotic properties of a value of a certain ζ or L-function.

(ii) Find an (arithmetic or geometric) interpretation of this value.

One knows that just like in the case of global fields in the d-dimensional situation zeta function $\zeta_X(s)$ of a variety X has a pole of order one at $s = d$. Thus the first idea would be to take the residue of $\zeta_X(s)$ at $s = d$ and study its asymptotic behaviour. In this direction we can indeed obtain a result. Let us proceed more formally.

Let X be a complete non-singular absolutely irreducible projective variety of dimension d defined over a finite field $\mathbb{F}_q$ with q elements, where q is a power of p. Denote by $|X|$ the set of closed points of X. We put $X_n = X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and $\bar{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Let Φ_{q^m} be the number of places of X having degree m, that is $\Phi_{q^m} = |\{\mathfrak{p} \in |X| \mid \deg(\mathfrak{p}) = m\}|$. Thus the number N_n of $\mathbb{F}_{q^n}$-points of the variety X_n is equal to

$$N_n = \sum_{m|n} m\Phi_{q^m}.$$

Let $b_s(X) = \dim_{\mathbb{Q}_l} H^s(\bar{X}, \mathbb{Q}_l)$ be the l-adic Betti numbers of X. We set $b(X) = \max_{i=1\ldots 2d} b_i(X)$. Recall that the zeta function of X is defined for $\mathrm{Re}(s) > d$ by the following Euler product :

$$\zeta_X(s) = \prod_{\mathfrak{p}\in|X|} \frac{1}{1 - N(\mathfrak{p})^{-s}} = \prod_{m=1}^{\infty} \left(\frac{1}{1 - q^{-sm}}\right)^{\Phi_{q^m}},$$

where $N(\mathfrak{p}) = q^{-\deg \mathfrak{p}}$. It is known that $\zeta_X(s)$ has an analytic continutation to a meromorphic function on the complex plane with a pole of order one at $s = d$. Furthermore, if we set $Z(X, q^{-s}) = \zeta_X(s)$ then the function $Z(X,t)$ is a rational function of $t = q^{-s}$.

Consider a family $\{X_j\}$ of complete non-singular absolutely irreducible d–dimensional projective varieties over $\mathbb{F}_q$. We assume that the families under consideration satisfy $b(X_j) \to \infty$ when $j \to \infty$. Recall (see [51]) that such a family is called asymptotically exact if the following limits exist :

$$\phi_{q^m}(\{X_j\}) = \lim_{j\to\infty} \frac{\Phi_{q^m}(X_j)}{b(X_j)}, \qquad m = 1, 2, \ldots$$

The invariants ϕ_{q^m} of a family $\{X_j\}$ are called the Tsfasman–Vlăduţ invariants of this family. One knows that any family of varieties contains an asymptotically exact subfamily.

Definition 3.3.1. *We define the Brauer–Siegel ratio for an asymptotically exact family as*

$$\mathrm{BS}(\{X_j\}) = \lim_{j\to\infty} \frac{\log|\varkappa(X_j)|}{b(X_j)},$$

where $\varkappa(X_j)$ is the residue of $Z(X_j, t)$ at $t = q^{-d}$.

In §3.4 we prove the following generalization of the classical Brauer–Siegel theorem :

Theorem 3.3.1. *For an asymptotically exact family $\{X_j\}$ the limit $\mathrm{BS}(\{X_j\})$ exists and the following formula holds :*

$$\mathrm{BS}(\{X_j\}) = \sum_{m=1}^{\infty} \phi_{q^m} \log \frac{q^{md}}{q^{md} - 1}. \tag{3.1}$$

However, we come across a problem when we trying to carry out the second part of the strategy sketched above. There seems to be no easy geometric interpretaion of the invariant $\varkappa(X)$ (apart from the case $d = 1$ where we have a formula relating $\varkappa_X$ to the number of $\mathbb{F}_q$-points on the Jacobian of X). See however [66] for a certain cohomological interpretation of $\varkappa(X)$.

Let us now switch our attention to the two other approaches by Kunyavskii–Tsfasman and by Hindry–Pacheko. Both of them have for their starting points the famous Birch–Swinnerton-Dyer conjecture which expresses the value at $s = 1$ of the L-function of an abelian variety in terms of certain arithmetic invariants related to this variety. Thus, in this case we have (at least conjecturally) an interpretation of the special value of the L-function at $s = 1$. However, the situation with the asymptotic behaviour of this value is much less clear. Let us begin with the approach of Kunyavskii–Tsfasman. To simplify our notation we restrict ourselves to the case of elliptic curves and refer for the general case of abelian varieties to the original paper [49].

Let K be a global field that is either a number field or $K = \mathbb{F}_q(X)$ where X is a smooth, projective, geometrically irreducible curve over a finite field $\mathbb{F}_q$. Let E/K be an elliptic curve over K. Let $\text{Ш} := |\text{Ш}(E)|$ be the order of the Shafarevich–Tate group of E, and Δ the determinant of the Mordell–Weil lattice of E (see [82] for definitions). Note that in a certain sense Ш and Δ are the analogues of the class number and of the regulator respectively. The goal of Kunyavskii and Tsfasman in [49] is to study the asymptotic behaviour of the product $\text{Ш} \cdot \Delta$ as $g \to \infty$. They are able to treat the so-called constant case :

Theorem 3.3.2 (Kunyavskii–Tsfasman)**.** *Let $E = E_0 \times_{\mathbb{F}_q} K$ where E_0 a fixed elliptic curve over $\mathbb{F}_q$. Let K vary in an asymptotically exact family $\{K_i\} = \{\mathbb{F}_q(X_i)\}$, and let $\phi_{q^m} = \phi_{q^m}(\{X_i\})$ be the corresponding Tsfasman–Vlăduţ invariants. Then*

$$\lim_{i\to\infty} \frac{\log_q(\text{Ш}_i \cdot \Delta_i)}{g_i} = 1 - \sum_{m=1}^{\infty} \phi_{q^m} \log_q \frac{N_m(E_0)}{q^m},$$

where $N_m(E_0) = |E_0(\mathbb{F}_{q^m})|$.

Note that there is no need to assume the above mentioned Birch and Swinnerton–Dyer conjecture as it was proven by Milne [65] in the constant case. The proof of the above theorem uses this result of Milne to get an explicit formula for $\text{Ш} \cdot \Delta$ thus reducing the proof of the theorem to the study of asymptotic properties of curves over finite fields the latter ones being much better known.

Kunyavskii and Tsfasman also make a conjecture in a certain non constant case. To formulate it we have to introduce some more notation. Let E be again an arbitrary elliptic K-curve. Denote by $\mathcal{E}$ the corresponding elliptic surface (this means that there is a proper connected smooth morphism $f \colon \mathcal{E} \to X$ with the generic fibre E). Assume that f fits into an infinite Galois tower, i.e. into a commutative diagram of the following form :

$$\begin{array}{ccccccccc} \mathcal{E} = \mathcal{E}_0 & \longleftarrow & \mathcal{E}_1 & \longleftarrow & \dots & \longleftarrow & \mathcal{E}_j & \longleftarrow & \dots \\ \big\downarrow f & & \big\downarrow & & & & \big\downarrow & & \\ X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_j & \longleftarrow & \dots, \end{array} \tag{3.2}$$

where each lower horizontal arrow is a Galois covering. For every $v \in X$ closed point in X, let $E_v = f^{-1}(v)$. Let $\Phi_{v,i}$ denote the number of points of X_i lying above v, $\phi_v = \lim_{i\to\infty} \Phi_{v,i}/g_i$ (we suppose the limits exist). Furthermore, denote by $f_{v,i}$ the residue degree of a point of X_i lying above v (the tower being Galois, this does not depend on the point), and let $f_v = \lim_{i\to\infty} f_{v,i}$. If $f_v = \infty$, we have $\phi_v = 0$. If f_v is finite, denote by $N(E_v, f_v)$ the number of $\mathbb{F}_{q^{f_v}}$-points of E_v. Finally, let τ denote the "fudge" factor in the Birch and Swinnerton-Dyer conjecture (see [82] for its precise definition). Under this setting Kunyavskii and Tsfasman formulate the following conjecture in [49] :

Conjecture 3.3.3 (Kunyavskii–Tsfasman)**.** *Assuming the Birch and Swinnerton-Dyer conjecture for elliptic curves over function fields, we have*

$$\lim_{i\to\infty} \frac{\log_q(\text{Ш}_i \cdot \Delta_i \cdot \tau_i)}{g_i} = 1 - \sum_{v\in X} \phi_v \log_q \frac{N(E_v, f_v)}{q^{f_v}}.$$

Let us finally turn our attention to the approach of Hindry and Pacheko. They treat the case in some sense "orthogonal" to that of Kunyavskii and Tsfasman. Here, contrary to the previous setting of this section, we consider the number field case as the more complete one. We refer to [32] for the function field case. As in the approach of Kunyavskii and Tsfasman we study elliptic curves over global fields. However, here the ground field K is fixed and we let vary the elliptic curve E. Denote by $h(E)$ the logarithmic height of an elliptic curve E (see [31] for the precise definition, asymptotically its properties are close to those of the conductor). Hindry in [31] formulates the following conjecture :

Conjecture 3.3.4 (Hindry–Pacheko)**.** *Let E_i run through a family of pairwise non-isomorphic elliptic curves over a fixed number field K. Then*

$$\lim_{i\to\infty}\frac{\log(\text{Ш}_i\cdot\Delta_i)}{h(E_i)}=1.$$

To motivate this conjecture, Hidry reduces it to a conjecture on the asymptotics of the special value of L-functions of elliptic curves at $s=1$ using the conjecture of Birch and Swinnerton-Dyer as well as that of Szpiro and Frey (the latter one is equivalent to the ABC conjecture when $K=\mathbb{Q}$).

Let us finally state some open questions that arise naturally from the above discussion.

– What is the number field analogue of theorem 3.3.1 ?

It seems not so difficult to prove the result corresponding to theorem 3.3.1 in the number field case assuming GRH. Without GRH the situation looks much more challenging. In particular, one has to be able to controll the so called Siegel zeroes of zeta functions of varieties (that is real zeroes close to $s=d$) which might turn out to be a difficult problem. The conjecture 3.3.3 can be easily written in the number field case. However, in this situation we have even less evidence for it since theorem 3.3.2 is a particular feature of the function field case.

– How can one unify the conjectures of Kunyavskii–Tsfasman and Hindry—Pacheko ?

In particular it is unclear which invariant of elliptic curves should play the role of genus from the case of global fields. It would also be nice to be able to formulate some conjectures for a more general type of L-functions, such as automorphic L-functions.

– Is it possible to justify any of the above conjectures in certain particular cases ? Can one prove some cases of these conjectures "on average" (in some appropriate sense) ?

For now the only case at hand is the one given by theorem 3.3.2.

3.4 The proof of the Brauer–Siegel theorem for varieties over finite fields : case $s=d$

Recall that the trace formula of Lefschetz–Grothendieck gives the following expression for N_n — the number of $\mathbb{F}_{q^n}$ points on a variety X :

$$N_n=\sum_{s=0}^{2d}(-1)^s q^{ns/2}\sum_{i=1}^{b_s}\alpha_{s,i}^n, \tag{3.3}$$

where $\{q^{s/2}\alpha_{s,i}\}$ is the set of of inverse eigenvalues of the Frobenius endomorphism acting on $H^s(\bar{X},\mathbb{Q}_l)$. By Poincaré duality one has $b_{2d-s}=b_s$ and $\alpha_{s,i}=\alpha_{2d-s,i}$. The conjecture of Riemann–Weil proven by Deligne states that the absolute values of $\alpha_{s,i}$ are equal to 1. One also knows that $b_0=1$ and $\alpha_{0,1}=1$.

One can easily see that for $Z(X,q^{-s})=\zeta_X(s)$ we have the following power series expansion :

$$\log Z(X,t)=\sum_{n=1}^{\infty}N_n\frac{t^n}{n}. \tag{3.4}$$

Combining (3.4) and (3.3) we obtain

$$Z(X,t) = \prod_{s=0}^{2d} (-1)^{s-1} P_s(X,t), \tag{3.5}$$

where $P_s(X,t) = \prod_{i=1}^{b_i}(1 - q^{s/2}\alpha_{s,i})$. Furthermore we note that $P_0(X,t) = 1-t$ and $P_{2d}(X,t) = 1 - q^d t$.

To prove theorem 3.3.1 we will need the following lemma.

Lemma 3.4.1. *For $c \to \infty$ we have*

$$\frac{\log|\varkappa(X_j)|}{b(X_j)} = \sum_{l=1}^{c} \frac{N_l(X_j) - q^{dl}}{l} q^{-dl} + R_c(X_j),$$

with $R_c(X_j) \to 0$ uniformly in j.

Proof of the Lemma. Using (3.5) one has

$$\begin{aligned}
\frac{\log|\varkappa(X_j)|}{b(X_j)} + d\frac{\log q}{b(X_j)} &= \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \log|P_s(X_j, q^{-d})| = \\
&= \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \sum_{k=1}^{b_s(X_j)} \log(1 - q^{(s-2d)/2}\alpha_{s,i}) = \\
&= -\frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \sum_{k=1}^{b_s(X_j)} \sum_{l=1}^{\infty} \frac{q^{(s-2d)l/2}\alpha_{s,i}^l}{l} = \\
&= \frac{1}{b(X_j)} \sum_{l=1}^{c} \frac{q^{-dl}}{l} \left(\sum_{s=0}^{2d} (-1)^s q^{sl/2} \sum_{k=1}^{b_s(X_j)} \alpha_{s,i}^l - q^{dl} \right) + \\
&\quad + \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^s \sum_{k=1}^{b_s(X_j)} \sum_{l=c+1}^{\infty} \frac{q^{(s-2d)l/2}\alpha_{s,i}^l}{l} = \\
&= \sum_{l=1}^{c} \frac{N_l(X_j) - q^{dl}}{l} q^{-dl} + R_c(X_j).
\end{aligned}$$

An obvious estimate gives

$$|R_c(X_j)| \leq \frac{\sum_{s=0}^{2d} b_s(X_j)}{b(X_j)} \sum_{l=c+1}^{\infty} \frac{q^{-l/2}}{l} \to 0$$

for $c \to \infty$ uniformly in j. □

Now let us note that

$$\frac{1}{b(X_j)} \sum_{l=1}^{c} \frac{1}{l} \leq \frac{2}{b(X_j)} \log c \to 0$$

when $\log c/b(X_j) \to 0$. Thus to prove the main theorem we are left to deal with the following sum :

$$\frac{1}{b(X_j)}\sum_{l=1}^{c}\frac{q^{-ld}}{l}N_l(X_j) =$$

$$= \frac{1}{b(X_j)}\sum_{l=1}^{c}\frac{q^{-dl}}{l}\sum_{m|l}m\Phi_{q^m} = \frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\sum_{k=1}^{\lfloor c/m\rfloor}\frac{q^{-mkd}}{k} =$$

$$= \frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\log\frac{q^{md}}{q^{md}-1} - \frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\sum_{\lfloor c/m\rfloor+1}^{\infty}\frac{q^{-mkd}}{k}.$$

Let us estimate the last term :

$$\frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\sum_{k=\lfloor c/m\rfloor+1}^{\infty}\frac{q^{-mkd}}{k} \le$$

$$\le \frac{1}{b(X_j)}\sum_{m=1}^{c}\frac{N_m(X_j)q^{-md(\lfloor c/m\rfloor+1)}}{m(\lfloor c/m\rfloor+1)(1-q^{-md})} \le \frac{1}{b(X_j)}\sum_{m=1}^{c}\frac{N_m(X_j)q^{-cd}}{c(1-q^{-md})} \le$$

$$\le \frac{1}{b(X_j)}\sum_{m=1}^{c}\left(q^{md}+1+\sum_{s=1}^{2d-1}b_s q^{ms/2}\right)\frac{q^{-dc}}{c(1-q^{-md})} \le$$

$$\le \frac{1}{b(X_j)}\left(q^{cd}+1+\sum_{s=1}^{2d-1}b_s q^{cs/2}\right)\frac{q^{-dc}}{(1-q^{-1})} \to 0$$

as both $b(X_j) \to \infty$ and $c \to \infty$.

Now, to finish the proof we will need an analogue of the basic inequality from [85]. In the higher dimensional case there are several versions of it. However, here the simplest one will suffice. Let us define for $i = 0 \dots 2d$ the following invariants :

$$\beta_i(\{X_j\}) = \limsup_j \frac{b_i(X_j)}{b(X_j)}.$$

Theorem 3.4.2. *For an asymptotically exact family $\{X_j\}$ we have the inequality :*

$$\sum_{m=1}^{\infty}\frac{m\phi_{q^m}}{q^{(2d-1)m/2}-1} \le (q^{(2d-1)/2}-1)\left(\sum_{i\equiv 1 \bmod 2}\frac{\beta_i}{q^{(i-1)/2}+1} + \sum_{i\equiv 0 \bmod 2}\frac{\beta_i}{q^{(i-1)/2}-1}\right).$$

Proof. See [51], Remark 8.8. □

Applying this theorem together with the fact that

$$\log\frac{q^{md}}{q^{md}-1} = O\left(\frac{1}{q^{dm}-1}\right) = O\left(\frac{m}{q^{(2d-1)m/2}-1}\right)$$

when $m \to \infty$, we conclude that the series on the right hand side of (3.1) converges. Thus the difference

$$\sum_{m=1}^{\infty}\phi_{q^m}\log\frac{q^{md}}{q^{md}-1} - \frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\log\frac{q^{md}}{q^{md}-1} =$$

$$= \sum_{m=1}^{c}\left(\phi_{q^m}-\frac{\Phi_{q^m}}{b(X_j)}\right)\log\frac{q^{md}}{q^{md}-1} - \sum_{m=c+1}^{\infty}\phi_{q^m}\log\frac{q^{md}}{q^{md}-1} \to 0$$

when $c \to \infty$, $j \to \infty$ and j is large enough compared to c. This concludes the proof of theorem 3.3.1.

Chapitre 4

Uniform distribution of zeroes of L-functions of modular forms

4.1 Introduction

It is well known that zeroes of L-functions contain an important information about the arithmetic properties of the objects to which these L-functions are associated. The question about the distribution of these zeroes on the critical line was studied by many authors. This problem can be looked upon from many angles (the proportion of zeroes on the critical line, low zeroes, zero spacing, etc.).

In this chapter we study the distribution of zeroes of L-functions on the critical line when we let vary the modular form to which the L-function is associated. The same question was considered by S. Lang in [53] and M. Tsfasman and S. Vlăduţ in [87] for the Dedekind zeta function of number fields.

Let $f(z)$ be a holomorphic cusp of weight $k = k_f$ for the group $\Gamma_0(N)$ such that $f(z) = \sum\limits_{n=1}^{\infty} a_n n^{(k-1)/2} e^{2\pi i n z}$ is its normalized Fourier expansion at the cusp ∞. We suppose that $f(z)$ is a primitive form in the sense of Atkin–Lehner, so $L_f(s)$ can be defined by the Euler product

$$L_f(s) = \prod_{p|N} (1 - a_p p^{-s})^{-1} \prod_{p\nmid N} (1 - a_p p^{-s} + p^{-2s})^{-1}.$$

We denote by α_p and $\bar{\alpha}_p$ the two conjugate roots of the polynomial $1 - a_p p^{-s} + p^{-2s}$. Deligne has shown (see [10]) that $|\alpha_p| = |\bar{\alpha}_p| = 1$ for $p \nmid N$ (the Ramanujan–Peterson conjecture). On the other hand, one knows (see [1]) that for $p \mid N$ we have $|a_p| \leq 1$.

If we define the gamma factor by

$$\gamma_f(s) = \pi^{-s} \Gamma\left(\frac{s + (k-1)/2}{2}\right) \Gamma\left(\frac{s + (k+1)/2}{2}\right) = c_k (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right)$$

with $c_k = 2^{(3-k)/2}\sqrt{\pi}$, then the function $\Lambda(s) = N^{s/2}\gamma_f(s) L_f(s)$ is entire and satisfies the functional equation $\Lambda(s) = w\Lambda(1-s)$ with $w = \pm 1$. The Generalized Riemann Hypothesis (GRH) for L-function of modular forms states that all the non-trivial zeroes of these L-functions lie on the critical line $\operatorname{Re} s = \frac{1}{2}$. Throughout the chapter we assume that GRH is true.

The analytic conductor q_f (see [42]) is defined as

$$q_f = N\left(\frac{k-1}{2} + 3\right)\left(\frac{k+1}{2} + 3\right) \sim \frac{Nk^2}{4},$$

when $k \to \infty$. We will use the last expression (or, more precisely, its logarithm minus a constant) as a weight in all the zero sums in this chapter.

To each $f(z)$ we can associate the measure

$$\Delta_f := \frac{2\pi}{\log q_f} \sum_{L_f(\rho)=0} \delta_{t(\rho)},$$

where $t(\rho) = \frac{1}{i}\left(\rho - \frac{1}{2}\right)$ and ρ runs through all non-trivial zeroes of $L_f(s)$; here δ_a denotes the atomic (Dirac) measure at a. Since we suppose that GRH is true, Δ_f is a discrete measure on $\mathbb{R}$. Moreover, it can easily be seen that Δ_f is a measure of slow growth (see below).

Our main result is the following one :

Theorem 4.1.1. *Assuming GRH, for any family $\{f_j(z)\}$ of primitive forms with $q_{f_j} \to \infty$ the limit*

$$\Delta = \lim_{j\to\infty} \Delta_j = \lim_{j\to\infty} \Delta_{f_j}$$

exists in the space of measures of slow growth on $\mathbb{R}$ and is equal to the measure with density 1.

