Unions of Reducibility Families

for λ-Calculus and Rewriting

Colin Riba

INRIA Sophia Antipolis – Méditerranée
Everest

7 Mai 2008
LDP – Marseille

General Motivations

Termination of extensions of typed λ-calculus.

- Proofs assistants (strong normalization).
- Functional programming.

λ-Calculus with Rewriting

λ-Calculus with Rewriting

- **Terms**

$$t, u \in \Lambda(\Sigma) \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad t\,u \quad | \quad \mathtt{f}(t_1, \ldots, t_n)\ ,$$

where $\mathtt{f} \in \Sigma_n$.

λ-Calculus with Rewriting

- **Terms**

$$t, u \in \Lambda(\Sigma) \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad t\,u \quad | \quad \mathtt{f}(t_1, \dots, t_n)\ ,$$

  where $\mathtt{f} \in \Sigma_n$.

- A type system (eg. simple types).

λ-Calculus with Rewriting

- **Terms**

$$t, u \in \Lambda(\Sigma) \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad t\, u \quad | \quad \mathtt{f}(t_1, \dots, t_n)\ ,$$

  where $\mathtt{f} \in \Sigma_n$.

- A type system (eg. simple types).
- **Rewrite rules** of the form

$$\mathtt{f}(l_1, \dots, l_n) \mapsto_{\mathcal{R}} r\ ,$$

  where

$$\frac{\Gamma \vdash l_1 : T_1 \quad \dots \quad \Gamma \vdash l_n : T_n}{\Gamma \vdash \mathtt{f}(l_1, \dots, l_n) : T} \qquad \text{and} \qquad \Gamma \vdash r : T$$

Strong Normalization

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Strong normalization** ($\mathcal{SN}$)
  No infinite sequence

$$t_1 \quad \rightarrow_R \quad \dots \quad \rightarrow_R \quad t_n \quad \rightarrow_R \quad \dots$$

Strong Normalization

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Strong normalization** ($\mathcal{SN}$)
  No infinite sequence

$$t_1 \quad \rightarrow_R \quad \dots \quad \rightarrow_R \quad t_n \quad \rightarrow_R \quad \dots$$

- Tools to prove that

$$\text{if} \quad \vdash t : T \quad \text{then} \quad t \in \mathcal{SN}$$

Type Interpretation

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Interpretation of types**

$$T \in \mathcal{T} \quad \mapsto \quad [\![T]\!] \subseteq \mathcal{SN}$$

- **Adequacy**

$$\vdash t : T \quad \Longrightarrow \quad t \in [\![T]\!]$$

Type Interpretation

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Reducibility family**

$$\mathcal{R}ed \subseteq \mathcal{P}(\Lambda(\Sigma)) \qquad \text{such that} \qquad \forall A \in \mathcal{R}ed. \quad \mathcal{V}ar \subseteq A \subseteq \mathcal{SN}$$

- **Interpretation of types**

$$T \in \mathcal{T} \quad \mapsto \quad [\![T]\!] \in \mathcal{R}ed$$

- **Adequacy**

$$\vdash t : T \quad \Longrightarrow \quad t \in [\![T]\!]$$

Type Interpretation

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Reducibility family**

$$\mathcal{R}ed \subseteq \mathcal{P}(\Lambda(\Sigma)) \qquad \text{such that} \qquad \forall A \in \mathcal{R}ed. \quad \mathcal{V}ar \subseteq A \subseteq \mathcal{SN}$$

- **Interpretation of types**

$$T \in \mathcal{T} \quad \mapsto \quad [\![T]\!] \in \mathcal{R}ed$$

- **Adequacy**

$$\vdash t : T \qquad \Longrightarrow \qquad t \in [\![T]\!]$$

- $\mathcal{R}ed$ is a complete lattice

Different reducibility families:

- Tait's Saturated Sets [Tai75]
- Girard's Reducibility Candidates [Gir72]
- Biorthogonals [Gir87]

Different reducibility families:

- Tait's Saturated Sets [Tai75]
- Girard's Reducibility Candidates [Gir72]
- Biorthogonals [Gir87]

Compare them wrt **stability by union**:

$$\emptyset \neq \mathcal{R} \subseteq \mathcal{R}ed \quad \Longrightarrow \quad \bigcup \mathcal{R} \in \mathcal{R}ed$$

Different reducibility families:

- Tait's Saturated Sets [Tai75]
- Girard's Reducibility Candidates [Gir72]
- Biorthogonals [Gir87]

Compare them wrt **stability by union**:

$$\emptyset \neq \mathcal{R} \subseteq \mathcal{R}ed \quad \Longrightarrow \quad \bigcup \mathcal{R} \in \mathcal{R}ed$$

Property used eg. in [BR06, Abe06, Tat07].