4.2 Proof of theorem 4.1.1

Our method of the proof will, roughly speaking, follow that of [87], where a similar question is treated in the case of Dedekind zeta functions. It will even be simplier in our case due to the fact that the family we consider is "asymptotically bad".

Let us recall a few facts and definitions from the theory of distribution. We will use [72] as our main reference. Recall that the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is the space of all real valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}$ (i. e. $\phi(x)$ and any its derivative go to 0 when $|x| \to \infty$ faster then any power of $|x|$). The space $\mathcal{D}(\mathbb{R})$ is defined to be the space of all real valued infinitely differentiable functions with compact support on $\mathbb{R}$. Both $\mathcal{S}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ are equipped with the structures of topological vector spaces.

The space $\mathcal{D}'$ (resp. $\mathcal{S}'$), topologically dual to $\mathcal{D}$ (resp. $\mathcal{S}$) is called the space of distribution (resp. tempered distributions). We also define the space of measures $\mathcal{M}$ as the topological dual of the space of real valued continuous functions with compact support on $\mathbb{R}$. The space $\mathcal{M}$ contains a cone of positive measures $\mathcal{M}_+$, i. e. of measures taking positive values on positive functions. One has the following inclusions : $\mathcal{S}' \subset \mathcal{D}'$ and $\mathcal{M}_+ \subset \mathcal{M} \subset \mathcal{D}'$. The intersection $\mathcal{M}_{sl} = \mathcal{M} \cap \mathcal{S}'$ is called the space of measures of slow growth. A measure μ of slow growth can be characterized by the property that for some positive integer k the integral

$$\int_{-\infty}^{+\infty} (x^2+1)^{-k} d\mu$$

converges (see [72, Thm. VII of Ch. VII]). In particular, from this criterion and the fact that the series $\sum\limits_{\rho\neq 0,1} |\rho|^{-2}$ converges ([42, Lemma 5.5]), we see that Δ_f is a measure of slow growth for any f.

Finally, we note that the Fourier transform $\hat{}$ is defined on $\mathcal{S}$ and $\mathcal{S}'$ and is a topological automorphism on these spaces. $\mathcal{D}$ is known to be dense in $\mathcal{S}$ and so $\hat{\mathcal{D}}$ is also dense in $\mathcal{S} = \hat{\mathcal{S}}$. To check that μ is a measure of slow growth it is enough to check that it is defined on a dense subset and that it is continuous on this dense subset in the topology of $\mathcal{S}$. In the same way, to check that a sequence of measures of slow growth converges to a measure of slow growth it is enough to check its convergence on a dense subset to a measure continuous on this dense subset. This follows from the definition of measures as linear functionals.

Our main tool will be a version of Weil explicit formula for L-functions of modular forms proven in [63] or in [42, Chap. V] (in the last source some extra conditions on test functions are imposed).

Suppose $F \in \mathcal{S}(\mathbb{R})$ satisfies for some $\epsilon > 0$ the following condition

$$|F(x)|, |F'(x)| \ll ce^{(-\frac{1}{2}+\epsilon)|x|} \text{ as } |x| \to \infty. \tag{4.1}$$

Let

$$\Phi(s) := \int_0^\infty F(x)e^{(s-\frac{1}{2})x}dx = \hat{F}(t),$$

where $s = \frac{1}{2} + it$. The next proposition gives us the explicit formula that we need to relate the sum over zeroes to the sum of coefficient of modular forms :

Proposition 4.2.1. *Let $f(z)$ be a primitive form of level N and weight k. Then the limit*

$$\sum_{L_f(\rho)=0} \Phi(\rho) = \lim_{T\to\infty} \sum_{\substack{L_f(\rho)=0 \\ |\rho|<T}} \Phi(\rho)$$

exists and we have the following formula :

$$\sum_{L_f(\rho)=0} \Phi(\rho) = -\sum_{p,m} b(p^m)(F(m\log p) + F(-m\log p))\frac{\log p}{p^{m/2}} +$$
$$+ F(0)(\log N - 2\log(2\pi)) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\Phi\left(\frac{1}{2}+it\right) + \Phi\left(\frac{1}{2}-it\right)}{2} \cdot \psi\left(\frac{k}{2}+it\right)dt,$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$, $b(p^m) = (a_p)^m$ if $p \mid N$ and $b(p^m) = (\alpha_p)^m + (\bar{\alpha}_p)^m$ otherwise.

Taking a subsequence of $\{f_j\}$ we can assume that the limit $\alpha = \lim\limits_{j\to\infty} \frac{\log N_j}{\log N_j + \log k_j}$ exists. We will check the convergence of measures on $\hat{\mathcal{D}}$. From the above discussion this is enough to prove the result. Let us take any $\phi \in \hat{\mathcal{D}}$, $\phi = \hat{F}$, $F \in \mathcal{D}$. We have $\phi(t) = \Phi\left(\frac{1}{2}+it\right)$. The function F satisfies the condition (4.1), so we can apply the explicit formula to it. We fix $\phi(t)$ and let vary f_j Then, we get the equality when $j \to \infty$.

$$\Delta(\phi) = 2\pi F(0)\alpha + 2\int_{-\infty}^{+\infty} \frac{\phi(t)+\phi(-t)}{2} \cdot \lim_{j\to\infty} \frac{\psi\left(\frac{k_j}{2}+it\right)}{\log N_j + \log k_j}dt, \tag{4.2}$$

since $|b(p^m)| \leq 2$ and the integral is uniformly convergent as $\phi(t) \in \mathcal{S}$. The limit under the integral sign can be evaluated using the Stirling formula $\psi(s) = \log s + O\left(\frac{1}{|s|}\right)$ (see [54, p. 332]). This gives us

$$\lim_{j\to\infty} \frac{\psi\left(\frac{k_j}{2}+it\right)}{\log N_j + \log k_j} = \frac{1}{2}(1-\alpha).$$

But $\int_{-\infty}^{+\infty} \psi(t)dt = 2\pi F(0)$ and so the right hand side of (4.2) equals

$$2\pi F(0)\alpha + 2\pi F(0)(1-\alpha) = 2\pi F(0) = \int_{-\infty}^{+\infty} \phi(t)dt.$$

This concludes the proof of the theorem. $\square$

Corollary 4.2.2. *Any fixed interval around $s = \frac{1}{2}$ contains zeroes of $L_f(s)$ if q_f is sufficiently large.*

Remark 4.2.1. One can prove a similar equidistribution statement for L-functions of bounded degree in the Selberg class, assuming suitable conjectures (like the Generalized Riemann Hypothesis or the Ramanujan Conjecture). It is an interesting question how zeroes of L-functions are distributed if the degree of these L-functions grows with the analytic conductor. Some examples of non-trivial distributions of zeroes for Dedekind zeta functions are considered in [87].

Chapitre 5

Asymptotic properties of zeta functions over finite fields

5.1 Introduction

The study of asymptotic properties of zeta functions of curves over finite fields was initiated by Tsfasman and Vlăduţ who had the so called Drinfeld – Vlăduţ inequality for the asymptotic number of points on curves over finite fields as initial motivation ([16], [85]). This work went far beyond this initial inequality and led to the introduction of the concept of limit zeta function which turned out to be very useful [86]. It also had quite numerous applications to coding theory (see, for example, the book [88] for some of them).

The above study of limit zeta functions involves three main topics :

1. The basic inequality, which can be regarded as a rather far reaching generalization of the Drinfeld – Vlăduţ inequality ;
2. Brauer–Siegel type results, in which the asymptotic properties of special values of zeta functions (such as the order of the Picard group) are studied ;
3. The distribution of zeroes of zeta functions in families.

There are at least two main directions in the further study of these topics. First, one may ask what are the number field counterparts of these results (for number fields and function fields are regarded by many as facets of a single gemstone). The translation of these results to the number field case is the subject of the paper [87]. The techniques turns out to be very analytically involved but the reward is no doubts significant as the authors managed to resolve some of the long standing problems (such as the generalization of the Brauer–Siegel theorem to an asymptotically good case) as well as to improve several difficult results (Odlyzko–Serre inequalities for the discriminant, Zimmert's bound for regulators).

Second, one may ask what happens with higher dimensional varieties over finite fields. Here the answers are less complete. The first topic (main inequalities) was extensively studied in [51]. The results obtained there are fairly complete, though they do not directly apply to L-functions (such as L-functions of elliptic curves over function fields). The second topic is considerably less developed though it received a particular attention in the recent years in the case of elliptic surfaces [32], [49] and in the case of zeta functions of varieties over finite fields [92]. The results concerning the third topic seem to be even scarcer. One can cite a paper by Michel [64] where the case of elliptic surfaces over $\mathbb{F}_q(t)$ is treated. Quite a considerable attention was devoted to some finer questions related to the distribution of zeroes [46]. However, to our knowledge, not a single result of this type for asymptotically good families of varieties has been obtained before.

The goal of this chapter is to study the above three topic in the case of more general zeta and L-functions. We take the axiomatic approach, defining a class of L-functions to which our results may be applicable. This can be regarded as the function field analogue of working with the Selberg class in characteristic zero, though obviously the analytic contents in the function case is much less substantial (and often times even negligible). In our investigations we devote more attention to the second and the third topics as being far less developed then the first one. So, while giving results on the generalizations of the basic inequality, we do not seek to prove them in utmost generality (like in the paper [51]). We hope that this allows us to gain in clarity of the presentation as well as to save a considerable amount of space.

We use families of elliptic curves over function fields as our motivating example. After each general statement concerning any of the three topics we specify what concrete results we get for curves and varieties over finite fields and elliptic curves over function fields. In the study of the second topic we actually manage prove something new even in the classical case of zeta functions of curves, namely we prove a statement on the limit behaviour of zeta functions of which the Brauer–Siegel theorem from [86] is a particular case (see theorem 5.5.2 and corollary 5.5.4). We also reprove and extend some of the Ihara's results on Euler–Kronecker constant of function fields [41] incorporating them in the same general framework of limit zeta functions (see corollary 5.5.5). Our statements about the distribution of zeroes (theorem 5.6.1 and corollary 5.6.4) imply in the case of elliptic curves over function fields a generalization of a result due to Michel [64] (however, unlike us, Michel also provides an estimate for the error term).

Here is the plan of the chapter. In section 5.2 we present the axiomatic framework for zeta and L-functions with which we will be working, then we prove an explicit formula for them. In the end of the section we introduce several particular examples coming from algebraic geometry (zeta functions of curves, zeta functions of varieties over finite fields, L-functions of elliptic curves over function fields) to which we will apply the general results. Each further section contains a subsection where we show what the results on abstract zeta and L-functions give in these concrete cases. In section 5.3 we outline the asymptotic approach to the study of zeta and L-functions, introducing the notions of asymptotically exact and asymptotically very exact families. Section 5.4 is devoted to the proof of several versions of the basic inequality. The study of the Brauer–Siegel type results is undertaken in section 5.5. In the same section we show how these results imply a formula for the asymptotic behaviour of the invariants of function fields generalizing the Euler–Kronecker constant (corollary 5.5.5) and a certain bound towards the conjectures of Kunyavskii, Tsfasman and Hindry (theorem 5.5.11). We prove the zero distribution results in section 5.6. There we also give some applications to the distribution of zeroes and the growth of ranks in families of elliptic surfaces (corollary 5.6.4 and corollary 5.6.6). Finally, in section 5.7 we discuss some possible further development as well as open questions.

5.2 Zeta and L-functions

5.2.1 Definitions

Let us define the class L-functions we will be working with. Let $\mathbb{F}_q$ be a finite field with q elements.

Definition 5.2.1. *An L-function $L(s)$ over a finite field $\mathbb{F}_q$ is a holomorphic function in s such that for $u = q^{-s}$ the function $\mathcal{L}(u) = L(s)$ is a polynomial with real coefficients, $\mathcal{L}(0) = 1$ and all the roots of $\mathcal{L}(u)$ are on the circle of radius $q^{-\frac{d}{2}}$ for some non-negative integer number d.*

We will refer to the last condition in the definition as the Riemann hypothesis for $L(s)$ since it is the finite field analogue of the classical Riemann hypothesis for the Riemann zeta function.

The number d in the definition of an L-function will be called its weight. We will also say that the degree g of the polynomial $\mathcal{L}(u)$ is the degree of the L-function $L(s)$ (it should not be confused with the degree of an L-function in the analytic number theory, where it is taken to be the degree of the polynomial in its Euler product).

The logarithm of an L-function has a Dirichlet series expansion

$$\log L(s) = \sum_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs},$$

which converges for $\operatorname{Re} s > \frac{d}{2}$. For the opposite of the logarithmic derivative we get the formula :

$$-\frac{L'(s)}{L(s)} = \sum_{f=1}^{\infty} (\Lambda_f \log q)\, q^{-fs} = u\, \frac{\mathcal{L}'(u)}{\mathcal{L}(u)} \log q.$$

There is a functional equation for $L(s)$ of the form

$$L(d-s) = \omega q^{(\frac{d}{2}-s)g} L(s), \tag{5.1}$$

where $g = \deg \mathcal{L}(u)$ and $\omega = \pm 1$ is the root number. This can be proven directly as follows. Let $\mathcal{L}(u) = \prod_{i=1}^{g} \left(1 - \frac{u}{\rho_i}\right)$. Then

$$\mathcal{L}\left(\frac{1}{uq^d}\right) = \prod_{\rho} \left(1 - \frac{1}{\rho u q^d}\right) = \prod_{\rho} \rho \cdot q^{dg} u^g \prod_{\rho} \left(\frac{u}{\bar{\rho}} - 1\right) = \pm q^{\frac{dg}{2}} u^g \prod_{\rho} \left(1 - \frac{u}{\rho}\right).$$

Here we used the fact that all coefficients of $\mathcal{L}(u)$ are real, so its complex roots come in pairs ρ and $\bar{\rho}$.

Definition 5.2.2. *A zeta function $\zeta(s)$ over a finite field $\mathbb{F}_q$ is a product of L-functions in powers ± 1 :*

$$\zeta(s) = \prod_{k=0}^{d} L_k(s)^{w_k},$$

where $w_k \in \{-1, 1\}$, $L_k(s)$ is an L-function of weight k.

For the logarithm of a zeta function we also have the Dirichlet series expansion :

$$\log \zeta(s) = \sum_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs}$$

which is convergent for $\operatorname{Re} s > \frac{d}{2}$.

5.2.2 Explicit formulae

In this subsection we will derive the analogues of Weil and Stark explicit formulae for our zeta and L-functions. The proofs of the Weil explicit formula can be found in [74] for curves and in [51] for varieties over finite fields. An explicit formula for L-functions of elliptic surfaces is proven in [7]. In our proof we will follow the latter exposition.

Recall that our main object of study is $\zeta(s) = \prod_{i=0}^{d} L_i(s)^{w_i}$ a zeta function with $L_i(s)$ given by

$$L_i(s) = \prod_{j=1}^{g_i} \left(1 - \frac{q^{-s}}{\rho_{ij}}\right).$$

As before, we define Λ_f via the relation $\log \zeta(s) = \sum\limits_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs}$.

Proposition 5.2.1. *Let $\mathbf{v} = (v_f)_{f\geq 1}$ be a sequence of real numbers and let $\psi_{\mathbf{v}}(t) = \sum\limits_{f=1}^{\infty} v_f t^f$. Let $\rho_{\mathbf{v}}$ be the radius of convergence of the series for $\psi_{\mathbf{v}}(t)$. Assume that $|t| < q^{-d/2}\rho_{\mathbf{v}}$, then*

$$\sum_{f=1}^{\infty} \Lambda_f v_f t^f = -\sum_{i=0}^{d} w_i \sum_{j=1}^{g_i} \psi_{\mathbf{v}}(q^i \rho_{ij} t).$$

Proof. Let us prove this formula for L-functions. The formula for zeta functions will follow by additivity.

The simplest is to work with $\mathcal{L}(u) = \prod\limits_{j=1}^{g} \left(1 - \frac{u}{\rho}\right)$. The coefficient of u^f in $-u\mathcal{L}'(u)/\mathcal{L}(u)$ is seen to be $\sum\limits_{\rho} \rho^{-f}$ for $f \geq 1$. From this we derive the equality :

$$\sum_{\rho} \rho^{-f} = -\Lambda_f.$$

The map $\rho \mapsto (q^d\rho)^{-1}$ permutes the zeroes $\{\rho\}$, thus for any $f \geq 1$ we have :

$$S_n = \sum_{\rho} (q^d\rho)^f = -\Lambda_f.$$

Multiplying the last identity by $v_f t^f$ and summing for $f = 1, 2, \ldots$ we get the statement of the theorem. $\square$

From this theorem one can easily get a more familiar version of the explicit formula (like the one from [74] in the case of curves over finite fields).

Corollary 5.2.2. *Let $L(s)$ be an L-function, with zeroes $\rho = q^{-d/2}e^{i\theta}$, $\theta \in [-\pi, \pi]$. Let $f : [-\pi, \pi] \to \mathbb{C}$ be an even trigonometric polynomial*

$$f(\theta) = v_0 + 2\sum_{n=1}^{Y} v_n \cos(n\theta).$$

Then we have the explicit formula :

$$\sum_{\theta} f(\theta) = v_0 g - 2\sum_{f=1}^{Y} v_f \Lambda_f q^{-\frac{df}{2}}.$$

Proof. We put $t = q^{-\frac{d}{2}}$ in the above explicit formula and notice that the sum over zeroes can be written using cos since all the non-real zeroes come in complex conjugate pairs. $\square$

In the next sections we will also make use of the so called Stark formula (which borrows its name from its number field counterpart from [80]).

Proposition 5.2.3. *For a zeta function $\zeta(s)$ we have :*

$$\frac{1}{\log q} \frac{\zeta'(s)}{\zeta(s)} = \sum_{i=0}^{d} w_i \sum_{j=1}^{g_i} \frac{1}{q^s \rho_{ij} - 1} = -\frac{1}{2}\sum_{i=0}^{d} w_i g_i + \frac{1}{\log q} \sum_{i=0}^{d} w_i \sum_{L_i(\theta_{ij})=0} \frac{1}{s - \theta_{ij}}.$$

Proof. The first equality is a trivial consequence of the formulae expressing $\mathcal{L}_i(u)$ as polynomials in u.

The second equality follows from the following series expansion :

$$\frac{\log q}{\rho^{-1}q^s - 1} + \frac{\log q}{2} = \lim_{T\to\infty} \sum_{\substack{q^\theta=\rho \\ |\theta|\le T}} \frac{1}{s-\theta}.$$

□

5.2.3 Examples

We have in mind three main types of examples : zeta functions of curves over finite fields, zeta functions of varieties over finite fields and L-functions of elliptic curves over function fields.

Example 5.2.1 (Curves over finite fields). Let X be an absolutely irreducible smooth projective curve of genus g over the finite field $\mathbb{F}_q$ with q elements. Let Φ_f be the number of points of degree f on X. The zeta function of X is defined for $\operatorname{Re} s > 1$ as

$$\zeta_X(s) = \prod_{f=1}^{\infty} (1 - q^{-fs})^{-\Phi_f}.$$

It is known that $\zeta_X(s)$ is a rational function in $u = q^{-s}$. Moreover,

$$\zeta_K(s) = \frac{\prod_{j=1}^{g} \left(1 - \frac{u}{\rho_j}\right)\left(1 - \frac{u}{\bar{\rho}_j}\right)}{(1-u)(1-qu)},$$

and $|\rho_j| = q^{-\frac{1}{2}}$. It can easily be seen that in this case $\Lambda_f = N_f(X)$ is the number of points on $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ over $\mathbb{F}_{q^n}$. A very important feature of this example which will be lacking in general is that $\Lambda_f \ge 0$ for all f.

Though $\zeta_X(s)$ is not an L-function, in all asymptotic considerations the denominator will be irrelevant and it will behave as an L-function.

This example will serve as a motivation in most of our subsequent considerations, for most (but not all, see section 5.5) of the results we derive for general zeta and L-functions are known in this setting.

Example 5.2.2 (Varieties over finite fields). Let X be a non-singular absolutely irreducible projective variety of dimension n defined over a finite field $\mathbb{F}_q$. Denote by $|X|$ the set of closed points of X. We put $X_f = X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^f}$ and $\bar{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Let Φ_f be the number of points of X having degree f, that is $\Phi_f = |\{v \in |X| \mid \deg(v) = f\}|$. The number N_f of $\mathbb{F}_{q^f}$-points of the variety X_f is equal to $N_f = \sum_{m|f} m\Phi_m$.

Let $b_s(X) = \dim_{\mathbb{Q}_l} H^s(\bar{X}, \mathbb{Q}_l)$ be the l-adic Betti numbers of X. The zeta function of X is defined for $\operatorname{Re}(s) > d$ by the following Euler product :

$$\zeta_X(s) = \prod_{v\in|X|} \frac{1}{1 - \mathrm{N}v^{-s}} = \prod_{f=1}^{\infty} (1 - q^{-fs})^{-\Phi_f},$$

where $\mathrm{N}v = q^{-\deg v}$. If we set $Z_X(u) = \zeta_X(s)$ with $u = q^{-s}$ then the function $Z_X(u)$ is a rational function of u and can be expressed as

$$Z_X(u) = \prod_{i=0}^{2n} (-1)^{i-1} \log P_i(X, u),$$

where

$$P_i(X,u) = \prod_{j=1}^{b_i} \left(1 - \frac{u}{\rho_{ij}}\right),$$

and $|\rho_{ij}| = q^{-i/2}$. Moreover, $P_0(X,u) = 1-u$ and $P_{2n}(X,u) = 1 - q^d u$. As before, we have that $\Lambda_f = N_f(X) \geq 0$.