Outline

Outline

Reducibility

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

Reducibility

Let $\rightarrow_{\mathrm{R}}$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Reducibility family**

$$\mathcal{R}ed \subseteq \mathcal{P}(\Lambda(\Sigma)) \qquad \text{such that} \qquad \forall A \in \mathcal{R}ed. \quad \mathcal{V}ar \subseteq A \subseteq \mathcal{SN}$$

Reducibility

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Reducibility family**

$$\mathcal{R}ed \subseteq \mathcal{P}(\Lambda(\Sigma)) \qquad \text{such that} \qquad \forall A \in \mathcal{R}ed. \quad \mathcal{V}ar \subseteq A \subseteq \mathcal{SN}$$

- **Interpretation of types**

$$T \in \mathcal{T} \quad \mapsto \quad [\![T]\!] \in \mathcal{R}ed$$

Reducibility

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$.

- **Reducibility family**

$$\mathcal{R}ed \subseteq \mathcal{P}(\Lambda(\Sigma)) \qquad \text{such that} \qquad \forall A \in \mathcal{R}ed. \quad \mathcal{V}ar \subseteq A \subseteq \mathcal{SN}$$

- **Interpretation of types**

$$T \in \mathcal{T} \quad \mapsto \quad [\![T]\!] \in \mathcal{R}ed$$

- Sufficient conditions on $\mathcal{R}ed$ to get an **adequate** interpretation:

$$\vdash t : T \qquad \Longrightarrow \qquad t \in [\![T]\!]$$

Saturated Sets (Pure λ-Calculus)

Saturated Sets (Pure λ-Calculus)

- **Elimination contexts**

$$E[\,] \in \mathcal{E}_{\Rightarrow} \quad ::= \quad [\,] \quad | \quad E[\,]\, t$$

Saturated Sets (Pure λ-Calculus)

- **Elimination contexts**

$$E[\,] \in \mathcal{E}_{\Rightarrow} \quad ::= \quad [\,] \quad | \quad E[\,]\,t$$

- **(Non) interaction properties**

if $E[x] \rightarrow_\beta v$ then the reduction is in $E[\,]$,
if $E[(\lambda x.t)u] \rightarrow_\beta v$ then the reduction is either in $E[\,]$ or in $(\lambda x.t)u$.

Saturated Sets (Pure λ-Calculus)

- **Elimination contexts**

$$E[\,] \in \mathcal{E}_{\Rightarrow} \quad ::= \quad [\,] \quad | \quad E[\,]\,t$$

- **(Non) interaction properties**

if $E[x] \rightarrow_\beta v$ then the reduction is in $E[\,]$,
if $E[(\lambda x.t)u] \rightarrow_\beta v$ then the reduction is either in $E[\,]$ or in $(\lambda x.t)u$.

- A β-reduct of $(\lambda x.t)u$ is either $t[u/x]$ or $(\lambda x.t')u'$ with $(t, u) \rightarrow_\beta (t', u')$.

Saturated Sets (Pure λ-Calculus)

- **Elimination contexts**

$$E[\,] \in \mathcal{E}_{\Rightarrow} \quad ::= \quad [\,] \quad | \quad E[\,]\,t$$

- **(Non) interaction properties**

$$\begin{array}{lrl} \text{if} & E[x] \to_\beta v & \text{then the reduction is in } E[\,], \\ \text{if} & E[(\lambda x.t)u] \to_\beta v & \text{then the reduction is either in } E[\,] \\ & & \text{or in } (\lambda x.t)u. \end{array}$$

- A β-reduct of $(\lambda x.t)u$ is either $t[u/x]$ or $(\lambda x.t')u'$ with $(t,u) \to_\beta (t',u')$.

- **Consequence:**
  If $E[t[u/x]] \in \mathcal{SN}_\beta$ and $u \in \mathcal{SN}_\beta$ then $E[(\lambda x.t)u] \in \mathcal{SN}_\beta$.

Saturated Sets (Pure λ-Calculus)

- **Elimination contexts**

$$E[\,] \in \mathcal{E}_{\Rightarrow} \quad ::= \quad [\,] \quad | \quad E[\,]\,t$$

- **(Non) interaction properties**

if $E[x] \to_\beta v$ then the reduction is in $E[\,]$,
if $E[(\lambda x.t)u] \to_\beta v$ then the reduction is either in $E[\,]$ or in $(\lambda x.t)u$.

- $S \subseteq \mathcal{SN}_\beta$ is a **saturated set** ($S \in \mathcal{SAT}$) iff

$(\mathcal{SAT}1)$ if $E[\,] \in \mathcal{SN}_\beta$ and $x \in \mathcal{V}ar$ then $E[x] \in S$,
$(\mathcal{SAT}2_\beta)$ if $E[t[u/x]] \in S$ and $u \in \mathcal{SN}_\beta$ then $E[(\lambda x.t)u] \in S$.