The previous example is obviously included in this one. However, it is better to separate them as in the case of zeta functions of general varieties over finite fields much less is known. One more reason to distinguish between these two examples is that, whereas zeta functions of curves asymptotically behave as L-functions, zeta functions of varieties are "real" zeta functions. Thus there is quite a number of properties that simply do not hold in general (for example, those connected to the distribution of zeroes).

Example 5.2.3 (Elliptic curves over function fields). Let E be a non-constant elliptic curve over a function field $K = \mathbb{F}_q(X)$ with finite constant field $\mathbb{F}_q$. The curve E can also be regarded as an elliptic surface over $\mathbb{F}_q$. Let g be the genus of X. Places of K (that is points of X) will be denoted by v. Let $d_v = \deg v$, $|v| = \mathrm{N}v = q^{\deg v}$ and let $\mathbb{F}_v = \mathbb{F}_{\mathrm{N}v}$ be the residue field of v.

For each place v of K we define a_v from $|E_v(\mathbb{F}_v)| = |v|+1-a_v$, where $|E_v(\mathbb{F}_v)|$ is the number of points on the reduction E_v of the curve E. The local factors $L_v(s)$ are defined by

$$L_v(s) = \begin{cases} (1 - a_v|v|^{-s} + |v|^{1-2s})^{-1}, & \text{if } E_v \text{ is non-singular}\,; \\ (1 - a_v|v|^{-s})^{-1}, & \text{otherwise.} \end{cases}$$

We define the global L-function $L_E(s) = \prod\limits_v L_v(s)$. The product converges for $\operatorname{Re} s > \frac{3}{2}$ and defines an analytic function in this half-plane. Define the conductor N_E of E as the divisor $\sum\limits_v n_v v$ with $n_v = 1$ at places of multiplicative reduction, $n_v = 2$ at places of additive reduction for $p > 3$ (and possibly larger for $p = 2$ or 3) and $n_v = 0$ otherwise. Let $n_E = \deg N_E = \sum\limits_v n_v \deg v$.

It is known (see [7]) that $L_E(s)$ is a polynomial $\mathcal{L}_E(u)$ in $u = q^{-s}$ of degree $n_E + 4g - 4$. The polynomial $\mathcal{L}_E(u)$ has real coefficients, satisfies $\mathcal{L}_E(0) = 1$ and all of its roots have absolute value q^{-1}.

Let $\alpha_v, \bar{\alpha}_v$ be the roots of the polynomial $1 - a_v t + |v|t^2$ for a place v of good reduction and let $\alpha_v = a_v$ and $\bar{\alpha}_v = 0$ for a place v of bad reduction. Then from the definition of $L_E(s)$ one can easily deduce that

$$\Lambda_f = \sum_{md_v = f} d_v(\alpha_v^m + \bar{\alpha}_v^m), \tag{5.2}$$

the sum being taken over all places v of K and $m \geq 1$ such that $m \deg v = f$.

This example will be the principal one in the sense that all our results on L-functions are established in the view to apply them to this particular case. These L-functions are particularly interesting from the arithmetic point of view, especially due to the connection between the special value at $s = 1$ and the arithmetic invariants of the elliptic curve (the order of the Shafarevich–Tate group and the regulator) provided by the Birch and Swinnerton-Dyer conjecture.

5.3 Families of zeta and L-functions

5.3.1 Definitions and basic properties

We are interested in studying sequences of zeta and L-functions. Let us fix the finite field $\mathbb{F}_q$.

Definition 5.3.1. *We will call a sequence $\{L_k(s)\}_{k=1\ldots\infty}$ of L-functions a family if they all have the same weight d and the degree g_k tends to infinity.*

Definition 5.3.2. *We will call a sequence $\{\zeta_k(s)\}_{k=1\ldots\infty} = \left\{\prod\limits_{i=0}^{d} L_{ki}(s)^{w_i}\right\}_{k=1\ldots\infty}$ of zeta functions a family if the total degree $g_k = \sum\limits_{i=0}^{d} g_{ki}$ tends to infinity. Here g_{ki} are the degrees of the individual L-functions $L_{ki}(s)$ in $\zeta_k(s)$.*

Remark 5.3.1. In the definition of a family of zeta functions we assume that $d = d_k$ and $w_i = w_{ki}$ are the same for all k. Obviously, any family of L-functions is at the same time a family of zeta functions.

Definition 5.3.3. *A family $\{\zeta_k(s)\}_{k=1\ldots\infty}$ of zeta or L-functions is called asymptotically exact if the limits*

$$\gamma_i = \gamma_i(\{\zeta_k(s)\}) = \lim_{k\to\infty} \frac{g_{ki}}{g_k} \quad \textit{and} \quad \lambda_f = \lambda_f(\{\zeta_k(s)\}) = \lim_{k\to\infty} \frac{\Lambda_{kf}}{g_k}$$

exist for each $i = 0, \ldots, d$ and each $f \in \mathbb{Z}$, $f \geq 1$. It is called asymptotically bad if $\lambda_f = 0$ for any f and asymptotically good otherwise.

The following (easy) proposition will be important.

Proposition 5.3.1. *Let $L(s)$ be an L-function. Then*

1. *for each f we have the bound $|\Lambda_f| \leq q^{\frac{df}{2}} g$;*
2. *there exists a number $C(q, d, s)$ depending on q, d and s but not on g such that $|\log L(s)| \leq C(q, d, s)g$ for any s with $\operatorname{Re} s \neq \frac{d}{2}$. The bound is uniform in each vertical strip $a \leq \operatorname{Re} s \leq b$, $\frac{d}{2} \notin [a, b]$.*

Proof. To prove the first part we use proposition 5.2.1. Applying it to the sequence consisting of one non-zero term we get :

$$\Lambda_f = -\sum_{\mathcal{L}_{ki}(\rho)=0} q^{df}\rho^f. \tag{5.3}$$

The absolute value of the right hand side of this equality is bounded by $q^{\frac{df}{2}} g$.

To prove the second part we assume first that $\operatorname{Re} s = \epsilon + \frac{d}{2} > \frac{d}{2}$. We have the estimate :

$$|\log L(s)| = \left|\sum_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs}\right| \leq \sum_{f=1}^{\infty} \frac{g}{f} \cdot q^{\frac{df}{2}} \cdot q^{-f\operatorname{Re} s} \leq g \sum_{f=1}^{\infty} \frac{1}{f q^{\epsilon f}}.$$

For $\operatorname{Re} s < \frac{d}{2}$ we use the functional equation (5.1). ☐

Proposition 5.3.2. *Any family of zeta and L-functions contains an asymptotically exact subfamily.*

Proof. We note that both $\frac{g_{ki}}{g_k}$ and $\frac{\Lambda_{kf}}{g_k}$ are bounded. For the first expression it is obvious and the second expression is bounded by proposition 5.3.1. Now we can use the diagonal method to choose a subfamily for which all the limits exist. ☐

As in the case of curves over finite fields we have to single out the factors in zeta functions which are asymptotically negligible. This can be done using proposition 5.3.1.

Definition 5.3.4. *Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Define the set $I \subset \{0 \dots d\}$ by the condition $i \in I$ if and only if $\gamma_i = 0$. We define $\zeta_{\mathbf{n},k}(s) = \prod_{i \in I} L_{ki}(s)^{w_i}$ the negligible part of $\zeta_k(s)$ and $\zeta_{\mathbf{e},k}(s) = \prod_{i \in \{0 \dots d\} - I} L_{ki}(s)^{w_i}$ the essential part of $\zeta_k(s)$. Define also $d_{\mathbf{e}} = \max\{i \mid i \notin I\}$.*

Remark 5.3.2. The functions $\zeta_{\mathbf{n},k}(s)$ and $\zeta_{\mathbf{e},k}(s)$ make sense only for families of zeta functions and not for individual zetas. We also note that the definitions of the essential and the negligible parts are obviously trivial for families of L-functions.

The following proposition, though being rather trivial, turns out to be useful.

Proposition 5.3.3. *For an asymptotically exact family of zeta functions $\{\zeta_i(s)\}$ we have $\lambda_f(\zeta_i(s)) = \lambda_f(\zeta_{\mathbf{e},i}(s))$.*

Proof. This is an immediate corollary of proposition 5.3.1. □

The condition on a family to be asymptotically exact suffices in the case of varieties over finite fields due to the positivity of coefficients Λ_f. However, in general we will have to impose somewhat stricter conditions on the family.

Definition 5.3.5. *We say that an asymptotically exact family of zeta or L-functions is asymptotically very exact if the series*

$$\sum_{f=1}^{\infty} |\lambda_f| q^{-\frac{f d_{\mathbf{e}}}{2}}$$

is convergent.

Example 5.3.1. An obvious example of a family which is asymptotically exact but not very exact is given by the family of L-functions $L_k(s) = (1 - q^{-s})^k$. We have $\lambda_f = -1$ for any k and the series $\sum_{f=1}^{\infty} (-1)$ is clearly divergent.

Proposition 5.3.4. *Assume that we have an asymptotically exact family of zeta functions $\{\zeta_k(s)\} = \left\{ \prod_{i=0}^{d} L_{ki}(s)^{w_i} \right\}_{k=1\dots\infty}$, such that all the families $\{L_{ki}(s)\}$ are also asymptotically exact. Then, the family $\{\zeta_k(s)\}$ is asymptotically very exact if and only if the family $\{L_{kd_{\mathbf{e}}}(s)\}$ is asymptotically very exact.*

Proof. This follows from proposition 5.3.1 together with proposition 5.3.3. □

In practice, this proposition means that the asymptotic behaviour of zeta functions at $s = \frac{d_{\mathbf{e}}}{2}$ is essentially the same as that of their weight $d_{\mathbf{e}}$ part. Thus, most asymptotic questions about zeta functions are reduced to the corresponding question about L-function.

5.3.2 Examples

As before we stick to three types of examples : curves over finite fields, varieties over finite fields and elliptic curves over function fields.

Example 5.3.2 (Curves over finite fields). Let $\{X_j\}$ be a family of curves over $\mathbb{F}_q$. Recall (see [86]) that an asymptotically exact family of curves was defined by Tsfasman and Vlăduţ as such that the limits

$$\phi_f = \lim_{j \to \infty} \frac{\Phi_f(X_j)}{g_j} \tag{5.4}$$

exist. This is equivalent to our definition since $\Lambda_f = N_f(X) = \sum_{m|f} m\Phi_m$. Note a little difference in the normalization of coefficients : in the case of curves we let $\lambda_f(\{X_j\}) = \lim_{j\to\infty} \frac{\Lambda_{jf}}{2g_j}$ since $2g_j$ is the degree of the corresponding polynomial in the numerator of $\zeta_{X_j}(s)$ and the authors of [86] choose to consider simply $\lim_{j\to\infty} \frac{\Lambda_{jf}}{g_j}$.

For any asymptotically exact family of zeta functions of curves the negligible part of $\zeta_X(s)$ is its denominator $(1-q^{-s})(1-q^{1-s})$ and the essential part is its numerator. Thus, zeta functions of curves asymptotically behave as L-functions. Any asymptotically exact family of curves is asymptotically very exact as shows the basic inequality from [86] (see also corollary 5.4.2 below), which is in fact due to positivity of λ_f.

Example 5.3.3 (Varieties over finite fields). In the case of varieties over finite fields we have an analogous notion of an asymptotically exact family [51], namely we ask for the existence of the limits

$$\phi_f = \lim_{j\to\infty} \frac{\Phi_f(X_j)}{b(X_j)} \quad \text{and} \quad \beta_i = \lim_{j\to\infty} \frac{b_i(X_j)}{b(X_j)},$$

where $b(X_j) = \sum_{i=0}^{2d} b_i(X_j)$ is the sum of Betti numbers. Again this definition and our definition 5.3.3 are equivalent.

In this case the factors $(1-q^{-s})$ and $(1-q^{d-s})$ of the denominator are also always negligible. However, we can have more negligible factors as the following example shows. Take the product $C \times C$, where C is a curve of genus $g \to \infty$. The dimension of the middle cohomology group H^2 grows as g^2 and $b_1 = b_3 = g$ (Kunneth formula). Thus $\zeta_{C\times C}(s)$ behaves like the inverse of an L-function. If for an asymptotically exact family we have $d_{\mathbf{e}} = 2d-1$ then it is asymptotically very exact as shows a form of the basic inequality [51, (8.8)] (it actually gives that the series $\sum_{f=1}^{\infty} \lambda_f q^{-f(d-1/2)}$ always converges), see also corollary 5.4.4 below.

Example 5.3.4 (Elliptic curves over function fields). In the last example we will be interested in two particular types of asymptotically exact families.

Asymptotically bad families. Let us fix a function field $K = \mathbb{F}_q(X)$ and let us take the sequence of all pairwise non-isomorphic elliptic curves E_i/K. We get a family of L-functions since $n_{E_i} \to \infty$. From (5.2) we deduce that $|\Lambda_f| \le 2\left(\sum_{d_v|f} d_v\right) q^{\frac{f}{2}}$ which is independent of n_{E_i}. Thus, this family is asymptotically exact and asymptotically bad, i. e. $\lambda_f = 0$ for any $f \ge 1$. This will be the only fact important for our asymptotic considerations. There will be no difference in the treatment of this particular family or in that of any other asymptotically bad family of L-functions.

This family was considered in [32] in the connection with the generalized Brauer–Siegel theorem. The main result of that paper is the reduction of the statement about the behaviour of the order of the Tate–Shafarevich group and the regulator of elliptic curves over function fields to a statement about the values of their L-functions at $s = 1$. See also [31] for a similar problem treated in the number field case.

Base change. Let us consider a family which is, in a sense, orthogonal to the previous one. Let $K = \mathbb{F}_q(X)$ be a function field and let E/K be an elliptic curve. Let $f : \mathcal{E} \to X$ be the corresponding elliptic surface. Consider a family of coverings of curves $X = X_0 \leftarrow X_1 \cdots \leftarrow$

$X_i \leftarrow \ldots$ and the family of elliptic surfaces $\mathcal{E}_i$, given by the base change :

$$\begin{array}{ccccccccc} \mathcal{E} = \mathcal{E}_0 & \longleftarrow & \mathcal{E}_1 & \longleftarrow & \ldots & \longleftarrow & \mathcal{E}_i & \longleftarrow & \ldots \\ \Big\downarrow f & & \Big\downarrow & & & & \Big\downarrow & & \\ X = X_0 & \longleftarrow & X_1 & \longleftarrow & \ldots & \longleftarrow & X_i & \longleftarrow & \ldots . \end{array}$$

Let $\Phi_{v,f}(X_i)$ be the number of points on X_i, lying above a closed point $v \in |X|$, such that their residue fields have degree f over $\mathbb{F}_v$.

Lemma 5.3.5. *The limits*

$$\phi_{v,f} = \phi_{v,f}(\{X_i\}) = \lim_{i\to\infty} \frac{\Phi_{v,f}(X_i)}{g(X_i)}$$

always exist.

Proof. We will follow the proof of the similar statement for Φ_f from [87, lemma 2.4]. Let $K_2 \supseteq K_1 \supseteq K$ be finite extension of function fields. From the Riemann–Hurwitz formula we deduce the inequality $g(K_2) - 1 \geq [K_2 : K_1](g(K_1) - 1)$, where $[K_2 : K_1]$ is the degree of the corresponding extension. Now, if we fix w a place of K_1 above v and consider its decomposition $\{w_1, \ldots, w_r\}$ in K_2, then we have $\sum_{i=1}^{r} \deg w_i \leq [K_2 : K_1]$. Thus we get for any $n \geq 1$ the inequality $\sum_{f=1}^{n} f\Phi_{v,f}(K_2) \leq [K_2 : K_1] \sum_{f=1}^{n} f\Phi_{v,f}(K_1)$. Dividing we see that

$$\frac{\sum_{f=1}^{n} f\Phi_{v,f}(K_2)}{g(K_2) - 1} \leq \frac{\sum_{f=1}^{n} f\Phi_{v,f}(K_1)}{g(K_1) - 1}.$$

It follows that the sequence $\sum_{f=1}^{n} \frac{f\Phi_{v,f}(X_i)}{g(X_i)-1}$ is non-increasing and bounded for any fixed n and so has a limit. Taking $n = 1$ we see that $\phi_{v,1}$ exists, taking $n = 2$ we derive the existence of $\phi_{v,2}$ and so on. $\square$

Let us remark that $\Phi_f = \sum_{m \deg v = f} \Phi_{v,m}$, the sum being taken over all places v of K and the same equality holds for ϕ_v.

For our family we can derive a concrete expression for the Dirichlet series coefficients of the logarithms of L-functions. Indeed, (5.2) gives us

$$\Lambda_f = \sum_{mkd_v = f} md_v \Phi_{v,m}(\alpha_v^{mk} + \bar{\alpha}_v^{mk}). \tag{5.5}$$

Lemma 5.3.6. *Let E_i/K_i be a family of elliptic curves obtained by a base change and let $n_i = n_{E_i/K_i}$ be the degree of the conductor of E_i/K_i. Then the ratio $\frac{n_i}{g_i}$ is bounded by a constant depending only on E_0/K_0.*

If, furthermore, $\operatorname{char} \mathbb{F}_q \neq 2, 3$ *or the extensions K_i/K_0 are Galois for all i then the limit $\nu = \lim_{i\to\infty} \frac{n_i}{g_i}$ exists.*

Proof. The proof basically consists of looking at the definition of the conductor and applying the same method as in the proof of lemma 5.3.5. Recall, that if E/K is an elliptic curve over a local field K, $T_l(E)$ is its Tate module, $l \neq \operatorname{char} \mathbb{F}_q$, $V_l(E) = T_l(E) \otimes \mathbb{Q}_l$, $I(\bar{K}/K)$ is the inertia subgroup of $\operatorname{Gal}(\bar{K}/K)$, then the tame part of the conductor is defined as

$$\varepsilon(E/K) = \dim_{\mathbb{Q}_l}(V_l(E)/V_l(E)^{I(\bar{K}/K)}).$$

It is easily seen to be non increasing in extensions of K, moreover it is known that $0 \leq \varepsilon(E/K) \leq 2$ (see [78, Chap. IV, §10]).

If we let $L = K(E[l])$, $g_i(L/K) = |G_i(L/K)|$, where $G_i(L/K)$ is the i^{th} ramification group of L/K, then the wild part of the conductor is defined as

$$\delta(E/K) = \sum_{i=1}^{\infty} \frac{g_i(L/K)}{g_0(L/K)} \dim_{\mathbb{F}_l}(E[l]/E[l]^{G_i(L/K)}).$$

One can prove [78, Chap. IV, §10] that $\delta(E/K)$ is zero unless the characteristic of the residue field of K is equal to 2 or 3. In any case, the definition shows that $\delta(E/M)$ can take only finitely many values if we fix E and let vary the extension M/K.

The exponent of the conductor of E over the local field K is defined to be $f(E/K) = \varepsilon(E/K) + \delta(E/K)$.

From the previous discussion we see that the ratio $\frac{n_i}{g_i}$ is bounded. If, furthermore, $\operatorname{char} \mathbb{F}_q \neq 2, 3$, then an argument similar to the one used in the proof of lemma 5.3.5 together with the fact that $n_w(E) \leq n_v(E)$ if w lies above v in an extension of fields gives us that the sequence $\frac{n_i}{g_i}$ is non-increasing and so it has a limit $\nu = \nu(\{E_i/K_i\})$.

In the case of Galois extensions we notice that $n_w(E)$ must stabilize in a tower, so the previous argument is applicable once again. □

Now we can prove the following important proposition :

Proposition 5.3.7. *Any family of elliptic curves obtained by a base change contains an asymptotically very exact subfamily. If, furthermore,* $\operatorname{char} \mathbb{F}_q \neq 2, 3$ *or the extensions* K_i/K_0 *are Galois for all* i *then it is itself asymptotically very exact.*

Proof. Recall that for each E_i/K_i the degree of the corresponding L-function is $n_i + 4g_i - 4$. It follows from the previous lemma that it is enough to prove the existence of the limits $\tilde{\lambda}_f = \lim\limits_{i\to\infty} \frac{\Lambda_f(E_i)}{g_i}$ and the convergence of the series $\sum\limits_{f=1}^{\infty} |\tilde{\lambda}_f| q^{-f}$.

The first statement is a direct corollary of lemma 5.3.5 and (5.5). As for the second statement, we have the following bound :

$$|\Lambda_f| \leq 2 \sum_{mkd_v = f} md_v \Phi_{v,m} q^{\frac{f}{2}} = 2 \sum_{lk=f} l\Phi_l q^{\frac{f}{2}} = 2N_f q^{\frac{f}{2}}.$$

Now, the convergence of the series $\sum\limits_{f=1}^{\infty} \nu_f q^{-\frac{f}{2}}$ with $\nu_f = \lim\limits_{i\to\infty} \frac{N_f(X_i)}{g_i}$ is a consequence of the basic inequality from [85, corollary 1]. □

Remark 5.3.3. It would be nice to know whether the statement of the previous proposition holds without any additional assumptions, i. e. whether a family obtained by a base change is always asymptotically very exact. This depends on lemma 5.3.6, which do not know how to prove in general.

The family of elliptic curves obtained by the base change was studied in [49] again in the attempts to obtain a generalization of the Brauer–Siegel theorem to this case. They formulate a conjecture on the asymptotic behaviour of the order of the Tate–Shafarevich group and the regulator in such families (see conjecture 5.5.9 below). They also treat the case of constant elliptic curves in more detail. Unfortunately, the proof of the main theorem [49, theorem 2.1] given there is not absolutely flawless (the change of limits remains to be justified, which seems to be very difficult if not inaccessible at present).