Rewriting

- Let $\mathcal{R}$ be a rewrite system on $\Lambda(\Sigma)$ whose rules are of the form

$$\mathtt{f}(l_1, \ldots, l_n) \mapsto_{\mathcal{R}} r$$

Rewriting

- Let $\mathcal{R}$ be a rewrite system on $\Lambda(\Sigma)$ whose rules are of the form

$$\mathtt{f}(l_1, \ldots, l_n) \mapsto_{\mathcal{R}} r$$

- **Saturated sets**
  In addition to $(\mathcal{SAT}1)$ and $(\mathcal{SAT}2_\beta)$ we need
  stability by reduction (if $t \in \mathcal{S}$ and $t \rightarrow_{\beta\mathcal{R}} u$ then $u \in \mathcal{S}$),

Rewriting

- Let $\mathcal{R}$ be a rewrite system on $\Lambda(\Sigma)$ whose rules are of the form

$$\mathsf{f}(l_1, \dots, l_n) \mapsto_{\mathcal{R}} r$$

- **Saturated sets**
  In addition to $(\mathcal{SAT}1)$ and $(\mathcal{SAT}2_\beta)$ we need
  stability by reduction (if $t \in S$ and $t \to_{\beta\mathcal{R}} u$ then $u \in S$),
  and that for all $E[\mathsf{f}(t_1, \dots, t_n)]$,

$$\begin{array}{ccc} & E[\mathsf{f}(t_1, \dots, t_n)] & \\ {}_{\beta\mathcal{R}}\swarrow & & \searrow_{\beta\mathcal{R}} \\ u_1 \in S & \dots & u_n \in S \end{array} \quad \Longrightarrow \quad E[\mathsf{f}(t_1, \dots, t_n)] \in S$$

Neutral Terms

Consider a set of contexts $E[\ ] \in \mathcal{E}$ and a rewrite relation $\rightarrow_R$.

- A term t is **neutral** $(t \in \mathcal{N})$ if it interacts with no contexts $E[\ ] \in \mathcal{E}$:

  if $\quad E[t] \rightarrow_R v \quad$ then the reduction is either in $E[\ ]$ or in t.

Neutral Terms

Consider a set of contexts $E[\ ] \in \mathcal{E}$ and a rewrite relation $\rightarrow_R$.

- A term t is **neutral** $(t \in \mathcal{N})$ if it interacts with no contexts $E[\ ] \in \mathcal{E}$:

  if $E[t] \rightarrow_R v$ then the reduction is either in $E[\ ]$ or in t.

- $\mathcal{E}$ is a set of **elimination contexts** if
  $\mathcal{E}$ is stable by reduction and by composition and
  all variables are neutral and
  if $t \in \mathcal{N}$ and $E[\ ] \in \mathcal{E}$ then $E[t] \in \mathcal{N}$.

- **Example:**
  $\mathcal{E}_{\Rightarrow}$ is a set of elimination contexts for $\rightarrow_\beta$.

Neutral Terms

Consider a set of contexts $E[\ ] \in \mathcal{E}$ and a rewrite relation $\rightarrow_R$.

- A term t is **neutral** $(t \in \mathcal{N})$ if it interacts with no contexts $E[\ ] \in \mathcal{E}$:

  if $E[t] \rightarrow_R v$ then the reduction is either in $E[\ ]$ or in t.

- $\mathcal{E}$ is a set of **elimination contexts** if
  $\mathcal{E}$ is stable by reduction and by composition and
  all variables are neutral and
  if $t \in \mathcal{N}$ and $E[\ ] \in \mathcal{E}$ then $E[t] \in \mathcal{N}$.

- **Example:**
  $\mathcal{E}_{\Rightarrow}$ is a set of elimination contexts for $\rightarrow_\beta$.

- A **value** $(t \in \mathcal{V})$ is an observable term, ie a term which interacts with some contexts $E[\ ] \in \mathcal{E}$.

Girard's Reducibility Candidates

Assume that $\mathcal{E}$ is a set of elimination contexts for $\rightarrow_{\mathrm{R}}$.

Girard's Reducibility Candidates

Assume that $\mathcal{E}$ is a set of elimination contexts for $\rightarrow_R$.

- $C \subseteq \mathcal{SN}$ is a **reducibility candidate** ($C \in \mathfrak{CR}$) iff
  C is stable by reduction (if $t \in C$ and $t \rightarrow_R u$ then $u \in C$) and

Girard's Reducibility Candidates

Assume that $\mathcal{E}$ is a set of elimination contexts for $\to_R$.