Remark 5.3.4. If, for a moment, we turn our attention to general families of elliptic surfaces the following natural question arises :

Question 5.3.1. Is it true that any family of elliptic surfaces contains an asymptotically very exact subfamily ?

The fact that it is true for two "orthogonal" cases makes us believe that this property might hold in general.

5.4 Basic inequalities

In this section we finally start carrying out our program to generalize asymptotic results from the case of curves over finite fields to the case of general zeta and L-functions. We will start with the case of L-functions, where a little more can be said. Next we will prove a weaker result in the case of zeta functions.

5.4.1 Basic inequality for L-functions

Our goal here is to prove the following theorem, generalizing the basic inequality from [85].

Theorem 5.4.1. *Let $\{L_i(s)\}$ be an asymptotically exact family of L-functions or an asymptotically exact family of zeta functions $\{\zeta_i(s)\}$ with $\zeta_{\mathbf{e},i}(s)$ being an L-function for any i. Let d be its weight. Then for any $b \in \mathbb{N}$ the following inequality holds :*

$$\sum_{j=1}^{b}\left(1-\frac{j}{b+1}\right)\lambda_j q^{-\frac{dj}{2}} \leq \frac{1}{2}. \tag{5.6}$$

Proof. Using proposition 5.3.3 one immediately sees that it is enough to prove the statement of the theorem for L-functions.

As in the proof for curves our main tool will be the so called Drinfeld inequality. We take an L-function $L(s)$ and let $\alpha_i = q^{\frac{d}{2}}\rho_i$, where ρ_i are the roots of $\mathcal{L}(u)$, so that $|\alpha_i| = 1$. For any α_i we have

$$0 \leq |\alpha_i^b + \alpha_i^{b-1} + \cdots + 1|^2 = (b+1) + \sum_{j=1}^{b}(b+1-j)(\alpha_i^j + \alpha_i^{-j}).$$

Thus $b+1 \geq -\sum_{j=1}^{b}(b+1-j)(\alpha_i^j + \alpha_i^{-j})$. We sum the inequalities for $i = 1, \ldots, g$. Since the coefficients of $\mathcal{L}(u)$ are real we note that $\sum_{i=1}^{g}\alpha_i^j = \sum_{i=1}^{g}\alpha_i^{-j}$. From (5.3) we see that $\Lambda_j = -q^{dj}\sum_{i=1}^{g}\rho_i^j$. Putting it together we get :

$$g(b+1) \geq 2\sum_{j=1}^{b}(b+1-j)\Lambda_j q^{-\frac{dj}{2}}.$$

Now, we let vary $L_i(s)$ so that $g_i \to \infty$ and obtain the stated inequality. $\square$

Unfortunately, we are unable to say anything more in general without the knowledge of some additional properties of λ_j. However, the next corollary shows that sometimes we can do better.

Corollary 5.4.2. *If a family $\{L_i(s)\}$ is asymptotically exact then*

$$\sum_{j=1}^{\infty} \lambda_j q^{-\frac{dj}{2}} \leq \frac{1}{2},$$

provided one of the following conditions holds :

1. *either it is asymptotically very exact or*
2. $\lambda_j \geq 0$ *for any* j.

Proof. To prove the statement of the corollary under the first condition we choose an $\epsilon > 0$ and $b' \in \mathbb{N}$ such that the sum $\sum_{j=b'+1}^{\infty} |\lambda_j| q^{-\frac{dj}{2}} < \epsilon$. Then we choose b'' such that $\frac{b'}{b''+1} < \epsilon$. Now we apply the inequality from theorem 5.4.1 with $b = b''$. We get :

$$\frac{1}{2} \geq \sum_{j=1}^{b''} \left(1 - \frac{j}{b''+1}\right) \lambda_j q^{-\frac{dj}{2}} \geq \sum_{j=1}^{b'} \left(1 - \frac{j}{b''+1}\right) \lambda_j q^{-\frac{dj}{2}} +$$

$$+ \sum_{j=b'+1}^{b''} \left(1 - \frac{j}{b''+1}\right) \lambda_j q^{-\frac{dj}{2}} \geq (1-\epsilon) \sum_{j=1}^{\infty} \lambda_j q^{-\frac{dj}{2}} - 2\epsilon.$$

So the first part of the corollary is true.

To prove the statement under the second condition we use the same trick. We take $b' \in \mathbb{N}$ such that $\frac{b}{b'+1} < \epsilon$. Then we apply theorem 5.4.1 with $b = b'$ and notice that the sum only decreases when we drop the part $\sum_{j=b+1}^{b'} \left(1 - \frac{j}{b'+1}\right) \lambda_j q^{-\frac{dj}{2}}$ since $\lambda_j \geq 0$. This gives the second part of the corollary. $\square$

Remark 5.4.1. We notice that the corollary implies that any asymptotically exact family satisfying $\lambda_j \geq 0$ for any j is asymptotically very exact. This and the statement of the corollary are still true if assume that $\lambda_j \geq 0$ for all but a finite number of $j \in \mathbb{N}$.

Remark 5.4.2. The methods from the section 5.6 allow us to prove a little stronger statement. See remark 5.6.2 for details.

5.4.2 Basic inequality for zeta functions

We have noticed before that even in the case of L-functions we do not get complete results unless we assume that our family is asymptotically very exact or all the coefficients λ_f are positive. While working with zeta functions we face the same problem. However, we will deal with it in a different way for no general lower bound on the sums of the type (5.6) seems to be available and such a lower bound would be definitely necessary since zeta functions are products of L-functions both in positive and in negative powers.

Theorem 5.4.3. *Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Then for any real s with $\frac{d_{\mathrm{e}}}{2} < s < \frac{d_{\mathrm{e}}+1}{2}$ we have :*

$$-\sum_{i=0}^{d_{\mathrm{e}}} \frac{\gamma_i}{q^{s-i/2} - w_i} \leq \sum_{j=1}^{\infty} \lambda_j q^{-sj} \leq \sum_{i=0}^{d_{\mathrm{e}}} \frac{\gamma_i}{q^{s-i/2} + w_i}.$$

Proof. First of all, proposition 5.3.1 implies that it is enough to prove the inequality in the case when $\zeta_k(s) = \zeta_{\mathbf{e},k}$ and $d = d_{\mathbf{e}}$.

Let us write the Stark formula from proposition 5.2.3 :

$$\frac{1}{\log q}\frac{\zeta'(s)}{\zeta(s)} = \sum_{i=1}^{n} w_i \sum_{j=1}^{g_i} \frac{1}{q^s \rho_{ij} - 1}.$$

We notice that all the terms are real for real s and

$$R(r,\theta) = \mathrm{Re}\, \frac{re^{i\theta}}{1 - re^{i\theta}} = \frac{r\cos\theta}{1 - 2r\cos\theta + r^2}.$$

Applying this relation we see that

$$\frac{1}{\log q}\frac{\zeta'(s)}{\zeta(s)} = \sum_{i=0}^{d} w_i \sum_{j=1}^{g_i} R(q^{i/2-s}, \theta_{ij}),$$

where $\rho_{kj} = q^{-\frac{k}{2}} e^{i\theta_{kj}}$.

For $0 < r < 1$ we have the bounds on $R(r,\theta)$:

$$-\frac{r}{r+1} \le R(r,\theta) \le \frac{r}{r-1}.$$

From this we deduce that for s with $\frac{d}{2} < s < \frac{d+1}{2}$ the following inequality holds

$$-\sum_{i=0}^{d} \frac{\gamma_i}{q^{s-i/2} - w_i} \le \frac{-1}{\log q}\frac{\zeta'(s)}{\zeta(s)} \le \sum_{i=0}^{d} \frac{\gamma_i}{q^{s-i/2} + w_i}. \tag{5.7}$$

The next step is to use theorem 5.5.2. For any s in the interval $\left(\frac{d}{2}, \frac{d+1}{2}\right)$ it gives that

$$\lim_{k\to\infty} \frac{-1}{g_k \log q} \cdot \frac{\zeta_k'(s)}{\zeta_k(s)} = \sum_{j=1}^{\infty} \lambda_j q^{-\frac{sj}{2}}.$$

Dividing (5.7) by g, passing to the limit and using the previous equality we get the statement of the theorem. □

Corollary 5.4.4. *1. If $w_{d_{\mathbf{e}}} = 1$ and either the family is asymptotically very exact or $\lambda_j \ge 0$ for all j then*

$$\sum_{j=1}^{\infty} \lambda_j q^{-\frac{d_{\mathbf{e}} j}{2}} \le \sum_{i=0}^{d_{\mathbf{e}}} \frac{\gamma_i}{q^{(d_{\mathbf{e}}-i)/2} + w_i}$$

2. If $w_{d_{\mathbf{e}}} = -1$ and either the family is asymptotically very exact or $\lambda_j \le 0$ for all j then

$$-\sum_{i=0}^{d_{\mathbf{e}}} \frac{\gamma_i}{q^{(d_{\mathbf{e}}-i)/2} - w_i} \le \sum_{j=1}^{\infty} \lambda_j q^{-\frac{d_{\mathbf{e}} j}{2}}.$$

Proof. Let us suppose that $w_{d_{\mathbf{e}}} = 1$ (the other case is treated similarly). For an asymptotically very exact family for any $\epsilon > 0$ we can choose $N > 0$ such that $\sum\limits_{j>N}^{\infty} |\lambda_j| q^{-\frac{d_{\mathbf{e}} j}{2}} < \epsilon$. Thus both for a very exact family and a family with $\lambda_j \ge 0$ for all j we have

$$\sum_{j=1}^{N} \lambda_j q^{-sj} \le \sum_{i=0}^{d_{\mathbf{e}}} \frac{\gamma_i}{q^{s-i/2} + w_i} + \epsilon$$

for any real s with $\frac{d_{\mathbf{e}}}{2} < s < \frac{d_{\mathbf{e}}+1}{2}$. Passing to the limit when $s \to \frac{d_{\mathbf{e}}}{2}$ we get the statement of the corollary. □

Remark 5.4.3. As before we see that any asymptotically exact family, such that $w_{d_{\mathrm{e}}}\,\mathrm{sign}(\lambda_j) = 1$ for any j, is asymptotically very exact.

Remark 5.4.4. Though the corollary 5.4.4 implies the corollary 5.4.2, the basic inequality for L-functions given by theorem 5.4.1 is different from the one obtained by application of theorem 5.4.3.

5.4.3 Examples

Example 5.4.1 (Curves over finite fields). For curves over finite fields we obtain once again the classical basic inequality from [85] :

$$\sum_{j=1}^{\infty} 2\lambda_j q^{-\frac{j}{2}} = \sum_{m=1}^{\infty} \frac{m\phi_m}{q^{m/2}-1} \leq 1.$$

Of course, this is not an interesting example, since we used this inequality as our initial motivation.

Example 5.4.2 (Varieties over finite fields). In a similar way, for varieties over finite fields we get the inequality from [51, (8.8)] :

$$\sum_{m=1}^{\infty} \frac{m\phi_m}{q^{(2d-1)m/2}-1} \leq (q^{(2d-1)/2}-1)\left(\frac{\beta_1}{2} + \sum_{2|i} \frac{\beta_i}{q^{(i-1)/2}+1} + \sum_{2\nmid i} \frac{\beta_i}{q^{(i-1)/2}-1}\right).$$

With more efforts one can reprove most (if not all) of the inequalities from [51, (8.8)] in the general context of zeta functions, since the main tools used in [51] are the explicit formulae. However, we don not do it here as for the moment we are unable see any applications it might have to particular examples of zeta functions.

Example 5.4.3 (Elliptic curves over function fields). The case of asymptotically bad families is trivial : we don not obtain any interesting results here since all $\lambda_j = 0$.

Let us consider the base change case. Let us take an asymptotically very exact family of elliptic curves obtained by a base change (by proposition 5.3.7 any family obtained by a base change is asymptotically very exact, provided $\mathrm{char}\,\mathbb{F}_q \neq 2, 3$). We can apply corollary 5.4.2 to obtain that $\sum\limits_{j=1}^{\infty} \lambda_j q^{-j/2} \leq \frac{1}{2}$. Using (5.5), one can rewrite it using $\phi_{v,m}$ as follows :

$$\sum_{v,m} \frac{md_v\phi_{v,m}(\alpha_v^m + \bar{\alpha}_v^m)q^{-md_v}}{1-(\alpha_v^m + \bar{\alpha}_v^m)q^{-md_v}} \leq \frac{\nu+4}{2}$$

(here $\nu = \lim\limits_{i\to\infty} \frac{n_{E_i/K_i}}{g_{K_i}}$).

5.5 Brauer–Siegel type results

5.5.1 Limit zeta functions and the Brauer–Siegel theorem

Our approach to the Brauer–Siegel type results will be based on limit zeta functions.

Definition 5.5.1. *Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Then the corresponding limit zeta function is defined as*

$$\zeta_{\mathbf{lim}}(s) = \exp\left(\sum_{f=1}^{\infty} \frac{\lambda_f}{f}\, q^{-fs}\right).$$

Remark 5.5.1. If $\zeta_k(s) = \zeta_{f_k}(s)$ are associated to some arithmetic or geometric objects f_k we will denote the limit zeta function simply by $\zeta_{\{f_k\}}(s)$.

Remark 5.5.2. The basic inequality from theorem 5.4.3 can be reformulated in terms of $\zeta_{\mathbf{lim}}(s)$ as

$$-\sum_{i=0}^{d_{\mathbf{e}}} \frac{\gamma_i}{q^{s-i/2} - w_i} \leq -\frac{1}{\log q} \frac{\zeta'_{\mathbf{lim}}(s)}{\zeta_{\mathbf{lim}}(s)} \leq \sum_{i=0}^{d_{\mathbf{e}}} \frac{\gamma_i}{q^{s-i/2} + w_i}.$$

Here are the first elementary properties of limit zeta functions :

Proposition 5.5.1. *1. For an asymptotically exact family of zeta functions $\{\zeta_k(s)\}$ the series for $\log \zeta_{\mathbf{lim}}(s)$ is absolutely and uniformly convergent on compacts in the domain $\operatorname{Re} s > \frac{d_{\mathbf{e}}}{2}$, defining an analytic function there.*

2. If a family is asymptotically very exact then $\zeta_{\mathbf{lim}}(s)$ is continuous for $\operatorname{Re} s \geq \frac{d_{\mathbf{e}}}{2}$

3. If for a family we have $\lambda_j \geq 0$ for any j and $w_{d_{\mathbf{e}}} = 1$, then the series for $\log \zeta_{\mathbf{lim}}(s)$ is absolutely and uniformly convergent in the domain $\operatorname{Re} s \geq \frac{d_{\mathbf{e}}}{2} - \delta$ for some $\delta > 0$.

Proof. The first part of the proposition obviously follows from proposition 5.3.1 together with proposition 5.3.3.

By the definition of an asymptotically very exact sequence, the series for $\zeta_{\mathbf{lim}}(s)$ is uniformly and absolutely convergent for $\operatorname{Re} s \geq \frac{d_{\mathbf{e}}}{2}$ so defines a continuous function in this domain. Thus the second part is proven.

To get the third part we apply corollary 5.4.4 to see that our family is asymptotically very exact. Then we use a well known fact that the domain of convergence of a Dirichlet series with positive coefficients is an open half-plane $\operatorname{Re} s > \sigma$. □

It is important to see to which extent limit zeta functions are the limits of the corresponding zeta functions over finite fields. The question is answered by the generalized Brauer–Siegel theorem. Before stating it let us give one more definition :

Definition 5.5.2. *For an asymptotically exact family of zeta functions $\{\zeta_k(s)\}$ the limit $\lim\limits_{k\to\infty} \frac{\log \zeta_k(s)}{g_k}$ is called the Brauer–Siegel ratio of this family.*

Theorem 5.5.2 (The generalized Brauer–Siegel theorem)**.** *For any asymptotically exact family $\{\zeta_k(s)\}$ and any s with $\operatorname{Re} s > \frac{d_{\mathbf{e}}}{2}$ we have*

$$\lim_{k\to\infty} \frac{\log \zeta_{\mathbf{e},k}(s)}{g_k} = \log \zeta_{\mathbf{lim}}(s).$$

If, moreover, $2\operatorname{Re} s \notin \mathbb{Z}$, then

$$\lim_{k\to\infty} \frac{\log \zeta_k(s)}{g_k} = \log \zeta_{\mathbf{lim}}(s).$$

The convergence is uniform in any domain $\frac{d_{\mathbf{e}}}{2} + \epsilon < \operatorname{Re} s < \frac{d_{\mathbf{e}}+1}{2} - \epsilon$, $\epsilon \in \left(0, \frac{1}{2}\right)$.

Proof. To get the first statement we apply proposition 5.3.3 and exchange the limit when $k \to \infty$ and the summation, which is legitimate since the series in question are absolutely and uniformly convergent in a small (but fixed) neighbourhood of s.

To get the second statement we apply proposition 5.3.1, which gives us :

$$\lim_{k\to\infty} \frac{\log \zeta_{\mathbf{n},k}(s)}{g_k} = 0.$$

Now the second part of the theorem follows from the first. □

Remark 5.5.3. It might be unclear, why we call such a statement the Brauer–Siegel theorem. We will see below in subsection 5.5.3 that the above theorem indeed implies a natural analogue of the Brauer–Siegel theorem for curves and varieties over finite fields. It is quite remarkable that the proof of theorem 5.5.2 is very easy once one gives proper definitions.

Remark 5.5.4. Let us sketch another way to prove the generalized Brauer–Siegel theorem. It might seem unnecessarily complicated but it has the advantage of being applicable in the number field case when we no longer have the convergence of $\log L_k(s)$ for $\operatorname{Re} s > \frac{d}{2}$. We will deal with L-functions to simplify the notation. The main idea is to prove using Stark formula (proposition 5.2.3 in the case of L-functions over finite fields) that $\frac{L_k'(s)}{L_k(s)} \leq C(\epsilon) g_k$ for any s with $\operatorname{Re} s \geq \frac{d}{2} + \epsilon$. Then we apply the Vitali theorem from complex analysis, which states that for a sequence of bounded holomorphic functions in a domain $\mathcal{D}$ it is enough to check the convergence at a set of points in $\mathcal{D}$ with a limit point in $\mathcal{D}$.

Remark 5.5.5. It is natural to ask, what is the behaviour of limit zeta or L-functions for $\operatorname{Re} s \leq \frac{d_{\mathbf{e}}}{2}$. Unfortunately nice properties of L-functions such as the functional equation or the Riemann hypothesis do not hold for $L_{\mathbf{lim}}(s)$. This can be seen already for families of zeta functions of curves. The point is that the behaviour of $L_{\mathbf{lim}}(s)$ might considerably differ from that of $\lim\limits_{k \to \infty} \frac{\log L_k(s)}{g_k}$ when we pass the critical line.

5.5.2 Behaviour at the central point

It seems reasonable to ask, what is the relation between limit zeta functions and the limits of zeta functions over finite fields on the critical line (that is for $\operatorname{Re} s = \frac{d_{\mathbf{e}}}{2}$). This relation seems to be rather complicated. For example, one can prove that the limit $\lim\limits_{k \to \infty} \frac{1}{g_k} \frac{\zeta_k'(1/2)}{\zeta_k(1/2)}$ is always 1 in families of curves (this can be seen from the functional equation), which is definitely not true for the value $\frac{\zeta_{\mathbf{lim}}'(1/2)}{\zeta_{\mathbf{lim}}(1/2)}$.

However, the knowledge of this relation is important for some arithmetic problems (see the example of elliptic surfaces in the next subsection). The general feeling is that for "good" families the statement of the generalized Brauer–Siegel theorem holds for $s = \frac{d_{\mathbf{e}}}{2}$. There are very few cases when we know it (see section 5.7 for a discussion) and we, actually, can not even formulate this statement as a conjecture, since it is not clear what conditions on L-functions we should impose.

Still, in general one can prove the "easy" inequality. The term is borrowed from the classical Brauer–Siegel theorem from the number field case, where the upper bound is known unconditionally (and is easy to prove) and the lower bound is not proven in general (one has to assume either GRH or a certain normality condition on the number fields in question). This analogy does not go too far though for in the classical Brauer–Siegel theorem we work far from the critical line and here we study the behaviour of zeta functions on the critical line itself.

Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. We define r_k and c_k using the Taylor series expansion

$$\zeta_k(s) = c_k \left(s - \frac{d_{\mathbf{e}}}{2}\right)^{r_k} + O\left(\left(s - \frac{d_{\mathbf{e}}}{2}\right)^{r_k+1}\right).$$

Theorem 5.5.3. *For an asymptotically very exact family of zeta functions $\{\zeta_k(s)\}$ such that $w_{d_{\mathbf{e}}} = 1$ we have :*

$$\lim_{k \to \infty} \frac{\log |c_k|}{g_k} \leq \log \zeta_{\mathbf{lim}}\left(\frac{d_{\mathbf{e}}}{2}\right).$$

Proof. Replacing the family $\{\zeta_k(s)\}$ by the family $\{\zeta_{\mathbf{e},k}(s)\}$ we can assume that $d = d_{\mathbf{e}}$.

Let us write

$$\zeta_k(s) = c_k \left(s - \frac{d}{2}\right)^{r_k} F_k(s),$$

where $F_k(s)$ is an analytic function in the neighbourhood of $s = \frac{d}{2}$ such that $F_k\left(\frac{d}{2}\right) = 1$. Let us put $s = \frac{d}{2} + \theta$, where $\theta > 0$ is a small positive real number. We have

$$\frac{\log \zeta_k(\frac{d}{2} + \theta)}{g_k} = \frac{\log c_k}{g_k} + r_k \frac{\log \theta}{g_k} + \frac{\log F_k(\frac{d}{2} + \theta)}{g_k}.$$

To prove the theorem we will construct a sequence θ_k such that

1. $\frac{1}{g_k} \log \zeta_k\left(\frac{d}{2} + \theta_k\right) \to \log \zeta_{\mathbf{lim}}\left(\frac{d}{2}\right)$;
2. $\frac{r_k}{g_k} \log \theta_k \to 0$;
3. $\liminf \frac{1}{g_k} \log F_k\left(\frac{d}{2} + \theta_k\right) \geq 0$.