- $C \subseteq \mathcal{SN}$ is a **reducibility candidate** ($C \in \mathfrak{CR}$) iff
  C is stable by reduction (if $t \in C$ and $t \to_R u$ then $u \in C$) and
  C has the neutral term property: for all neutral term t,

$$
\begin{array}{ccccc}
 & & t & & \\
 & \swarrow_{R} & & \searrow^{R} & \\
u_1 \in C & & \cdots & & u_n \in C
\end{array}
\qquad \Longrightarrow \qquad t \in C
$$

Girard's Reducibility Candidates

Assume that $\mathcal{E}$ is a set of elimination contexts for $\rightarrow_R$.

- $C \subseteq \mathcal{SN}$ is a **reducibility candidate** ($C \in \mathfrak{CR}$) iff
  C is stable by reduction (if $t \in C$ and $t \rightarrow_R u$ then $u \in C$) and
  C has the neutral term property: for all neutral term t,

$$t \xrightarrow{R} u_1 \in C \quad \cdots \quad t \xrightarrow{R} u_n \in C \qquad \Longrightarrow \qquad t \in C$$

$$\Downarrow$$

$$E[u_1] \in \mathcal{SN} \quad \cdots \quad E[u_n] \in \mathcal{SN} \qquad \Longrightarrow \qquad E[t] \in \mathcal{SN}$$

Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

- Choose a pole

$$t \perp\!\!\!\perp E[\ ] \quad \Longleftrightarrow_{\text{def}} \quad E[t] \in \mathcal{SN}$$

Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

- Choose a pole

$$t \perp\!\!\!\perp E[\,] \quad \Longleftrightarrow_{\text{def}} \quad E[t] \in \mathcal{SN}$$

- Given $A \subseteq \Lambda(\Sigma)$ and $P \subseteq \mathcal{E}$, let

$$\begin{array}{lcl} A^{\perp\!\!\!\perp} & =_{\text{def}} & \{E[\,] \in \mathcal{E} \mid \forall t \in A.\ \ t \perp\!\!\!\perp E[\,]\} \\ P^{\perp\!\!\!\perp} & =_{\text{def}} & \{t \in \Lambda(\Sigma) \mid \forall E[\,] \in P.\ \ t \perp\!\!\!\perp E[\,]\} \end{array}$$

Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

- Choose a pole

$$t \perp\!\!\!\perp E[\,] \quad \Longleftrightarrow_{\text{def}} \quad E[t] \in \mathcal{SN}$$

- Given $A \subseteq \Lambda(\Sigma)$ and $P \subseteq \mathcal{E}$, let

$$\begin{array}{lcl} A^{\perp\!\!\!\perp} & =_{\text{def}} & \{E[\,] \in \mathcal{E} \mid \forall t \in A.\ \ t \perp\!\!\!\perp E[\,]\} \\ P^{\perp\!\!\!\perp} & =_{\text{def}} & \{t \in \Lambda(\Sigma) \mid \forall E[\,] \in P.\ \ t \perp\!\!\!\perp E[\,]\} \end{array}$$

$A \subseteq \Lambda(\Sigma)$

Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

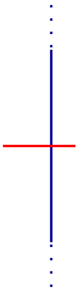
- Choose a pole

$$t \perp\!\!\!\perp E[\,] \quad \Longleftrightarrow_{\text{def}} \quad E[t] \in \mathcal{SN}$$

- Given $A \subseteq \Lambda(\Sigma)$ and $P \subseteq \mathcal{E}$, let

$$\begin{array}{lcl} A^{\perp\!\!\!\perp} & =_{\text{def}} & \{E[\,] \in \mathcal{E} \mid \forall t \in A.\ \ t \perp\!\!\!\perp E[\,]\} \\ P^{\perp\!\!\!\perp} & =_{\text{def}} & \{t \in \Lambda(\Sigma) \mid \forall E[\,] \in P.\ \ t \perp\!\!\!\perp E[\,]\} \end{array}$$

$$A \subseteq \Lambda(\Sigma)$$
$$\downarrow_{\perp\!\!\!\perp}$$
$$A^{\perp\!\!\!\perp} \subseteq \mathcal{E}$$



Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

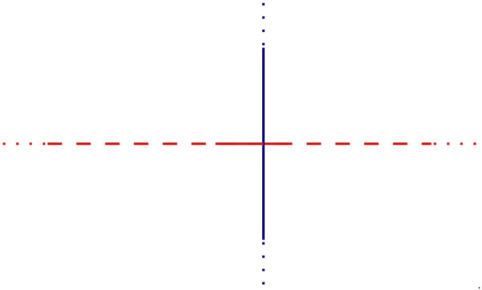
- Choose a pole

$$t \perp\!\!\!\perp E[\,] \quad \Longleftrightarrow_{\text{def}} \quad E[t] \in \mathcal{SN}$$

- Given $A \subseteq \Lambda(\Sigma)$ and $P \subseteq \mathcal{E}$, let

$$\begin{array}{lll} A^{\perp\!\!\!\perp} & =_{\text{def}} & \{E[\,] \in \mathcal{E} \mid \forall t \in A.\ \ t \perp\!\!\!\perp E[\,]\} \\ P^{\perp\!\!\!\perp} & =_{\text{def}} & \{t \in \Lambda(\Sigma) \mid \forall E[\,] \in P.\ \ t \perp\!\!\!\perp E[\,]\} \end{array}$$

$$\begin{array}{c} A \subseteq \Lambda(\Sigma) \\ \downarrow_{\perp\!\!\!\perp} \\ A^{\perp\!\!\!\perp} \subseteq \mathcal{E} \\ \downarrow_{\perp\!\!\!\perp} \\ A^{\perp\!\!\!\perp\perp\!\!\!\perp} \subseteq \Lambda(\Sigma) \end{array}$$



Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

- Choose a pole

$$t \perp\!\!\!\perp E[\,] \quad \Longleftrightarrow_{\text{def}} \quad E[t] \in \mathcal{SN}$$

- Given $A \subseteq \Lambda(\Sigma)$ and $P \subseteq \mathcal{E}$, let

$$\begin{array}{lcll} A^{\perp\!\!\!\perp} & =_{\text{def}} & \{E[\,] \in \mathcal{E} \mid & \forall t \in A. \quad t \perp\!\!\!\perp E[\,]\} \\ P^{\perp\!\!\!\perp} & =_{\text{def}} & \{t \in \Lambda(\Sigma) \mid & \forall E[\,] \in P. \quad t \perp\!\!\!\perp E[\,]\} \end{array}$$

- $(\_)^{\perp\!\!\!\perp\perp\!\!\!\perp}$ is a closure operator.

Biorthogonals [Gir87, Par97, DK00, Pit00, MV05]

- Choose a pole

$$t \perp\!\!\!\perp E[\,] \quad \Longleftrightarrow_{\text{def}} \quad E[t] \in \mathcal{SN}$$

- Given $A \subseteq \Lambda(\Sigma)$ and $P \subseteq \mathcal{E}$, let

$$\begin{array}{lclcl} A^{\perp\!\!\!\perp} & =_{\text{def}} & \{E[\,] \in \mathcal{E} & \mid & \forall t \in A. \quad t \perp\!\!\!\perp E[\,]\} \\ P^{\perp\!\!\!\perp} & =_{\text{def}} & \{t \in \Lambda(\Sigma) & \mid & \forall E[\,] \in P. \quad t \perp\!\!\!\perp E[\,]\} \end{array}$$

- $(\_)^{\perp\!\!\!\perp\perp\!\!\!\perp}$ is a closure operator.

Lemma

$$\emptyset \neq A \subseteq \mathcal{SN} \quad \Longrightarrow \quad A^{\perp\!\!\!\perp\perp\!\!\!\perp} \in \mathfrak{CR}$$

Outline

Given a typed rewrite system $\mathcal{R}$,

find a reducibility family $\mathfrak{Red}$
which leads to an adequate type interpretation
and such that

$$\emptyset \neq \mathfrak{R} \subseteq \mathfrak{Red} \quad \Longrightarrow \quad \bigcup \mathfrak{R} \in \mathfrak{Red}$$

Union Types

$$T_1, T_2 \in \mathcal{T} \quad ::= \quad \ldots \quad | \quad T_1 \sqcup T_2$$

- We put

$$[\![T_1 \sqcup T_2]\!] \quad =_{\text{def}} \quad \mathcal{R}ed([\![T_1]\!] \cup [\![T_2]\!])$$

  This validates

$$(\sqcup \mathsf{I})\,\frac{\Gamma \vdash t : T_i}{\Gamma \vdash t : T_1 \sqcup T_2}$$

Union Types

$$T_1, T_2 \in \mathcal{T} \quad ::= \quad \dots \quad | \quad T_1 \sqcup T_2$$

- We put

$$[\![T_1 \sqcup T_2]\!] \quad =_{\text{def}} \quad \mathcal{R}ed([\![T_1]\!] \cup [\![T_2]\!])$$

  This validates

$$(\sqcup \mathsf{I})\,\frac{\Gamma \vdash t : T_i}{\Gamma \vdash t : T_1 \sqcup T_2}$$

- If $\mathcal{R}ed$ stable by union, we have

$$[\![T_1 \sqcup T_2]\!] \quad = \quad [\![T_1]\!] \cup [\![T_2]\!]$$

  This is sufficient to validate

$$(\sqcup \mathsf{E})\,\frac{\Gamma \vdash t : T_1 \sqcup T_2 \qquad \begin{array}{c}\Gamma, x : T_1 \vdash c : C \\ \Gamma, x : T_2 \vdash c : C\end{array}}{\Gamma \vdash c[t/x] : C}$$

Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

$$t_1 \quad =_{\text{def}} \quad \lambda x.x\mathsf{a}\delta \qquad\qquad t_2 \quad =_{\text{def}} \quad \lambda y.\delta$$

Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

$$t_1 \quad =_{\text{def}} \quad \lambda x.xa\delta \qquad\qquad t_2 \quad =_{\text{def}} \quad \lambda y.\delta$$

$$\dfrac{\dfrac{t_1 : T_1 \qquad t_2 : T_2}{t_1 + t_2 : T_1 \sqcup T_2} \qquad \begin{array}{c} x : T_1 \vdash xx : C \\ x : T_2 \vdash xx : C \end{array}}{}$$

Because

$$t_1 t_1 \in \mathcal{SN} \qquad \text{and} \qquad t_2 t_2 \in \mathcal{SN}$$

Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

$$t_1 \quad =_{\text{def}} \quad \lambda x.x\mathsf{a}\delta \qquad\qquad t_2 \quad =_{\text{def}} \quad \lambda y.\delta$$

$$(\sqcup E)\ \frac{\dfrac{t_1 : T_1 \qquad t_2 : T_2}{t_1 + t_2 : T_1 \sqcup T_2} \qquad \begin{array}{c} x : T_1 \vdash xx : C \\ x : T_2 \vdash xx : C \end{array}}{(t_1 + t_2)(t_1 + t_2) : C}$$

While

$$(t_1 + t_2)(t_1 + t_2) \ \rightarrow\ t_1 t_2 \ \rightarrow\ (\lambda y.\delta)\,\mathsf{a}\,\delta \ \rightarrow\ \delta\,\delta \notin \mathcal{SN}$$

Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

$$t_1 \quad =_{\text{def}} \quad \lambda x.x\,\mathsf{a}\,\delta \qquad\qquad t_2 \quad =_{\text{def}} \quad \lambda y.\delta$$

$$(\sqcup E)\ \dfrac{\dfrac{t_1 : T_1 \qquad t_2 : T_2}{t_1 + t_2 : T_1 \sqcup T_2} \qquad \begin{array}{c} x : T_1 \vdash xx : C \\ x : T_2 \vdash xx : C \end{array}}{(t_1 + t_2)(t_1 + t_2) : C}$$

While

$$(t_1 + t_2)(t_1 + t_2) \ \to\ t_1 t_2 \ \to\ (\lambda y.\delta)\,\mathsf{a}\,\delta \ \to\ \delta\,\delta \notin \mathcal{SN}$$

Similar example with a **confluent** rewrite system.

Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

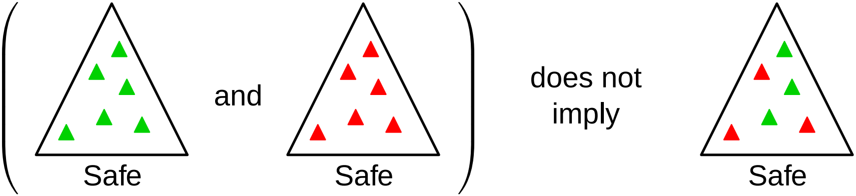
$$(\sqcup E)\ \frac{\dfrac{t_1 : T_1 \qquad t_2 : T_2}{t_1 + t_2 : T_1 \sqcup T_2} \qquad \begin{array}{c} x : T_1 \vdash c : C \\ x : T_2 \vdash c : C \end{array}}{c[(t_1 + t_2)/x] : C}$$

Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

$$(\sqcup \mathsf{E})\ \frac{\dfrac{t_1 : T_1 \qquad t_2 : T_2}{t_1 + t_2 : T_1 \sqcup T_2} \qquad \begin{array}{c} x : T_1 \vdash c : C \\ x : T_2 \vdash c : C \end{array}}{c[(t_1 + t_2)/x] : C}$$

But

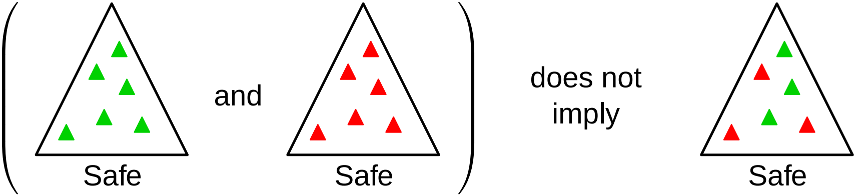


Unsafe Interaction [Rib07b]

$$t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_1 \qquad\qquad t_1 + t_2 \quad \mapsto_{\mathcal{R}} \quad t_2$$

$$(\sqcup E)\ \frac{\dfrac{t_1 : T_1 \qquad t_2 : T_2}{t_1 + t_2 : T_1 \sqcup T_2} \qquad \begin{array}{c} x : T_1 \vdash c : C \\ x : T_2 \vdash c : C \end{array}}{c[(t_1 + t_2)/x] : C}$$

But



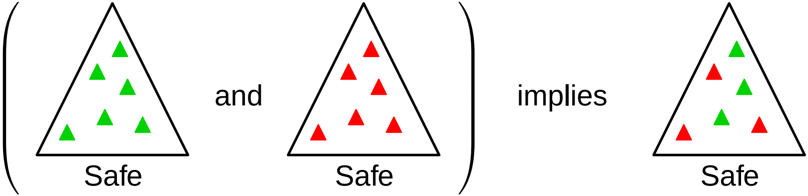
Prevents from having $[\![T_1 \sqcup T_2]\!] = [\![T_1]\!] \cup [\![T_2]\!]$.