For each natural number N we choose $\theta(N)$ a decreasing sequence such that

$$\left|\zeta_{\mathbf{lim}}\left(\frac{d}{2}\right) - \zeta_{\mathbf{lim}}\left(\frac{d}{2} + \theta(N)\right)\right| < \frac{1}{2N}.$$

This is possible since $\zeta_{\mathbf{lim}}(s)$ is continuous for $\operatorname{Re} s \geq \frac{d}{2}$ by proposition 5.5.1. Next we choose a sequence $k'(N)$ with the property :

$$\left|\frac{1}{g_k} \log \zeta_k\left(\frac{d}{2} + \theta\right) - \log \zeta_{\mathbf{lim}}\left(\frac{d}{2} + \theta\right)\right| < \frac{1}{2N}$$

for any $\theta \in [\theta(N+1), \theta(N)]$ and any $k \geq k'(N)$. This is possible by theorem 5.5.2. Then we choose $k''(N)$ such that

$$\frac{r_k \log \theta(N+1)}{g_k} \leq \frac{\theta(N)}{N}$$

for any $k \geq k''(N)$, which can be done thanks to corollary 5.6.2 that gives us for an asymptotically very exact family $\frac{r_k}{g_k} \to 0$. Finally, we choose an increasing sequence $k(N)$ such that $k(N) \geq \max(k'(N), k''(N))$ for any N.

Now, if we define $N = N(k)$ by the condition $k(N) \leq k \leq k(N+1)$ and let $\theta_k = \theta(N(k))$, then from the conditions imposed while defining θ_k we automatically get (1) and (2). The delicate point is (3). We apply the Stark formula from proposition 5.2.3 to get an estimate on $\left(\log F_k\left(\frac{d}{2} + \theta\right)\right)'$:

$$\frac{1}{g_k}\left(\log \zeta_k\left(\frac{d}{2} + \theta\right) + r_k \log \theta\right)' = -\frac{\log q}{2g_k} \sum_{i=0}^{d} w_i g_i + \\ + \frac{1}{g_k} \sum_{i=0}^{d-1} w_i \sum_{L_i(\theta_{ij})=0} \frac{1}{\frac{d}{2} + \theta - \theta_{ij}} + \frac{1}{g_k} \sum_{L_d(\theta_{dj})=0} \frac{1}{\frac{d}{2} + \theta - \theta_{dj}}.$$

The first term on the right hand side is clearly bounded by $-\log q$ from below. The first sum involving L-functions is also bounded by a constant C_1 as can be seen applying the Stark formula to individual L-functions and then using proposition 5.3.1. The last sum is non-negative. Thus, we see that $\frac{1}{g_k}\left(\log F_k\left(\frac{d}{2}+\theta\right)\right)' \geq C$ for any small enough θ. From this and from the fact that $F_k\left(\frac{d}{2}\right) = 1$ we deduce that

$$\frac{1}{g_k}\log F_k\left(\frac{d}{2}+\theta_k\right) \geq C\theta_k \to 0.$$

This proves (3) as well as the theorem. $\square$

Remark 5.5.6. The proof of the theorem shows the importance of "low" zeroes of zeta functions (that is zeroes close to $s = \frac{d}{2}$) in the study of the Brauer–Siegel ratio at $s = \frac{d}{2}$. The lack of control of these zeroes is the reason why we can not prove a lower bound on $\lim\limits_{k\to\infty} \frac{\log|c_k|}{g_k}$.

Remark 5.5.7. If we restrict our attention to L-functions with integral coefficients (i. e. such that $\mathcal{L}(u)$ has integral coefficients), then we can see that the ratio $\frac{\log|c_k|}{g_k}$ is bounded from below by $-d\log q$, at least for even d. This follows from a simple observation that if a polynomial with integral coefficients has a non-zero value at an integer point then this value is greater then or equal to one. One may ask whether there is a lower bound for arbitrary d and whether anything similar holds in the number field case.

5.5.3 Examples

Example 5.5.1 (Curves over finite fields). First of all, let us show that the generalized Brauer–Siegel theorem 5.5.2 implies the standard Brauer–Siegel theorem for curves over finite fields from [86].

Let h_X be the number of $\mathbb{F}_q$ rational pints on the Jacobian of X, i. e. $h_X = |\operatorname{Pic}^0_{\mathbb{F}_q}(X)|$.

Corollary 5.5.4. *For an asymptotically exact family of curves $\{X_i\}$ over a finite field $\mathbb{F}_q$ we have :*

$$\lim_{i\to\infty}\frac{\log h_i}{g_i} = \log q + \sum_{f=1}^{\infty}\phi_f \log\frac{q^f}{q^f-1}. \tag{5.8}$$

Proof. It is well known that for a curve X the number h_X can be expressed as $h_X = \mathcal{L}_X(1)$, where $\mathcal{L}_X(u)$ is the numerator of the zeta function of X. Using the functional equation for $\zeta_X(s)$ we see that this expression is equal to $L_X(0) = L_X(1) + g\log q$.

The right hand side of (5.8) can be written as $\log q + 2\log\zeta_{\{X_i\}}(1)$, where $\zeta_{\{X_i\}}(s)$ is the limit zeta function (the factor 2 appears from the definition of $\log\zeta_{\{X_i\}}(s)$, in which we divide by $2g$ and not by g). Thus it is enough to prove that

$$\lim_{i\to\infty}\frac{\log L_{X_i}(1)}{2g_i} = \log\zeta_{\{X_i\}}(1).$$

This follows immediately from the first equality of theorem 5.5.2. $\square$

Using nearly the same proof we can obtain one more statement about the asymptotic behaviour of invariants of function fields. To formulate it we will need to define the so called Euler–Kronecker constants of a curve X (see [41]) :

Definition 5.5.3. *Let X be a curve over a finite field $\mathbb{F}_q$ and let*

$$\frac{\zeta_X'(s)}{\zeta_X(s)} = -(s-1)^{-1} + \gamma_X^0 + \gamma_X^1(s-1) + \gamma_X^2(s-1)^2 + \dots$$

be the Taylor series expansion of $\frac{\zeta_X'(s)}{\zeta_X(s)}$ at $s=1$. Then $\gamma_X = \gamma_X^0$ is called the Euler–Kronecker constant of X and γ_X^k, $k \geq 1$ are be called the higher Euler-Kronecker constants.

We also define the asymptotic Euler-Kronecker constants $\gamma^k_{\{X_i\}}$ from :

$$\frac{\zeta'_{\{X_i\}}(s)}{\zeta_{\{X_i\}}(s)} = \gamma^0_{\{X_i\}} + \gamma^1_{\{X_i\}}(s-1) + \gamma^2_{\{X_i\}}(s-1)^2 + \dots$$

($\zeta_{\{X_i\}}(s)$ is holomorphic and non-zero at $s = 1$ so its logarithmic derivative has no pole at this point).

The following result generalizes theorem 2 from [41] :

Corollary 5.5.5. *For an asymptotically exact family of curves $\{X_i\}$ we have*

$$\lim_{i\to\infty} \frac{\gamma^k_i}{g_i} = \gamma^k_{\{X_i\}}$$

for any positive integer k. In particular,

$$\lim_{i\to\infty} \frac{\gamma_i}{g_i} = -\sum_{f=1}^{\infty} \frac{\phi_f f \log q}{q^f - 1}.$$

Proof. . We apply the first equality from theorem 5.5.2. Using the explicit expression for the negligible part of zetas as $(1 - q^{-s})(1 - q^{1-s})$, we see that

$$\lim_{i\to\infty} \frac{\log \zeta_{X_i}(s)}{2g_i} = \log \zeta_{\{X_i\}}(s)$$

for any s, such that $\operatorname{Re} s > \frac{1}{2}$ and $s \neq 1 + \frac{2\pi k}{\log q}$, $k \in \mathbb{Z}$ and the convergence is uniform in $a < |s - 1| < b$ for small enough a and b. We take the derivative of both sides and use the Cauchy integral formula to get the statement of the corollary. □

Remark 5.5.8. It seems not completely uninteresting to study the behaviour of γ^k_X "on the finite level", i.e. to try to obtain bounds on γ^k_X for an individual curve X. This was done in [41] for γ_X. In the general case the explicit version of the generalized Brauer–Siegel theorem from chapter 2 might be useful.

Remark 5.5.9. It is worth noting that the above corollaries describe the relation between $\log \zeta_{X_i}(s)$ and $\log \zeta_{\{X_i\}}(s)$ near the point $s = 1$. The original statement of theorem 5.5.2 is stronger since it gives this relation for all s with $\operatorname{Re} s > \frac{1}{2}$.

Example 5.5.2 (Varieties over finite fields). Just as for curves, for varieties over finite fields we can get similar corollaries concerning the asymptotic behaviour of $\zeta_{X_i}(s)$ close to $s = d$. We give just the statements, since the proofs are nearly the same as before.

The following result is the Brauer–Siegel theorem for varieties proven in [92].

Corollary 5.5.6. *For an asymptotically exact family of varieties $\{X_i\}$ over a finite field $\mathbb{F}_q$ we have :*

$$\lim_{i\to\infty} \frac{\log |\kappa_i|}{b(X_i)} = \sum_{f=1}^{\infty} \phi_f \log \frac{q^{fd}}{q^{fd} - 1},$$

where $\kappa_i = \operatorname{Res}_{s=d} \zeta_{X_i}(s)$.

In the next corollary we use the same definition of the Euler–Kronecker constants for varieties over finite fields as in the previous example for curves :

Corollary 5.5.7. *For an asymptotically exact family of varieties $\{X_i\}$ we have $\lim\limits_{i\to\infty} \frac{\gamma_i^k}{b(X_i)} = \gamma_{\{X_i\}}^k$ for any k. In particular, $\lim\limits_{i\to\infty} \frac{\gamma_i}{b(X_i)} = -\sum\limits_{f=1}^{\infty} \frac{\phi_f f \log q}{q^{fd}-1}$.*

Example 5.5.3 (Elliptic curves over function fields). Let us recall first the Brauer–Siegel type conjectures for elliptic curves over function fields due to Hindry–Pacheko [32] and Kunyavskii–Tsfasman [49].

For an elliptic curve E/K, $K = \mathbb{F}_q(X)$ we define $c_{E/K}$ and $r_{E/K}$ from $L_{E/K}(s) = c_{E/K}(s-1)^{r_{E/K}} + o\left((s-1)^{r_{E/K}}\right)$. The invariants $r_{E/K}$ and $c_{E/K}$ are important from the arithmetical point of view, since the Birch and Swinnerton-Dyer conjecture predicts that $r_{E/K}$ is equal to the rank of the Mordell–Weil group of E/K and $c_{E/K}$ can be expressed via the order of the Shafarevich–Tate group, the covolume of the Mordell–Weil lattice (the regulator) and some other quantities related to E/K which are easier to control.

Conjecture 5.5.8 (Hindry–Pacheko). *Let E_i run through a family of pairwise non-isomorphic elliptic curves over a fixed function field K. Then*

$$\lim_{i\to\infty} \frac{\log |c_{E_i/K}|}{h(E_i)} = 0,$$

where $h(E_i)$ is the logarithmic height of E_i.

Remark 5.5.10. We could have divided $\log |c_{E_i/K}|$ by n_{E_i} in the statement of the above conjecture since $h(E_i)$ and n_{E_i} have essentially the same order of growth.

Conjecture 5.5.9 (Kunyavskii–Tsfasman). *For a family of elliptic curves $\{E_i/K_i\}$ obtained by a base change we have :*

$$\lim_{i\to\infty} \frac{\log |c_{E_i/K_i}|}{g_{K_i}} = -\sum_{v\in X, f\geq 1} \phi_{v,f} \log \frac{|E_v(\mathbb{F}_{\mathrm{N}v^f})|}{\mathrm{N}v^f},$$

where $E = E_0, X = X_0$.

One can see that the above conjectures are both the statements of the type considered in the subsection 5.5.2. It is quite obvious for the first conjecture and for the second conjecture we have to use the explicit expression for the limit L-functions :

$$\log L_{\{E_i/K_i\}}(s) = -\frac{1}{\nu+4} \sum_{v,f} \phi_{v,f} \log \left(1 - (\alpha_v^f + \bar{\alpha}_v^f)\mathrm{N}v^{-fs} + \mathrm{N}v^{f(1-2s)}\right).$$

One can unify these two conjectures as follows :

Conjecture 5.5.10. *For an asymptotically very exact family of elliptic curves over function fields $\{E_i/K_i\}$ we have :*

$$\lim_{i\to\infty} \frac{\log |c_{E_i/K_i}|}{g_i} = \log L_{\{E_i/K_i\}}(1),$$

where g_i is the degree of $L_{E_i/K_i}(s)$.

Theorems 5.5.2 and 5.5.3 imply the following result in the direction of the above conjectures :

Theorem 5.5.11. *For an asymptotically very exact family of elliptic curves $\{E_i/K_i\}$ the following identity holds :*

$$\lim_{i\to\infty} \frac{\log L_{E_i/K_i}(s)}{g_i} = \log L_{\{E_i/K_i\}}(s),$$

for $\operatorname{Re} s > 1$ *(here $g_i = n_{E_i} + 4g_{K_i} - 4$). Moreover,*

$$\lim_{i\to\infty} \frac{\log |c_{E_i/K_i}|}{g_i} \leq \log L_{\{E_i/K_i\}}(1).$$

Remark 5.5.11. If we consider split families of elliptic curves (i.e. $E_i = E \times X_i$, where $E/\mathbb{F}_q$ is a fixed elliptic curve) then the proof of theorem 2.1 from [49] gives us that the question about the behaviour of $L_{E_i/X_i}(s)$ at $s = 1$ translates into the same question concerning the behaviour of $\zeta_{X_i}(s)$ on the critical line. More precisely,

Proposition 5.5.12. *Let $\psi = q^{1/2+i\theta}, \bar{\psi} = q^{1/2-i\theta}$ be the eigenvalues of the Frobenius acting on $H^1(E)$. Then conjecture 5.5.10 holds provided that $\lim\limits_{i\to\infty} \frac{\log |\zeta_{X_i}(\pm\theta)|}{g_i} = \log \zeta_{\{X_i\}}(\pm\theta)$ (where $\zeta_{X_i}(\theta)$ is understood as the first non-zero coefficient in the Taylor expansion at $s = \theta$). The above statements are equivalent if the curve E is supersingular, i.e. $\theta = 0$.*

So, to prove the simplest case of conjecture 5.5.10 we have to understand the asymptotic behaviour of zeta functions of curves over finite fields on the critical line. Unfortunately, this seems to be inaccessible at the moment.

5.6 Distribution of zeroes

5.6.1 Main results

In this section we will prove certain results about the limit distribution of zeroes in families of L-functions. As a corollary we will see that the multiplicities of zeroes in asymptotically very exact families of L-functions can not grow too fast.

Let $C = C[-\pi, \pi]$ be the space of real continuous functions on $[-\pi, \pi]$ with topology of uniform convergence. The space of measures μ on $[-\pi, \pi]$ is by definition the space $\mathcal{M}$, which is topologically dual to C. The topology on $\mathcal{M}$ is the $*$-weak one : $\mu_i \to \mu$ if and only if $\mu_i(f) \to \mu(f)$ for any $f \in C$.

The space C can be considered as a subspace of $\mathcal{M}$: if $\phi(x) \in C$ then $\mu_\phi(f) = \int_{-\pi}^{\pi} f(x)\phi(x)\,dx$. The subspace C is dense in $\mathcal{M}$ in $*$-weak topology.

Let $L(s)$ be an L-function and let $\rho_1, \ldots, \rho_g$ be the zeroes of the corresponding polynomial $\mathcal{L}(u)$. Define $\theta_k \in (-\pi, \pi]$ by $\rho_k = q^{-d/2}e^{i\theta_k}$. One can associate a measure to $L(s)$:

$$\mu_L(f) = \frac{1}{g}\sum_{k=1}^{g} \delta_{\theta_k}(f), \tag{5.9}$$

where δ_{θ_k} is the Dirac measure supported at θ_k, i.e. $\delta_{\theta_k}(f) = f(\theta_k)$ for an $f \in C$.

The main result of this section is the following one :

Theorem 5.6.1. *Let $\{L_j(s)\}$ be an asymptotically very exact family of L-functions. Then the limit $\mathcal{M}_{\mathbf{lim}} = \lim\limits_{j\to\infty} \mathcal{M}_j$ exists. Moreover, $\mathcal{M}_{\mathbf{lim}}$ is a nonnegative continuous function given by an absolutely and uniformly convergent series :*

$$\mathcal{M}_{\mathbf{lim}}(x) = 1 - 2\sum_{k=1}^{\infty} \lambda_k \cos(kx) q^{-\frac{dk}{2}}.$$

Proof. The absolute and uniform convergence of the series follows from the definition of an asymptotically very exact family. It is sufficient to prove the convergence of measures on the space C.

The linear combinations of $\cos(mx)$ and $\sin(mx)$ are dense in the space of continuous functions C, so it is enough to prove that for any $m = 0, 1, 2, \ldots$ we have :

$$\lim_{j\to\infty} \mathcal{M}_j(\cos(mx)) = \mathcal{M}_{\mathbf{lim}}(\cos(mx)), \tag{5.10}$$

and

$$\lim_{j\to\infty} \mathcal{M}_j(\sin(mx)) = \mathcal{M}_{\mathbf{lim}}(\sin(mx)). \tag{5.11}$$

The corollary 5.2.2 shows that :

$$\mathcal{M}_j(\cos(mx)) = \sum_{k=1}^{g_j} \cos(m\theta_{kj}) = -2\Lambda_m q^{-\frac{dm}{2}}$$

for $m \neq 0$ and $\mathcal{M}_j(1) = g_j$. Dividing by g_j and passing to the limit when $j \to \infty$ we get (5.10).

Now, we note, that if $\rho = e^{i\theta}$, with $\theta \neq k\pi$ is a zero of $\mathcal{L}(u)$ then $\rho = e^{i(\theta+\pi)}$ is also a zero of $\mathcal{L}(u)$ with the same multiplicity. Thus $\mathcal{M}_j(\sin(mx)) = 0 = \mathcal{M}_{\mathbf{lim}}(\sin(mx))$ for any j and m. So we get (5.11) and the theorem is proven. □

Corollary 5.6.2. *Let $\{\zeta_j(s)\}$ be an asymptotically very exact family of zeta functions with $w_{d_{\mathbf{e}}} = 1$ and let r_j be the order of zero of $\zeta_j(s)$ at $s = \frac{d_{\mathbf{e}}}{2}$. Then*

$$\lim_{j\to\infty} \frac{r_j}{g_j} = 0.$$

Proof. Suppose that $\limsup \frac{r_j}{g_j} = \epsilon > 0$. Taking a subsequence we can assume that $\lim\limits_{j\to\infty} \frac{r_j}{g_j} = \epsilon$. Taking a subsequence once again and using proposition 5.3.4 we can assume that we are working with an asymptotically very exact sequence of L-functions $\{L_j(s)\} = \{L_{jd_{\mathbf{e}}}(s)\}$ for which the same property concerning r_j holds.

By the previous theorem $\lim\limits_{j\to\infty} \mathcal{M}_j = \mathcal{M}_{\mathbf{lim}}$. Let us take an even continuous non-negative function $f(x) \in C[-\pi, \pi]$ with the support contained in $(-\frac{\epsilon}{\alpha}, \frac{\epsilon}{\alpha})$, where $\alpha = 4\max\{\int_{-\pi}^{\pi} \mathcal{M}_{\mathbf{lim}}(x)\, dx, 1\}$ and such that $f(0) = 1$. We see that

$$\epsilon \leq \lim_{j\to\infty} \mathcal{M}_j(f(x)) = \int_{-\pi}^{\pi} f(x)\mathcal{M}_{\mathbf{lim}}(x)\, dx \leq \frac{\epsilon}{2},$$

so we get a contradiction. Thus the theorem is proven. □

Remark 5.6.1. It is easy to see that the same proof gives that the multiplicity of the zero at any particular point of the critical line grows asymptotically slower than g.

Remark 5.6.2. Using theorem 5.6.1 one can give another proof of the basic inequality for asymptotically very exact families of L-functions(corollary 5.4.2). Indeed all the measures defined by (5.9) are non-negative. Thus the limit measure $\mathcal{M}_{\mathbf{lim}}$ must have a non-negative density at any point, in particular at $x = 0$. This gives us exactly the basic inequality. In this way we get an interpretation of the difference between the right hand side and the left hand side of the basic inequality as "the asymptotic number of zeroes of $L_j(s)$, accumulating at $s = \frac{d}{2}$".

In fact, using the same reasoning as before, we get a family of inequalities (which are interesting when not all the coefficients λ_f are positive) :

$$\sum_{k=1}^{\infty} \lambda_k \cos(kx) q^{-\frac{dk}{2}} \leq \frac{1}{2}$$

for any $x \in \mathbb{R}$.

5.6.2 Examples

Example 5.6.1 (Curves over finite fields). In the case of curves over finite fields we recover the theorem 2.1 from [86] :

Corollary 5.6.3. *For an asymptotically exact family $\{X_i\}$ of curves over a finite field $\mathbb{F}_q$ the limit $\mathcal{M}_{\{X_i\}} = \lim\limits_{i\to\infty} \mathcal{M}_{X_i}$ is a continuous function given by an absolutely and uniformly convergent series :*

$$\mathcal{M}_{\{X_i\}}(x) = 1 - \sum_{k=1}^{\infty} k\phi_k h_k(x),$$

where

$$h_k(x) = \frac{q^{k/2}\cos(kx) - 1}{q^k + 1 - 2q^{k/2}\cos(kx)}.$$

Proof. This follows from theorem 5.6.1 together with the following series expansion :

$$\sum_{l=1}^{\infty} t^{-l}\cos(lkx) = \frac{t\cos(kx) - 1}{t^2 + 1 - 2t\cos(kx)}.$$

□

Example 5.6.2 (Varieties over finite fields). We can not say much in this case since the zero distribution theorem 5.6.1 applies only to L-functions. The only thing we get is that the multiplicity of zeroes on the line $\operatorname{Re} s = d - \frac{1}{2}$ divided by the sum of Betti numbers tends to zero (corollary 5.6.2).

Example 5.6.3 (Elliptic curves over function fields). Let us consider first asymptotically bad families of elliptic curves. We have the following corollary of theorem 5.6.1.

Corollary 5.6.4. *For an asymptotically bad family of elliptic curves $\{E_i\}$ over function fields the zeroes of $L_{E_i}(s)$ become uniformly distributed on the critical line when $i \to \infty$.*

This result in the particular case of elliptic curves over the fixed field $\mathbb{F}_q(t)$ was obtained in [64]. In fact, unlike us, Michel gives an estimate for the difference $\mathcal{M}_i - \mathcal{M}_{\{E_i\}}$ in terms of the conductor n_{E_i}. It would be interesting to have such a bound in general.

Corollary 5.6.5. *For an asymptotically very good family of elliptic curves $\{E_i/K_i\}$ obtained by a base change the limit $\mathcal{M}_{\{E_i/K_i\}} = \lim\limits_{i\to\infty} \mathcal{M}_{E_i/K_i}$ is a continuous function given by an absolutely and uniformly convergent series :*

$$\mathcal{M}_{\{E_i/K_i\}}(x) = 1 - \frac{2}{\nu + 4} \sum_{v,f} \phi_{v,f} f d_v \sum_{k=1}^{\infty} \frac{\alpha_v^k + \bar{\alpha}_v^k}{q^{f d_v k}} \cos(f d_v k x).$$

Corollary 5.6.6. *For a family of elliptic curves $\{E_i/K_i\}$ obtained by a base change*

$$\lim_{i\to\infty} \frac{r_i}{g_i} = 0.$$

Proof. By proposition 5.3.7 any such family contains an asymptotically exact subfamily so we can apply corollary 5.6.2. □

Remark 5.6.3. For a fixed field K and elliptic curves over it a similar statement can be deduced from the bounds in [7]. However, in the case of the base change Brumer's bounds do not imply corollary 5.6.2. It would be interesting to see, what bounds one can get for the ranks of individual elliptic curves when we vary the ground field K. Getting such a bound should be possible with a proper choice of a test function in the explicit formulae.

5.7 Open questions and further research directions

In this section we would like to gather together the questions which naturally arose in the previous sections. Let us start with some general questions. First of all :

Question 5.7.1. To which extent the formal zeta and L-functions defined in section 5.2 come from geometry ?

One can make it precise in several ways. For example, it is possible to ask whether any L-function of weight d, such that $\mathcal{L}(u)$ has integral coefficients is indeed the characteristic polynomial of the Frobenius automorphism acting on the d-th cohomology group of some variety $V/\mathbb{F}_q$. A partial answer to this question when $d = 1$ is provided by the Honda–Tate theorem on abelian varieties [81]. The same question can be asked about motives over $\mathbb{F}_q$. This is similar to what is conjectured about L-functions from the so called Selberg class in the number field case (modularity, Galois representations side, etc.) [83].

Question 5.7.2. Describe the set $\{(\lambda_1, \lambda_2, \dots)\}$ for asymptotically exact (very exact) families of zeta functions (L-functions).

There are definitely some restrictions on this set, namely those given by various basic inequalities (theorems 5.4.1 and 5.4.3, corollary 5.6.2). It would be interesting to see whether there are any others. We emphasize that the problem is not of arithmetic nature since we do not assume that the coefficients of polynomials, corresponding to L-functions, are integral. It would be interesting to see what additional restrictions the integrality condition on the coefficients of $\mathcal{L}(u)$ might give. Note that, using geometric constructions, Tsfasman and Vlăduţ [86] proved that the families satisfying $\lambda_f \geq 0$ for any f and the basic inequality are all realized when q is a square and $d = 1$. This implies the same statement for formal L-functions and any q and d. However, our new L-function might no longer have integral coefficients.

Question 5.7.3. How many asymptotically good (very good) families are there among all asymptotically exact (very exact) families ?

The "how many" part of the question should definitely be made more precise. One way to do this is to consider the set V_g of the vectors of coefficients of polynomials corresponding to L-functions of degree g and its subset $V_g(f, a, b)$ consisting of the vectors of coefficients of polynomials corresponding to L-functions with $a < \frac{\Lambda_f}{g} < b$. A natural question is whether the ratio of the volume of $V_g(f, a, b)$ to the volume of V_g has a limit when $g \to \infty$ and what this limit is. See [13] for some information about V_g. The question is partly justified by the fact that it is difficult to construct asymptotically good families of curves. We would definitely like to know why.

Let us now ask some questions concerning the concrete results on zeta and L-functions proven in the previous sections.

Question 5.7.4. Is it true that the generalized Brauer–Siegel theorem 5.5.2 holds on the line $\operatorname{Re} s = \frac{1}{2}$ for some (most) asymptotically very exact families ?

It is probable that without the additional arithmetic conditions on the family the statement does not hold. The most interesting families here are the families of elliptic curves over function fields considered in subsection 5.5.3 due to the arithmetic applications. For the moment the author is not aware of the existence of a single family of geometric origin for which we know the result. One might try to look at particular examples of families of curves over finite fields where the zeta function is more or less explicitly known. These include the Fermat curves [46] and the Jacobi curves [48].

The only examples we know that support the conjecture come from the number field case. It is known that there exists a sequence $\{d_i\}$ in $\mathbb{N}$ of density at least $\frac{1}{2}$ such that

$$\lim_{i\to\infty} \frac{\log \zeta_{\mathbb{Q}(\sqrt{d_i})}(\frac{1}{2})}{\log d_i} = 0$$

(cf. [43]). The techniques of the evaluation of mollified moments of Dirichlet L- functions used in that paper is rather involved. It would be interesting to know whether one can obtain analogous results in the function field case. The related questions in the function field case are studied in [46]. It is not clear whether the results on the one level densities obtained there can be applied to the question of finding a lower bound on $\frac{\log|c_i|}{g_i}$ for some positive proportion of fields (both in the number field and in the function field cases).

Question 5.7.5. Prove the generalized Brauer–Siegel theorem 5.5.2 with an explicit error term.

This was done for curves over finite fields in chapter 2 and looks quite feasible in general. It is also worth looking at particular applications that such a result might have, in particular one may ask what bounds on the Euler–Kronecker constants it gives.

Question 5.7.6. How to characterize measures corresponding to asymptotically very exact families ?

This was done in [86] for families such that $\lambda_f \geq 0$ for all f. The general case remains open.

Question 5.7.7. Estimate the error term in theorem 5.6.1.

As it was mentioned before, in the case of elliptic curves over $\mathbb{F}_q(t)$ the estimates were carried out in [64].

Question 5.7.8. Find explicit bounds on the orders r of zeroes of L-functions on the line $\operatorname{Re} s = \frac{d}{2}$.

The corollary 5.6.2 gives that the ratio $\frac{r_i}{g_i} \to 0$ for asymptotically very exact families (here r_i is the multiplicity of the zero). In a particular case of elliptic curves over a fixed function field Brumer in [7] gives a bound which grows asymptotically slower than the conductor. Using explicit formulae with a proper choice of test functions, it should be possible to give such upper bounds for families obtained by a base change if not in general.

Let us finally ask a few more general questions.

Question 5.7.9. How can one apply the results of this chapter to get the information about the arithmetic or geometric properties of the objects to which L-functions are associated ?

We carried out this task (to a certain extent) in the case of curves and varieties over finite fields and elliptic curves over function fields. Additional examples are more than welcome.

The last but not least :

Question 5.7.10. What are the number field analogues of the results obtained in this chapter ?

It seems that most of the results can be generalized to the framework of the Selberg class (as described, for example, in [42, Chapter 5]), subject to imposing some additional hypothesis (such as the Generalized Riemann Hypothesis, the Generalized Ramanujan Conjectures, etc.). Of course, one will have to overcome quite a lot of analytical difficulties on the way (compare, for example, [86] and [87]).

We hope to return to this interesting and promising subject later on.

Deuxième partie

Variétés abéliennes de dimension 3

Chapitre 6

Jacobians among abelian threefolds : a formula of Klein and a question of Serre (with G. Lachaud and C. Ritzenthaler)

6.1 Introduction

6.1.1 Torelli theorem

Let k be an algebraically closed field and $g \geq 1$ be an integer. If X is a (nonsingular irreducible projective) curve of genus g over k, Torelli's theorem states that the map $X \mapsto (\operatorname{Jac} X, j)$, associating to X its Jacobian together with the canonical polarization j, is injective. The determination of the image of this map is a long time studied question.

When $g = 3$, the moduli space A_g of principally polarized abelian varieties of dimension g and the moduli space M_g of nonsingular algebraic curves of genus g are both of dimension $g(g+1)/2 = 3g - 3 = 6$. According to Hoyt [34] and Oort and Ueno [69], the image of M_3 is exactly the space of indecomposable principally polarized abelian threefolds. Moreover if $k = \mathbb{C}$, Igusa [39] characterized the locus of decomposable abelian threefolds and that of hyperelliptic Jacobians, making use of two particular modular forms Σ_{140} and χ_{18} on the Siegel upper half space of degree 3. A similar characterization also exists in case of $g = 2$ (cf. [58]).

Assume now that k is any field and $g \geq 1$. J.-P. Serre noticed in [55] that a precise form of Torelli's theorem reveals a mysterious obstruction for a geometric Jacobian to be a Jacobian over k. More precisely, he proved the following :

Theorem 6.1.1. *Let (A, a) be a principally polarized abelian variety of dimension $g \geq 1$ over k, and assume that (A, a) is isomorphic over $\overline{k}$ to the Jacobian of a curve X_0 of genus g defined over $\overline{k}$. The following alternative holds :*

1. *If X_0 is hyperelliptic, there is a curve X/k isomorphic to X_0 over $\overline{k}$ such that (A, a) is k-isomorphic to $(\operatorname{Jac} X, j)$.*
2. *If X_0 is not hyperelliptic, there is a curve X/k isomorphic to X_0 over $\overline{k}$, and a quadratic character*

$$\varepsilon : \operatorname{Gal}(k^{sep}/k) \longrightarrow \{\pm 1\}$$

such that the twisted abelian variety $(A, a)_\varepsilon$ is k-isomorphic to $(\operatorname{Jac} X, j)$. The character ε is trivial if and only if (A, a) is k-isomorphic to a Jacobian.

Thus, only case (1) occurs if $g = 1$ or $g = 2$, with all curves being elliptic or hyperelliptic. In this chapter we completely resolve for fields of characteristic zero the first previously unknown case $g = 3$.

6.1.2 Curves of genus 3

Assume now again $g = 3$. Let there be given an indecomposable principally polarized abelian threefold (A, a) defined over k. In a letter to J. Top [75], J.-P. Serre asked a twofold question :

– *How to decide, knowing only (A, a), that X is hyperelliptic ?*

– *If X is not hyperelliptic, how to compute the quadratic character ε ?*

Assume that $k \subset \mathbb{C}$. The first question can easily be answered using the forms Σ_{140} and χ_{18}. As for the second question, roughly speaking, Serre suggested that ε is trivial if and only if χ_{18} is a square in $k^\times$ (see Th.6.4.2 for a more precise formulation). This assertion was motivated by a formula of Klein [47] relating the modular form χ_{18} (in the notation of Igusa) to the square of the discriminant of plane quartics, which more or less gives the 'only if' part of the claim. In [50] Serre's assertion was justified for a three dimensional family of abelian varieties and in particular the absolute constant involved in Klein's formula was determined.

In this chapter we prove that Serre's assertion is valid for any abelian threefold, thus giving an algorithm which allows to determine whether a given principally polarized abelian threefold over k is a Jacobian over k. In order to do so, we start by taking a broader point of view, valid for any $g > 1$.

1. We look at the action of $\overline{k}$-isomorphisms on Siegel modular forms defined over k and we define invariants of k-isomorphism classes of abelian varieties over k.
2. We link Siegel modular forms, Teichmüller modular forms and invariants of plane curves.

Once these two goals are achieved, Serre's assertion can be rephrased as the following strategy :

– use (2) to prove that a certain Siegel modular form f is a suitable n-th power with $n > 1$ on the Jacobian locus ;

– use (1) to distinguish between Jacobians and their twists. Indeed, the action of a twist on f may change its value by a non n-th power and then we have a criterion to distinguish Jacobians according to (2) of Th.6.1.1.

For $g = 3$, Klein's formula shows that the form χ_{18} is a square on the Jacobian locus and that this is enough to characterize this locus. On the other hand, we show that the natural generalization χ_h, $h = 2^{g-2}(2^g + 1)$ no longer gives such a characterization when $g > 3$.
The relevance of Klein's formula in this problem was one of Serre's insights. We would like to point out that we do not actually need the full strength of Klein's formula to work out our strategy. Indeed, we do not go all the way from Siegel modular form to invariants. We use instead a formula due to Ichikawa relating χ_{18} to the square of a Teichmüller modular form, denoted $\mu_{3,9}$. However we think that the connection between Siegel modular forms and invariants is interesting enough in its own, besides the fact that it gives a new rigorous proof of Klein's formula. Note that, if the relation between $\mu_{3,9}$ and χ_{18} is quite present in the literature, the relation between χ_{18} and the discriminant was somehow lost, apart from Serre's remark in the letter to J. Top. It eventually give rise to the question of a direct proof of the relation between $\mu_{3,9}$ and the discriminant (see Rem.6.4.1).

The chapter is organized as follows. In §6.2, we review the necessary elements from the theory of Siegel and Teichmüller modular forms. Only §6.2.4 is original : we introduce the action of isomorphisms and see how the action of twists is reflected on the values of modular forms. In §6.3, we link modular forms and certain invariants of ternary forms. Finally in §6.4 we deal with the case $g = 3$. We first give a proof of Klein's formula and then we justify the validity of Serre's assertion. Finally we explain the reasons behind the failure of the obvious generalization of the theory in higher dimensions and state some natural questions.

6.2 Siegel and Teichmüller modular forms

6.2.1 Geometric Siegel modular forms

The references are [8], [11], [17], [20]. Let $g > 1$ and $n > 0$ be two integers and $\mathsf{A}_{g,n}$ be the moduli stack of principally polarized abelian schemes of relative dimension g with symplectic level n structure. Let $\pi : \mathsf{V}_{g,n} \longrightarrow \mathsf{A}_{g,n}$ be the universal abelian scheme, fitted with the zero section $\varepsilon : \mathsf{A}_{g,n} \longrightarrow \mathsf{V}_{g,n}$, and

$$\pi_* \Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}} = \varepsilon^* \Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}} \longrightarrow \mathsf{A}_{g,n}$$

the rank g bundle induced by the relative regular differential forms of degree one on $\mathsf{V}_{g,n}$ over $\mathsf{A}_{g,n}$. The relative canonical bundle over $\mathsf{A}_{g,n}$ is the line bundle

$$\boldsymbol{\omega} = \bigwedge^{g} \varepsilon^* \Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}}.$$

For a projective nonsingular variety X defined over a field k, we denote by

$$\Omega^1_k[X] = H^0(X, \Omega^1_X \otimes k)$$

the finite dimensional k-vector space of regular differential forms on X defined over k. Hence, the fibre of the bundle $\Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}}$ over $A \in \mathsf{A}_{g,n}(k)$ is equal to $\Omega^1_k[A]$, and the fibre of $\boldsymbol{\omega}$ is the one-dimensional vector space

$$\boldsymbol{\omega}[A] = \bigwedge^{g} \Omega^1_k[A].$$

We put $\mathsf{A}_g = \mathsf{A}_{g,1}$ and $\mathsf{V}_g = \mathsf{V}_{g,1}$. Let R be a commutative ring and h be a positive integer. A *geometric Siegel modular form* of genus g and weight h over R is an element of the R-module

$$\mathbf{S}_{g,h}(R) = \Gamma(\mathsf{A}_g \otimes R, \boldsymbol{\omega}^{\otimes h}).$$

Note that for any $n \geq 1$, we have an isomorphism

$$\mathsf{A}_g \simeq \mathsf{A}_{g,n} / \operatorname{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z}).$$

If $n \geq 3$, as shown in [67], from the rigidity lemma of Serre [73] we can deduce that the moduli space $\mathsf{A}_{g,n}$ can be represented by a smooth scheme over $\mathbb{Z}[\zeta_n, 1/n]$. Hence, for any algebra R over $\mathbb{Z}[\zeta_n, 1/n]$, the module $\mathbf{S}_{g,h}(R)$ is the submodule of

$$\Gamma(\mathsf{A}_{g,n} \otimes_{\mathbb{Z}[\zeta_n, 1/n]} R, \boldsymbol{\omega}^{\otimes h})$$

consisting of the elements invariant under $\operatorname{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$.

Assume now that $R = k$ is a field. If $f \in \mathbf{S}_{g,h}(k)$, A is a principally polarized abelian variety of dimension g defined over k and ω is a basis of $\boldsymbol{\omega}_k[A]$, define

$$f(A, \omega) = f(A)/\omega^{\otimes h}. \tag{6.1}$$

In this way such a modular form defines a rule which assigns the element $f(A, \omega) \in k$ to every such pair (A, ω) and such that :

1. $f(A, \lambda\omega) = \lambda^{-h} f(A, \omega)$ for any $\lambda \in k^\times$.
2. $f(A, \omega)$ depends only on the $\overline{k}$-isomorphism class of the pair (A, ω).

Conversely, such a rule defines a unique $f \in \mathbf{S}_{g,h}(k)$. This definition is a straightforward generalization of that of Deligne-Serre [12] and Katz [45] if $g = 1$.

6.2.2 Complex uniformisation

Assume $R = \mathbb{C}$. Let

$$\mathbb{H}_g = \left\{\tau \in \mathbf{M}_g(\mathbb{C}) \mid {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\right\}$$

be the Siegel upper half space of genus g and $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z})$. As explained in [8, §2], The complex orbifold $\mathsf{A}_g(\mathbb{C})$ can be expressed as the quotient $\Gamma\backslash\mathbb{H}_g$ where Γ acts by

$$M.\tau = (a\tau + b)\cdot(c\tau+d)^{-1} \quad \text{if} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The group $\mathbb{Z}^{2g}$ acts on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$v.(\tau, z) = (\tau, z + \tau m + n) \quad \text{if} \quad v = \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^{2g}.$$

If $\mathbb{U}_g = \mathbb{Z}^{2g}\backslash(\mathbb{H}_g \times \mathbb{C}^g)$, the projection

$$\pi : \mathbb{U}_g \longrightarrow \mathbb{H}_g$$

defines a universal principally polarized abelian variety with fibres

$$A_\tau = \pi^{-1}(\tau) = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g).$$

Let $j(M,\tau) = c\tau + d$ and define the action of Γ on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$M.(\tau, (z_1, \ldots, z_g)) = (M.\tau, {}^tj(M,\tau)^{-1}\cdot(z_1,\ldots,z_g)) \quad \text{if } M \in \Gamma.$$

The map ${}^tj(M,\tau)^{-1} : \mathbb{C}^g \to \mathbb{C}^g$ induces an isomorphism :

$$\varphi_M : A_\tau \longrightarrow A_{M.\tau}.$$

Hence, $\mathsf{V}_g(\mathbb{C}) \simeq \Gamma\backslash\mathbb{U}_g$ and the following diagram is commutative :

$$\begin{array}{ccc} \Gamma\backslash\mathbb{U}_g & \xrightarrow{\sim} & \mathsf{V}_g(\mathbb{C}) \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \Gamma\backslash\mathbb{H}_g & \xrightarrow{\sim} & \mathsf{A}_g(\mathbb{C}) \end{array}$$

As in [17, p. 141], let

$$\zeta = \frac{dq_1}{q_1}\wedge\cdots\wedge\frac{dq_g}{q_g} = (2i\pi)^g dz_1\wedge\cdots\wedge dz_g \in \Gamma(\mathbb{H}_g, \boldsymbol{\omega})$$

with $(z_i, \ldots, z_g) \in \mathbb{C}^g$ and $(q_i, \ldots, q_g) = (e^{2i\pi z_1}, \ldots e^{2i\pi z_g})$. This section of the canonical bundle is a basis of $\boldsymbol{\omega}[A_\tau]$ for all $\tau \in \mathbb{H}_g$ and the relative canonical bundle of $\mathbb{U}_g/\mathbb{H}_g$ is trivialized by ζ :

$$\boldsymbol{\omega}_{\mathbb{U}_g/\mathbb{H}_g} = \bigwedge^g \Omega^1_{\mathbb{U}_g/\mathbb{H}_g} \simeq \mathbb{H}_g \times \mathbb{C}\cdot\zeta.$$

The group Γ acts on $\boldsymbol{\omega}_{\mathbb{U}_g/\mathbb{H}_g}$ by

$$M.(\tau,\zeta) = (M.\tau, \det j(M,\tau)\cdot\zeta) \quad \text{if } M \in \Gamma,$$

in such a way that

$$\varphi_M^*(\zeta_{M.\tau}) = \det j(M,\tau)^{-1}\zeta_\tau.$$

Thus, a geometric Siegel modular form f of weight h becomes an expression

$$f(A_\tau) = \widetilde{f}(\tau) \cdot \zeta^{\otimes h},$$

where $\widetilde{f}$ belongs to the well-known vector space $\mathbf{R}_{g,h}(\mathbb{C})$ of *analytic Siegel modular forms* of weight h on $\mathbb{H}_g$, consisting of complex holomorphic functions $\phi(\tau)$ on $\mathbb{H}_g$ satisfying

$$\phi(M.\tau) = \det j(M.\tau)^h \phi(\tau)$$

for any $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Note that by Koecher principle [20, p. 11], the condition of holomorphy at ∞ is automatically satisfied since $g > 1$. The converse is also true [17, p. 141] :

Proposition 6.2.1. *If $f \in \mathbf{S}_{g,h}(\mathbb{C})$ and $\tau \in \mathbb{H}_g$, let*

$$\widetilde{f}(\tau) = f(A_\tau)/\zeta^{\otimes h} = (2i\pi)^{-gh} f(A_\tau)/(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h}.$$

Then the map $f \mapsto \widetilde{f}$ is an isomorphism $\mathbf{S}_{g,h}(\mathbb{C}) \xrightarrow{\sim} \mathbf{R}_{g,h}(\mathbb{C})$. □

6.2.3 Teichmüller modular forms

Let $g > 1$ and $n > 0$ be positive integers and let $\mathsf{M}_{g,n}$ denote the moduli stack of smooth and proper curves of genus g with symplectic level n structure [11]. Let $\pi : \mathsf{C}_{g,n} \longrightarrow \mathsf{M}_{g,n}$ be the universal curve, and let $\boldsymbol{\lambda}$ be the invertible sheaf associated to the *Hodge bundle*, namely

$$\boldsymbol{\lambda} = \bigwedge^g \pi_* \Omega^1_{\mathsf{C}_{g,n}/\mathsf{M}_{g,n}}.$$

For an algebraically closed field k the fibre over $C \in \mathsf{M}_{g,n}(k)$ is the one dimensional vector space $\boldsymbol{\lambda}[C] = \bigwedge^g \Omega^1_k[C]$.