Sufficient Conditions for $[\![T_1 \sqcup T_2]\!] = [\![T_1]\!] \cup [\![T_2]\!]$

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$ and $\mathcal{E}$ be a set of elimination contexts.

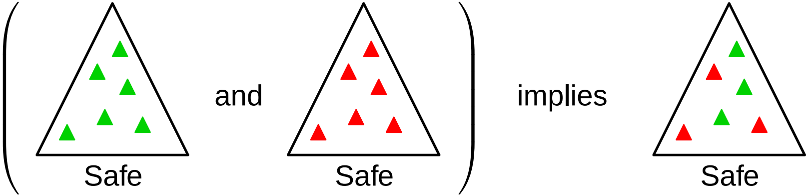
Sufficient Conditions for $[\![T_1 \sqcup T_2]\!] = [\![T_1]\!] \cup [\![T_2]\!]$

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$ and $\mathcal{E}$ be a set of elimination contexts.

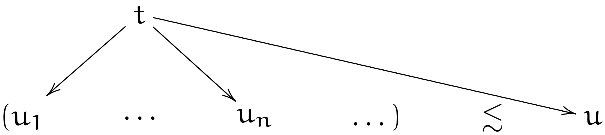


Sufficient Conditions for $[\![T_1 \sqcup T_2]\!] = [\![T_1]\!] \cup [\![T_2]\!]$

Let $\rightarrow_R$ be a rewrite relation on $\Lambda(\Sigma)$ and $\mathcal{E}$ be a set of elimination contexts.



OK if "▲ $\lesssim$ ▲" i.e. if t neutral has a "principal reduct" u:



Reducibility Candidates

Let $\emptyset \neq \mathcal{A} \subseteq \mathfrak{CR}$. We want

$$\bigcup \mathcal{A} \quad \in \quad \mathfrak{CR}$$

- $\mathcal{SN}$ and $(\mathfrak{CR}0)$ are OK.
- For $(\mathfrak{CR}1)$, let $\mathrm{t} \in \mathcal{N}$ with $(\mathrm{t})_{\rightarrow} \subseteq \bigcup \mathcal{A}$.

Reducibility Candidates

Let $\emptyset \neq \mathcal{A} \subseteq \mathfrak{CR}$. We want

$$\bigcup \mathcal{A} \quad \in \quad \mathfrak{CR}$$

- $\mathcal{SN}$ and $(\mathfrak{CR}0)$ are OK.
- For $(\mathfrak{CR}1)$, let $t \in \mathcal{N}$ with $(t)_{\rightarrow} \subseteq \bigcup \mathcal{A}$.
  If t is not normal, then

  $$\forall u \, (t \rightarrow u \quad \Longrightarrow \quad \text{there is some } A_u \in \mathcal{A} \text{ such that } u \in A_u) \; .$$

  That is

  $$t \in \mathcal{N}$$
  $$\swarrow \qquad\qquad \searrow$$
  $$u_1 \in A_{u_1} \qquad \cdots \qquad u_n \in A_{u_n}$$

- We are looking for some i such that $t \in A_{u_i}$.

Reducibility Candidates

- Let $u \lesssim_{\mathcal{CR}} t$ iff every value of u is a value of t, that is

$$\forall v \in \mathcal{V}. \qquad u \to_{R}^{*} v \quad \Longrightarrow \quad t \to_{R}^{*} v$$

Reducibility Candidates

- Let $\mathrm{u} \lesssim_{\mathfrak{CR}} \mathrm{t}$ iff every value of u is a value of t, that is

$$\forall v \in \mathcal{V}. \qquad \mathrm{u} \to_{\mathrm{R}}^{*} v \quad \Longrightarrow \quad \mathrm{t} \to_{\mathrm{R}}^{*} v$$

Lemma

Given $\mathrm{C} \in \mathfrak{CR}$ *and* $\mathrm{t} \in \mathcal{SN}$, *we have* $\mathrm{t} \in \mathrm{C}$ *iff*

$$\forall v \in \mathcal{V}. \qquad \mathrm{t} \to^{*} v \quad \Longrightarrow \quad v \in \mathrm{C}\ .$$

Reducibility Candidates

- Let $u \lesssim_{\mathfrak{CR}} t$ iff every value of u is a value of t, that is