Let R be a commutative ring and h a positive integer. A *Teichmüller modular form* of genus g and weight h over R is an element of

$$\mathbf{T}_{g,h}(R) = \Gamma(\mathsf{M}_g \otimes R, \boldsymbol{\lambda}^{\otimes h}).$$

These forms have been thoroughly studied by Ichikawa [35], [36], [37], [38]. As in the case of the moduli space of abelian varieties, for any $n \geq 1$ we have

$$\mathsf{M}_g \simeq \mathsf{M}_{g,n}/\mathrm{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z}),$$

and $\mathsf{M}_{g,n}$ can be represented by a smooth scheme over $\mathbb{Z}[\zeta_n, 1/n]$ if $n \geq 3$. Then, for any algebra R over $\mathbb{Z}[\zeta_n, 1/n]$, the module $\mathbf{T}_{g,h}(R)$ is the submodule of

$$\Gamma(\mathsf{M}_{g,n} \otimes_{\mathbb{Z}[\zeta_n, 1/n]} R, \boldsymbol{\lambda}^{\otimes h})$$

invariant under $\mathrm{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$.

Let C/k be a genus g curve. Let $\lambda_1, \ldots, \lambda_g$ be a basis of $\Omega^1_k[C]$ and $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_g$ a basis of $\boldsymbol{\lambda}[C]$. As for Siegel modular forms in (6.1), for a Teichmüller modular form $f \in \mathbf{T}_{g,h}(k)$ we define

$$f(C, \lambda) = f(C)/\lambda^{\otimes h} \in k.$$

Ichikawa asserts the following proposition :

Proposition 6.2.2. *The Torelli map $t : \mathsf{M}_g \longrightarrow \mathsf{A}_g$, associating to a curve C its Jacobian $\mathrm{Jac}\, C$ with the canonical polarization j, satisfies $t^*\boldsymbol{\omega} = \boldsymbol{\lambda}$, and induces for any commutative ring R a linear map*

$$t^* : \mathbf{S}_{g,h}(R) = \Gamma(\mathsf{A}_g \otimes R, \boldsymbol{\omega}^{\otimes h}) \longrightarrow \mathbf{T}_{g,h}(R) = \Gamma(\mathsf{M}_g \otimes R, \boldsymbol{\lambda}^{\otimes h}),$$

such that $[t^* f](C) = t^*[f(\mathrm{Jac}\, C)]$. Fixing a basis λ of $\boldsymbol{\lambda}[C]$, this is

$$f(\mathrm{Jac}\, C, \omega) = [t^* f](C, \lambda) \qquad \text{if } t^*\omega = \lambda.$$

□

6.2.4 Action of isomorphisms

Suppose $\phi : (A', a') \longrightarrow (A, a)$ is a $\overline{k}$-isomorphism of principally polarized abelian varieties. Let $\omega_1, \ldots, \omega_g \in \Omega^1{}_{\overline{k}}[A]$ and $\omega = \omega_1 \wedge \cdots \wedge \omega_g \in \boldsymbol{\omega}[A]$. Then by definition

$$f(A, \omega) = f(A', \gamma)$$

where $\gamma_i = \phi^*(\omega_i)$ is a basis of $\Omega^1{}_{\overline{k}}[A']$ and $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_g \in \boldsymbol{\omega}[A']$. If $\omega'_1, \ldots, \omega'_g$ is another basis of $\Omega^1{}_{\overline{k}}[A']$ and $\omega' = \omega'_1 \wedge \cdots \wedge \omega'_g$, we denote by $M_\phi \in \mathrm{GL}_g(\overline{k})$ the matrix of the basis (γ_i) in the basis (ω'_i). We can easily see that

Proposition 6.2.3. *In the above notation,*

$$f(A, \omega) = \det(M_\phi)^h \cdot f(A', \omega'). \quad \square$$

First of all, from this formula applied to the action of -1, we deduce that, if k is a field of characteristic different from 2, then $\mathbf{S}_{g,h}(k) = \{0\}$ if gh is odd. From now on we assume that gh is even and $\mathrm{char}\, k \neq 2$.

Corollary 6.2.4. *Let (A, a) be a principally polarized abelian variety of dimension g defined over k and $f \in \mathbf{S}_{g,h}(k)$. Let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega^1_k[A]$, and let $\omega = \omega_1 \wedge \cdots \wedge \omega_g \in \boldsymbol{\omega}[A]$. Then the quantity*

$$\bar{f}(A) = f(A, \omega) \bmod^\times k^{\times h} \in k/k^{\times h}$$

does not depend on the choice of the basis of $\Omega^1_k[A]$. In particular $\bar{f}(A)$ is an invariant of the k-isomorphism class of A. $\square$

Corollary 6.2.5. *Assume that g is odd. Let $f \in \mathbf{S}_{g,h}(k)$ and $\phi : A' \longrightarrow A$ a non trivial quadratic twist. There exists $c \in k \setminus k^2$ such that $\bar{f}(A) = c^{h/2} \bar{f}(A')$. Thus, if $\bar{f}(A) \neq 0$ then $\bar{f}(A)$ and $\bar{f}(A')$ do not belong to the same class in $k^\times/k^{\times h}$.*

Proof. Assume that ϕ is given by the quadratic character ε of $\mathrm{Gal}(\overline{k}/k)$. Then

$$d^\sigma = \varepsilon(\sigma)^g \cdot d, \text{ where } \quad d = \det(M_\phi) \in \overline{k}, \quad \sigma \in \mathrm{Gal}(\overline{k}/k).$$

Assume that g is odd. Then by our assumption h is even, and $d^2 = \varepsilon(\sigma) d d^\sigma \in k$. But $d \notin k$ since there exists σ such that $\varepsilon(\sigma) = -1$. Using Prop.6.2.3 we find that

$$f(A, \omega) = (d^2)^{h/2} f(A', \omega').$$

Since d^2 is not a square in k, if $\bar{f}(A) \neq 0$ then $\bar{f}(A)$ and $\bar{f}(A')$ belong to two different classes. $\square$

Let now (A, a) be a principally polarized abelian variety of dimension g defined over $\mathbb{C}$. Let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega^1_{\mathbb{C}}[A]$ and $\omega = \omega_1 \wedge \cdots \wedge \omega_g \in \boldsymbol{\omega}[A]$. Let $\gamma_1, \ldots \gamma_{2g}$ be a symplectic basis (for the polarization a). The period matrix

$$\Omega = [\Omega_1 \ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}$$

belongs to the set $\mathcal{R}_g \subset \mathbf{M}_{g,2g}(\mathbb{C})$ of period matrices, and $\tau = \Omega_2^{-1} \Omega_1 \in \mathbb{H}_g$.

Proposition 6.2.6. *In the above notation,*

$$f(A, \omega) = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h}.$$

Proof. The abelian variety A is isomorphic to $A_\Omega = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ and $\omega \in \boldsymbol{\omega}[A]$ maps to $\xi = dz_1 \wedge \cdots \wedge dz_g \in \boldsymbol{\omega}[A_\Omega]$ under this isomorphism. The linear map $z \mapsto \Omega_2^{-1}z = z'$ induces the isomorphism

$$\varphi : A_\Omega \longrightarrow A_\tau = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g).$$

Let us denote $\xi' = dz'_1 \wedge \cdots \wedge dz'_g = (2i\pi)^{-g}\zeta$ in $\boldsymbol{\omega}[A_\tau]$. Thus, using Prop.6.2.3, Equation (6.1) and Prop.6.2.1, we obtain

$$\begin{aligned} f(A,\omega) = f(A_\Omega,\xi) &= \det \Omega_2^{-h} f(A_\tau,\xi') \\ &= \det \Omega_2^{-h} f(A_\tau)/\xi'^{\otimes h} = (2i\pi)^{gh} \det \Omega_2^{-h} f(\tau)/\zeta^{\otimes h} = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h}, \end{aligned}$$

from which the proposition follows. □

6.3 Invariants and modular forms

Let $d > 0$ be an integer and in this section k is an algebraically closed field of characteristic coprime with d.

6.3.1 Invariants

We review some classical invariant theory. Let E be a vector space of dimension n over k. The left regular representation r of $\mathrm{GL}(E)$ on the vector space $\mathsf{X}_d = \mathrm{Sym}^d(E^*)$ of homogeneous polynomials of degree d on E is given by

$$r(u) : F(x) \mapsto (u \cdot F)(x) = F(ux)$$

for $u \in \mathrm{GL}(E)$, $F \in \mathsf{X}_d$ and $x \in E$. If U is an open subset of X_d stable under r, we still denote by r the left regular representation of $\mathrm{GL}(E)$ on the k-algebra $\mathcal{O}(U)$ of regular functions on U, in such a way that

$$r(u) : \Phi(F) \mapsto (u \cdot \Phi)(F) = \Phi(u \cdot F),$$

if $u \in \mathrm{GL}(E)$, $\Phi \in \mathcal{O}(U)$ and $F \in U$. If $h \in \mathbb{Z}$, we denote by $\mathcal{O}_h(U)$ the subspace of homogeneous elements of degree h, satisfying $\Phi(\lambda F) = \lambda^h \Phi(F)$ for $\lambda \in k^\times$ and $F \in U$. The subspaces $\mathcal{O}_h(U)$ are stable under r. An element $\Phi \in \mathcal{O}_h(U)$ is an *invariant of degree h on U* if

$$u \cdot \Phi = \Phi \quad \text{for every } u \in \mathrm{SL}(E),$$

and we denote by $\mathsf{Inv}_h(U)$ the subspace of $\mathcal{O}_h(U)$ of invariants of degree h on U. If $\mathsf{Inv}_h(U) \neq \{0\}$, then $hd \equiv 0 (\mathrm{mod}\, n)$, since the group $\boldsymbol{\mu}_n$ of n-th roots of unity is in the kernel of r. Hence, if $\Phi \in \mathcal{O}(U)$, and if w and n are two integers such that $hd = nw$, the following statements are equivalent :

1. $\Phi \in \mathsf{Inv}_h(U)$;
2. $u \cdot \Phi = (\det u)^w \Phi \quad$ for every $u \in \mathrm{GL}(E)$.

If these conditions are satisfied, we call w the *weight* of Φ.

The *multivariate resultant* $\mathrm{Res}(f_1, \ldots, f_n)$ of n forms $f_1, \ldots f_n$ in n variables with coefficients in k is an irreducible polynomial in the coefficients of $f_1, \ldots f_n$ which vanishes whenever $f_1, \ldots f_n$ have a common non-zero root. One requires that the resultant is irreducible over $\mathbb{Z}$, *i. e.* it has coefficients in $\mathbb{Z}$ with greatest common divisor equal to 1, and moreover

$$\mathrm{Res}(x_1^{d_1}, \ldots, x_n^{d_n}) = 1$$

for any $(d_1, \ldots, d_n) \in \mathbb{N}^n$. The multivariate resultant exists and is unique. Now, let $F \in \mathsf{X}_d$, and denote $q_1, \ldots, q_n$ the partial derivatives of F. The *discriminant* of F is

$$\operatorname{Disc} F = c_{n,d}^{-1} \operatorname{Res}(q_1, \ldots, q_n), \quad \text{with} \quad c_{n,d} = d^{((d-1)^n - (-1)^n)/d},$$

the coefficient $c_{n,d}$ being chosen according to [75]. Hence, the projective hypersurface which is the zero locus of $F \in \mathsf{X}_d$ is nonsingular if and only if $\operatorname{Disc} F \neq 0$. The discriminant is an irreducible polynomial in the coefficients of F, see for instance [21, Chap. 9, Ex. 1.6(a)]. From now on we restrict ourselves to the case $n = 3$, *i. e.* we consider invariants of ternary forms of degree d, and summarize the results that we shall need.

Proposition 6.3.1. *If $F \in \mathsf{X}_d$ is a ternary form, the discriminant*

$$\operatorname{Disc} F = d^{-(d-1)(d-2)-1} \cdot \operatorname{Res}(q_1, q_2, q_3)$$

where q_1, q_2, q_3 are the partial derivatives of F, is given by an irreducible polynomial over $\mathbb{Z}$ in the coefficients of F, and vanishes if and only if the plane curve C_F defined by F is singular. The discriminant is an invariant of X_d of degree $3(d-1)^2$ and weight $d(d-1)^2$. □

We refer to [21, p. 118] and [50] for an explicit formula for the discriminant, found by Sylvester.

Example 6.3.1 (Ciani quartics). We recall some results whose proofs are given in [50]. Let

$$m = \begin{bmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{bmatrix} \in \operatorname{Sym}_3(k),$$

and for $1 \leq i \leq 3$, let $c_i = a_j a_k - b_i^2$ be the cofactor of a_i. If

$$\det(m) \neq 0, \; a_1 a_2 a_3 \neq 0 \text{ and } c_1 c_2 c_3 \neq 0$$

then

$$F_m(x, y, z) = a_1 x^4 + a_2 y^4 + a_3 z^4 + 2(b_1 y^2 z^2 + b_2 x^2 z^2 + b_3 x^2 y^2)$$

defines a non singular plane quartic. Moreover

$$\operatorname{Disc} F_m = 2^{40}\, a_1\, a_2\, a_3\, (c_1\, c_2\, c_3)^2\, \det(m)^4.$$

Note that the discrepancy between the powers of 2 here and in [50, Prop.2.2.1] comes from the normalization by $c_{n,d}$.

6.3.2 Geometric invariants for plane curves

Let E be a vector space of dimension 3 over k and $G = \operatorname{GL}(E)$. The *universal curve* over the affine space $\mathsf{X}_d = \operatorname{Sym}^d(E)$ is the variety

$$\mathsf{Y}_d = \left\{(F, x) \in \mathsf{X}_d \times \mathbb{P}^2 \;\middle|\; F(x) = 0\right\}.$$

The *nonsingular locus* of X_d is the principal open set

$$\mathsf{X}_d^0 = (\mathsf{X}_d)_{\operatorname{Disc}} = \{F \in \mathsf{X}_d \mid \operatorname{Disc}(F) \neq 0\}.$$

If Y_d^0 is the universal curve over the nonsingular locus X_d^0, the projection is a smooth surjective k-morphism

$$\pi : \mathsf{Y}_d^0 \longrightarrow \mathsf{X}_d^0$$

whose fibre over F is the non singular plane curve C_F.

We recall the classical way to write down an explicit k-basis of $\Omega^1[C_F] = H^0(C_F, \Omega^1)$ for $F \in \mathsf{X}_d^0(k)$ (see [6, p. 630]). Let

$$\eta_1 = \frac{f(x_2dx_3 - x_3dx_2)}{q_1}, \quad \eta_2 = \frac{f(x_3dx_1 - x_1dx_3)}{q_2}, \quad \eta_3 = \frac{f(x_1dx_2 - x_2dx_1)}{q_3},$$

where q_1, q_2, q_3 are the partial derivatives of F, and where f belongs to the space X_{d-3} of ternary forms of degree $d-3$. The forms η_i glue together and define a regular differential form $\eta_f(F) \in \Omega^1[C_F]$. Since $\dim \mathsf{X}_{d-3} = (d-1)(d-2)/2 = g$, the linear map $f \mapsto \eta_f(F)$ defines an isomorphism

$$\mathsf{X}_{d-3} \xrightarrow{\ \sim\ } \Omega^1[C_F].$$

This implies that the sections $\eta_f \in \Gamma(\mathsf{X}_d^0, \pi_*\Omega^1_{\mathsf{Y}_d^0/\mathsf{X}_d^0})$ lead to a trivialization

$$\mathsf{X}_d^0 \times \mathsf{X}_{d-3} \xrightarrow{\ \sim\ } \pi_*\Omega^1_{\mathsf{Y}_d^0/\mathsf{X}_d^0}.$$

We denote $\eta_1, \dots, \eta_g$ the sequence of sections obtained by substituting for f in η_f the g members of the canonical basis of X_{d-3}, enumerated according to the lexicographic order. Then

$$\eta = \eta_1 \wedge \cdots \wedge \eta_g$$

is a section of

$$\boldsymbol{\alpha} = \bigwedge^g \pi_*\Omega^1_{\mathsf{Y}_d^0/\mathsf{X}_d^0},$$

the Hodge bundle of the universal curve over X_d^0.

Since the map $u : x \mapsto ux$ induces an isomorphism

$$u : C_{u\cdot F} \xrightarrow{\ \sim\ } C_F$$

it has a natural action $u^* : \Omega^1[C_F] \to \Omega^1[C_{u\cdot F}]$ on the differentials and hence, on the sections of $\boldsymbol{\alpha}^h$, for $h \in \mathbb{Z}$. More specifically, if $s \in \Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha}^{\otimes h})$, one can write $s = \Phi \cdot \eta^{\otimes h}$ with $\Phi \in \mathcal{O}(\mathsf{X}_d^0)$; for $F \in \mathsf{X}_d^0$, one has

$$u^*s(F) = \Phi(F) \cdot (u^*\eta(F))^{\otimes h}.$$

Lemma 6.3.2. *The section $\eta \in \Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha})$ satisfies for $u \in G$ and $F \in \mathsf{X}_d^0$*

$$u^*\eta(F) = \det(u)^{w_0} \cdot \eta(u \cdot F), \quad \textit{with } w_0 = \binom{d}{3} = \frac{dg}{3} \in \mathbb{N}.$$

Proof. Since $\dim \boldsymbol{\alpha}[F] = \dim \boldsymbol{\alpha}[u \cdot F] = 1$, there is $c(u, F) \in k^\times$ such that

$$u^*\eta(F) = c(u, F) \cdot \eta(u \cdot F).$$

and c is a "crossed character", satisfying

$$c(uu', F) = c(u, F)\, c(u', u \cdot F).$$

Now the regular function $F \mapsto c(u, F)$ does not vanishes on X_d^0. By Lemma 6.3.3 below and the irreducibility of the discriminant (Prop. 6.3.1), we have

$$c(u, F) = \chi(u)(\operatorname{Disc} F)^{n(u)}$$

with $\chi(u) \in k^\times$ and $n(u) \in \mathbb{Z}$. The group G being connected, the function $n(u) = n$ is constant. Since $c(\mathbf{I}_3, F) = 1$, we have $(\operatorname{Disc} F)^n = \chi(\mathbf{I}_3)^{-1}$, and this implies $n = 0$. Hence, $c(u, F)$ is independent of F and χ is a character of G. Since the group of commutators of G is $\mathrm{SL}_3(k)$, we have

$$\chi(u) = \det(u)^{w_0}$$

for some $w_0 \in \mathbb{Z}$. It is therefore enough to compute $\chi(u)$ when $u = \lambda \mathbf{I}_3$, with $\lambda \in k^\times$. In this case $u \cdot F = \lambda^d F$. Moreover, for all $f \in X_{d-3}$, since the section η_f is homogeneous of degree -1

$$\eta_f(\lambda^d F) = \lambda^{-d} \cdot \eta_f(F), \text{ and } \eta(\lambda^d F) = \lambda^{-dg} \cdot \eta(F).$$

Hence, as u is the identity on the curve $C_F = C_{u \cdot F}$,

$$u^* \eta(F) = \eta(F) = \lambda^{dg} \cdot \eta(u \cdot F) = \det(u)^{w_0} \cdot \eta(u \cdot F).$$

This implies

$$\det(u)^{w_0} = \lambda^{3w_0} = \lambda^{dg},$$

and the result is proven. $\square$

We made use of the following elementary lemma :

Lemma 6.3.3. *Let $f \in k[T_1, \ldots, T_n]$ be irreducible and let $g \in k(T_1, \ldots, T_n)$ be a rational function which has neither zeroes nor poles outside the set of zeroes of f. Then there is an $m \in \mathbb{Z}$ and $c \in k^\times$ such that $g = cf^m$.*

Proof. This is an immediate consequence of Hilbert's Nullstellensatz, together with the fact that the ring $k[T_1, \ldots, T_n]$ is factorial. $\square$

For any $h \in \mathbb{Z}$, we denote by $\Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha}^{\otimes h})^G$ the subspace of sections $s \in \Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha}^{\otimes h})$ such that

$$u^* s(F) = s(u \cdot F) \quad \text{for every } u \in G, F \in \mathsf{X}_d^0.$$

Proposition 6.3.4. *Let $h \geq 0$ be an integer. The linear map*

$$\Phi \mapsto \rho(\Phi) = \Phi \cdot \eta^{\otimes h}$$

is an isomorphism

$$\rho : \mathsf{Inv}_{gh}(\mathsf{X}_d^0) \xrightarrow{\sim} \Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha}^{\otimes h})^G.$$

Proof. Let $\Phi \in \mathsf{Inv}_{gh}(\mathsf{X}_d^0)$, $s = \rho(\Phi) = \Phi \cdot \eta^{\otimes h}$, and $w = dgh/3$, the weight of Φ. Then using Lem.6.3.2,

$$\begin{aligned} u^* s(F) &= \Phi(F) \cdot (u^* \eta(F))^{\otimes h} \\ &= \Phi(F) \cdot \det(u)^{w_0 h} \cdot \eta(u \cdot F)^{\otimes h} \\ &= \det(u)^w \Phi(F) \cdot \eta(u \cdot F)^{\otimes h} \\ &= \Phi(u \cdot F) \cdot \eta(u \cdot F)^{\otimes h} = s(u \cdot F). \end{aligned}$$

Hence, $\rho(\Phi) \in \Gamma(\mathsf{X}_d^0, \boldsymbol{\lambda}^{\otimes h})^G$. Conversely, the inverse of ρ is the map $s \mapsto s/\eta^{\otimes h}$, and this proves the proposition. $\square$

6.3.3 Modular forms as invariants

Let $d > 2$ be an integer and $g = \binom{d}{2}$. Since the fibres of $\mathsf{Y}_d^0 \longrightarrow \mathsf{X}_d^0$ are nonsingular non hyperelliptic plane curves of genus g, by the universal property of M_g we get a morphism

$$p : \mathsf{X}_g^0 \longrightarrow \mathsf{M}_g^0,$$

where M_g^0 is the moduli stack of nonhyperelliptic curves of genus g and $p^*\boldsymbol{\lambda} = \boldsymbol{\alpha}$ by construction. This induces a morphism

$$p^* : \Gamma(\mathsf{M}_g^0, \boldsymbol{\lambda}^{\otimes h}) \longrightarrow \Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha}^{\otimes h}).$$

Moreover, for $u \in G$, since $u : C_{u\cdot F} \to C_F$ is an isomorphism, we get the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{\lambda}[C_F] & \xrightarrow{u^*} & \boldsymbol{\lambda}[C_{u\cdot F}] \\ {\scriptstyle p^*}\big\downarrow & & {\scriptstyle p^*}\big\downarrow \\ \boldsymbol{\alpha}[F] & \xrightarrow{u^*} & \boldsymbol{\alpha}[u\cdot F]. \end{array}$$

For any $f \in \Gamma(\mathsf{M}_g^0, \boldsymbol{\lambda}^{\otimes h})$, the modular invariance of f means that

$$u^*f(C_F) = f(C_{u\cdot F}).$$

Then

$$u^*[(p^*f)(F)] = u^*[p^*(f(C_F))] = p^*[u^*f(C_F)] = p^*[f(C_{u\cdot F})] = (p^*f)(u\cdot F),$$

and this means that $p^*f \in \Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha}^{\otimes h})^G$. Combining this result with Prop.6.3.4, we obtain :

Proposition 6.3.5. *For any integer $h \geq 0$, the linear map $\sigma = \rho^{-1} \circ p^*$ is a homomorphism :*

$$\Gamma(\mathsf{M}_g^0, \boldsymbol{\lambda}^{\otimes h}) \longrightarrow \mathsf{Inv}_{gh}(\mathsf{X}_d^0)$$

such that

$$\sigma(f)(F) = f(C_F, \lambda)$$

with $\lambda = (p^*)^{-1}\eta$, for any $F \in \mathsf{X}_d^0$ and any section $f \in \Gamma(\mathsf{M}_g^0, \boldsymbol{\lambda}^{\otimes h})$. $\square$

We finally make a link between invariants and analytic Siegel modular forms. Let $F \in X_d^0(\mathbb{C})$ and let $\eta_1, \dots, \eta_g$ be the basis of regular differentials on C_F defined in Sec.6.3.2. Let $\gamma_1, \dots \gamma_{2g}$ be a symplectic basis of $H_1(C, \mathbb{Z})$ (for the intersection pairing). The matrix

$$\Omega = [\Omega_1\ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \eta_1 & \cdots & \int_{\gamma_{2g}} \eta_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \eta_g & \cdots & \int_{\gamma_{2g}} \eta_g \end{pmatrix}$$

belongs to the set $\mathcal{R}_g \subset \mathbf{M}_{g,2g}(\mathbb{C})$ of Riemann matrices, and $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g$.

Corollary 6.3.6. *Let $f \in \mathbf{S}_{g,h}(\mathbb{C})$ be a geometric Siegel modular form, $\widetilde{f} \in \mathbf{R}_{g,h}(\mathbb{C})$ the corresponding analytic modular form, and $\Phi = \sigma(t^*f)$ the corresponding invariant. In the above notation,*

$$\Phi(F) = (2i\pi)^{gh}\frac{\widetilde{f}(\tau)}{\det \Omega_2^h}\,.$$

Proof. Let $\lambda = (p^*)^{-1}(\eta)$ and $\omega = (t^*)^{-1}(\lambda)$. From Prop.6.2.2 and 6.3.5, we deduce

$$\Phi(F) = (t^* f)(C_F, \lambda) = f(\operatorname{Jac} C_F, \omega).$$

On the other hand, by the canonical identifications

$$\Omega^1[C_F] = \Omega^1[\operatorname{Jac} C_F], \quad H_1(C_F, \mathbb{Z}) = H_1(\operatorname{Jac} C_F, \mathbb{Z})$$

and Prop.6.2.6 we get

$$f(\operatorname{Jac} C_F, \omega) = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h},$$

from which the result follows. $\square$

6.4 The case of genus 3

6.4.1 Klein's formula

We recall the definition of theta functions with (entire) characteristics

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \in \mathbb{Z}^g \oplus \mathbb{Z}^g,$$

following [5]. The *(classical) theta function* is given, for $\tau \in \mathbb{H}_g$ and $z \in \mathbb{C}^g$, by

$$\theta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}^g} q^{(n+\varepsilon_1/2)\tau(n+\varepsilon_1/2)+2(n+\varepsilon_1/2)(z+\varepsilon_2/2)}.$$

The *Thetanullwerte* are the values at $z = 0$ of these functions, and we write

$$\theta[\boldsymbol{\varepsilon}](\tau) = \theta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (\tau) = \theta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (0, \tau).$$

Recall that a characteristic is *even* if $\varepsilon_1 \cdot \varepsilon_2 \equiv 0 \pmod 2$ and *odd* otherwise. Let S_g be the set of even characteristics with coefficients in $\{0, 1\}$. For $g \geq 2$, we put $h = |S_g|/2 = 2^{g-2}(2^g + 1)$ and

$$\widetilde{\chi}_h(\tau) = \prod_{\boldsymbol{\varepsilon} \in S_g} \theta[\boldsymbol{\varepsilon}](\tau).$$

In his beautiful paper [39], Igusa proves the following result [*loc. cit.*, Lem. 10 and 11]. Denote by $\widetilde{\Sigma}_{140}$ the modular form defined by the thirty-fifth elementary symmetric function of the eighth power of the even Thetanullwerte. Recall that a principally polarized abelian variety (A, a) is decomposable if it is a product of principally polarized abelian varieties of lower dimension, and indecomposable otherwise.

Theorem 6.4.1. *If $g \geq 3$, then $\widetilde{\chi}_h(\tau) \in \mathbf{R}_{g,h}(\mathbb{C})$. Moreover, If $g = 3$ and $\tau \in \mathbb{H}_3$, then :*

1. *A_τ is decomposable if $\widetilde{\chi}_{18}(\tau) = \widetilde{\Sigma}_{140}(\tau) = 0$.*
2. *A_τ is a hyperelliptic Jacobian if $\widetilde{\chi}_{18}(\tau) = 0$ and $\widetilde{\Sigma}_{140}(\tau) \neq 0$.*
3. *A_τ is a non hyperelliptic Jacobian if $\widetilde{\chi}_{18}(\tau) \neq 0$.* $\square$

Using Prop. 6.2.1, we define the geometric Siegel modular form of weight h

$$\chi_h(A_\tau) = (2i\pi)^{gh}\, \widetilde{\chi}_h(\tau)(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h}.$$

Ichikawa [37], [38] proved that $\chi_h \in \mathbf{S}_{g,h}(\mathbb{Q})$. For $g = 3$, one finds

$$\chi_{18}(A_\tau) = -(2\pi)^{54}\, \widetilde{\chi}_{18}(\tau)(dz_1 \wedge dz_2 \wedge dz_3)^{\otimes 18}.$$

Now we are ready to give a proof of the following result [47, Eq. 118, p. 462] :

Theorem 6.4.2 (Klein's formula)**.** *Let $F \in \mathsf{X}_4^0(\mathbb{C})$ and C_F be the corresponding smooth plane quartic. Let η_1, η_2, η_3 be the classical basis of $\Omega^1[C_F]$ from Sec.6.3.2 and $\gamma_1, \ldots \gamma_6$ be a symplectic basis of $H_1(C_F, \mathbb{Z})$ for the intersection pairing. Let*

$$\Omega = [\Omega_1\ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \eta_1 & \cdots & \int_{\gamma_6} \eta_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \eta_3 & \cdots & \int_{\gamma_6} \eta_3 \end{pmatrix}$$

be a period matrix of $\mathrm{Jac}(C)$ *and $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_3$. Then*

$$\mathrm{Disc}(F)^2 = \frac{1}{2^{28}}(2\pi)^{54}\frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}.$$

Proof. Cor.6.3.6 shows that for any $F \in \mathsf{X}_4^0$ the invariant $I = \sigma \circ t^*(\chi_{18})$ satisfies

$$I(F) = -(2\pi)^{54}\,\frac{\widetilde{\chi}_{18}(\tau)}{\det \Omega_2^{18}}.$$

Moreover Th. 6.4.1(3) shows that $I(F) \neq 0$ for all $F \in \mathsf{X}_4^0$. Thus I is a non-zero invariant of weight 54. Applying Lem. 6.3.3 for the discriminant, we find by comparison of the weights that $I = c\,\mathrm{Disc}^2$ with $c \in \mathbb{C}$ a constant. But if F_m is the Ciani quartic associated to a matrix $m \in \mathrm{Sym}_3(k)$ as in Example 6.3.1, it is proven in [50, Cor. 4.2] that Klein's formula is true for F_m and $c = -2^{28}$. $\square$

Remark 6.4.1. The morphism t^* defines an injective morphism of graded k-algebras

$$\mathbf{S}_3(k) = \oplus_{h \geq 0}\mathbf{S}_{3,h}(k) \longrightarrow \mathbf{T}_3(k) = \oplus_{h\geq 0}\mathbf{T}_{3,h}(k).$$

In [36], Ichikawa proves that if k is a field of characteristic 0, then $\mathbf{T}_3(k)$ is generated by the image of $\mathbf{S}_3(k)$ and a primitive Teichmüller form $\mu_{3,9} \in \mathbf{T}_{3,9}(\mathbb{Z})$ of weight 9, which is not of Siegel modular type. He also proves in [38] that

$$t^*(\chi_{18}) = -2^{28} \cdot (\mu_{3,9})^2. \tag{6.2}$$

Th. 6.4.2 implies that $\mu_{3,9}$ is actually equal to the discriminant up to a sign. This might probably be deduced from the definition of $\mu_{3,9}$, although it seems that this fact was not observed before (see also [44, Sec. 2.4]).

Remark 6.4.2. Besides [58] and [26] where an analogue of Klein's formula is derived in the hyperelliptic case, there exists a beautiful algebraic Klein's formula for genus 3 curves, linking the discriminant with irrational invariants [22, Th.11.1].

6.4.2 Jacobians among abelian threefolds

Let $k \subset \mathbb{C}$ be a field and let $g = 3$. We prove the following theorem which allows us to determine whether a given abelian threefold defined over k is k-isomorphic to a Jacobian of a curve defined over the same field. This settles the question of Serre recalled in the introduction.

Theorem 6.4.3. *Let (A, a) be a principally polarized abelian threefold defined over $k \subset \mathbb{C}$. Let $\omega_1, \omega_2, \omega_3$ be any basis of $\Omega^1_k[A]$ and $\gamma_1, \dots \gamma_6$ a symplectic basis (for the polarization a) of $H_1(A, \mathbb{Z})$, in such a way that*

$$\Omega = [\Omega_1 \ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_6} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_3 & \cdots & \int_{\gamma_6} \omega_3 \end{pmatrix}$$

is a period matrix of (A, a). Put $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_3$.

1. *If $\widetilde{\Sigma}_{140}(\tau) = 0$ and $\widetilde{\chi}_{18}(\tau) = 0$ then (A, a) is decomposable over $\bar{k}$. In particular it is not a Jacobian.*
2. *If $\widetilde{\Sigma}_{140}(\tau) \neq 0$ and $\widetilde{\chi}_{18}(\tau) = 0$ then there exists a hyperelliptic curve X/k such that $(\mathrm{Jac}\, X, j) \simeq (A, a)$.*
3. *If $\widetilde{\chi}_{18}(\tau) \neq 0$ then (A, a) is isomorphic to a Jacobian if and only if*

$$-\chi_{18}(A, \omega_1 \wedge \omega_2 \wedge \omega_3) = (2\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}$$

is a square in k.

Proof. The first and second points follow from Th.6.4.1 and Th.6.1.1. Suppose now that (A, a) is isomorphic over k to the Jacobian of a non hyperelliptic genus 3 curve C/k. Let $\omega = \omega_1 \wedge \omega_2 \wedge \omega_3$ Using Prop.6.2.2, we get

$$-\chi_{18}(A, \omega) = t^*(-\chi_{18})(C, \lambda)$$

with $\lambda = t^*\omega$. The left hand side is (Prop.6.2.6)

$$-\chi_{18}(A, \omega) = -(2i\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}} = (2\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}.$$

According to Rem.6.4.1, the right hand side of the equality is

$$t^*(-\chi_{18})(C, \lambda) = 2^{28} \cdot \mu^2_{3,9}(C, \lambda) = (2^{14} \cdot \mu_{3,9}(C, \lambda))^2$$

so the desired expression is a square in k. On the contrary, Cor.6.2.5 shows that if (A, a) is a quadratic twist of a Jacobian (A', a') then there exists a non square $c \in k$ such that

$$-\bar{\chi}_{18}(A) = c^9 \cdot (-\bar{\chi}_{18}(A')).$$

As we have just proved that $-\bar{\chi}_{18}(A')$ is a non-zero square in $k/k^{\times 18}$ so $-\chi_{18}(A, \omega)$ is not. $\square$

Corollary 6.4.4. *In the notation of Th.6.4.3, the quadratic character ε of $\mathrm{Gal}(k_{sep}/k)$ introduced in Theorem 6.1.1 is given by $\varepsilon(\sigma) = d/d^\sigma$, where*

$$d = \sqrt{(2\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}},$$

with an arbitrary choice of the square root.

6.4.3 Beyond genus 3

It is natural to try to extend our results to the case $g > 3$. The first question to ask is

– Does there exist an analogue of Klein's formula for $g > 3$?

Here we can give a partial answer. Using Sec.6.2.3, we can consider the Teichmüller modular form $t^*(\chi_h)$ with $h = 2^{g-2}(2^g + 1)$. In [38, Prop.4.5] (see also [84]), it is proven that for $g > 3$ the element

$$t^*(\chi_h)/2^{2^{g-1}(2^g-1)}$$

has as a square root a primitive element $\mu_{g,h/2} \in \mathbf{T}_{g,h/2}(\mathbb{Z})$. If $g = 4$, in the footnote, p. 462 in [47] we find the following amazing formula

$$\frac{\widetilde{\chi}_{68}(\tau)}{\det(\Omega_2)^{68}} = c \cdot \Delta(X)^2 \cdot T(X)^8. \tag{6.3}$$

Here $\tau = \Omega_2^{-1}\Omega_1$, with $\Omega = [\Omega_1\ \Omega_2]$ a period matrix of a genus 4 non hyperelliptic curve X given in $\mathbb{P}^3$ as an intersection of a quadric Q and a cubic surface E. The elements $\Delta(X)$ and $T(X)$ are defined in the classical invariant theory as, respectively, the discriminant of Q and the tact invariant of Q and E (see [71, p.122]). Unfortuantely, no proof of this formula is given in [47], so the problem appears to deserve further study. No such formula seems to be known in the non hyperelliptic case for $g > 4$.

Let us now look at what happens when we try to apply Serre's approach for $g > 3$. To begin with, when g is even, we cannot use Cor.6.2.4 to distinguish between quadratic twists. In particular, using the previous result, we see that $\chi_h(A, \omega_k)$ is a square when A is a principally polarized abelian variety defined over k which is geometrically a Jacobian. A natural question is :

– What is the relation between this condition and the locus of geometric Jacobians over k?

Let us assume now that g is odd. Corollary 6.2.5 shows that there exists $c \in k \setminus k^2$ such that

$$\overline{\chi}_h(A') = c^{h/2} \cdot \overline{\chi}_h(A)$$

for a Jacobian A and a quadratic twist A'. What enabled us to distinguish between A and A' when $g = 3$ is that $h/2 = 9$ is odd. However as soon as $g > 3$, $2 \mid 2^{g-3}$, the power $g - 3$ being the maximal power of 2 dividing $h/2$, so it is not enough for $\bar{\chi}_{18}(A)$ to be a square in k to make a distinction between A and A'. It must rather be an element of $k^{2^{g-2}}$. It can be easily seen from the proof of [84, Th.1] that $t^*(\chi_h)$ does not admit a fourth root. This implies[1] that $\overline{\chi}_h(A) \notin k^{2^{g-2}}$ for infinitely many Jacobians A defined over number fields k. So we can no longer use the modular form χ_h to characterize Jacobians over k. The question is :

– Is it possible to find a modular form to replace χ_h in our strategy when $g > 3$?

[1] Personal communication by Y. F. Bilu and X. Xarles

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RESUME en français

Deux parties principales constituent le sujet de cette thèse. La première partie est consacrée à l'étude des propriétés asymptotiques des fonctions zêta, des fonctions L, des corps globaux et des variétés sur ces corps. Dans le premier chapitre, nous démontrons une généralisation du théorème de Brauer–Siegel au cas des suites de corps presque normaux. Dans le deuxième chapitre, nous étudions le comportement asymptotique des dérivées logarithmiques des fonctions zêta dans des familles de corps globaux. Dans le troisième chapitre, nous donnons un panorama des généralisations du théorème de Brauer–Siegel classique. Dans le même chapitre nous démontrons une version du théorème de Brauer–Siegel pour les variétés sur les corps finis. Le quatrième chapitre est consacré à l'étude de la distribution des zéros des fonctions L des formes modulaires. Dans le cinquième chapitre, nous étudions des propriétés asymptotiques des familles de fonctions zêta et de fonctions L sur les corps finis dans le contexte des trois problèmes suivants : l'inégalité principale, les résultats de type Brauer–Siegel et la distribution des zéros. Le but de la deuxième partie est d'obtenir une caractérisation des jacobiennes parmi les variétés abéliennes principalement polarisées de dimension 3, ce qui fournit une réponse à une question de J.-P. Serre. Nous obtenons aussi une nouvelle démonstration de la formule de Klein qui relie une certaine forme modulaire de Siegel au discriminant des quartiques planes.

TITRE en anglais

ASYMPTOTIC PROPERTIES OF GLOBAL FIELDS

RESUME en anglais

There are two main parts in this thesis. The first part is devoted to the study of asymptotic properties of zeta functions, L-functions, global fields and varieties over these fields. In the first chapter, we prove a generalization of the Brauer–Siegel theorem to the case of families of almost normal number fields. In the second chapter, we study the asymptotic behaviour of the logarithmic derivatives of zeta functions in families of global fields. In the third, chapter we give an overview of possible generalizations of the classical Brauer–Siegel theorem. In the same chapter, we prove a version of the Brauer–Siegel theorem for varieties over finite fields. The fourth chapter is devoted to the study of the distribution of zeroes of L-functions of modular forms. In the fifth chapter, we study the asymptotic properties of families of zeta and L-functions over finite fields in the context of the following problems : the basic inequality, the results of the Brauer–Siegel type and the distribution of zeroes. The aim of the second part is to obtain a characterization of Jacobians among principally polarized abelian varieties of dimension 3, which gives an answer to a question asked by J.-P. Serre. We also obtain a new proof of Klein's formula which connects a certain Siegel modular form to the discriminant of plane quartics.

DISCIPLINE MATHÉMATIQUES

MOTS-CLES
Corps globaux, variétés algébriques sur les corps finis, formules explicites, fonctions zêta, formes modulaires, théorème de Brauer–Siegel, variétés abéliennes, jacobiennes.

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