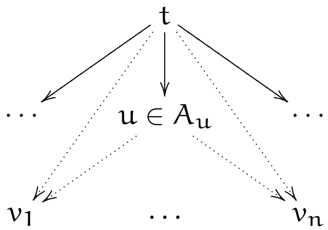
$$\forall v \in \mathcal{V}. \qquad u \to_R^* v \quad \Longrightarrow \quad t \to_R^* v$$

Lemma

Given $C \in \mathfrak{CR}$ and $t \in \mathcal{SN}$, we have $t \in C$ iff

$$\forall v \in \mathcal{V}. \qquad t \to^* v \quad \Longrightarrow \quad v \in C\,.$$

- If there is u s.t. $t \to u$ and $t \lesssim_{\mathfrak{CR}} u$, then we are done:



$$\Longrightarrow \qquad t \;\in\; A_u \;\subseteq\; \bigcup \mathcal{A}$$

Stability by Union of Reducibility Candidates [Rib07a]

Theorem

The following are equivalent:

(i) $\mathfrak{CR}$ *is stable by union,*

(ii) $\mathfrak{CR}$ *is the set of all non-empty subsets* C *of* $\mathcal{SN}$ *that are downward closed wrt.* $\lesssim_{\mathfrak{CR}}$*.*

(iii) *for every* t *which is non-normal, strongly normalizing and neutral, there is a term* u *such that*

$$t \rightarrow_R u \qquad \text{and} \qquad t \lesssim_{\mathfrak{CR}} u$$

Stability by Union of Reducibility Candidates [Rib07a]

Theorem

The following are equivalent:

(i) $\mathfrak{CR}$ *is stable by union,*

(ii) $\mathfrak{CR}$ *is the set of all non-empty subsets* C *of* $\mathcal{SN}$ *that are downward closed wrt.* $\lesssim_{\mathfrak{CR}}$.

(iii) *for every* t *which is non-normal, strongly normalizing and neutral, there is a term* u *such that*

$$t \rightarrow_R u \qquad \textit{and} \qquad t \lesssim_{\mathfrak{CR}} u$$

- u is a strong principal reduct of t.

Stability by Union of Reducibility Candidates [Rib07a]

Theorem

The following are equivalent:

(i) $\mathfrak{CR}$ *is stable by union,*

(ii) $\mathfrak{CR}$ *is the set of all non-empty subsets* C *of* $\mathcal{SN}$ *that are downward closed wrt.* $\lesssim_{\mathfrak{CR}}$.

(iii) *for every* t *which is non-normal, strongly normalizing and neutral, there is a term* u *such that*

$$t \to_R u \qquad \textit{and} \qquad t \lesssim_{\mathfrak{CR}} u$$

- u is a strong principal reduct of t.
- This holds for the λ-calculus with products and sums (also [Tat07]).

Biorthogonals

- Biorthogonals are reducibility candidates.

Biorthogonals

- Biorthogonals are reducibility candidates.
- In general, biorthogonals are not stable by union.

Biorthogonals

- Biorthogonals are reducibility candidates.
- In general, biorthogonals are not stable by union.
- Is the closure by union $\mathcal{O}$ of biorthogonals a reducibility family ?

Biorthogonals [Rib07b]

- Is the closure by union $\mathcal{O}$ of biorthogonals a reducibility family ?

Biorthogonals [Rib07b]

- Is the closure by union $\mathcal{O}$ of biorthogonals a reducibility family ?
- Let $u \lesssim_{\mathcal{SN}} t$ iff

$$\text{for all} \quad E[\ ] \in \mathcal{E}, \qquad \text{if} \quad E[u] \in \mathcal{SN} \quad \text{then} \quad E[t] \in \mathcal{SN}$$

Biorthogonals [Rib07b]

- Is the closure by union $\mathcal{O}$ of biorthogonals a reducibility family ?
- Let $u \lesssim_{\mathcal{SN}} t$ iff

$$\text{for all} \quad E[\ ] \in \mathcal{E}, \qquad \text{if} \quad E[u] \in \mathcal{SN} \quad \text{then} \quad E[t] \in \mathcal{SN}$$

Theorem

The following are equivalent:

(i) *unions of biorthogonals are reducibility candidates,*

(ii) *for every t which is non-normal, strongly normalizing and neutral, there is a term u such that*

$$t \rightarrow_R u \qquad \textit{and} \qquad u \lesssim_{\mathcal{SN}} t$$

Biorthogonals [Rib07b]

- Is the closure by union $\mathcal{O}$ of biorthogonals a reducibility family ?
- Let $u \lesssim_{\mathcal{SN}} t$ iff

$$\text{for all} \quad E[\ ] \in \mathcal{E}, \qquad \text{if} \quad E[u] \in \mathcal{SN} \quad \text{then} \quad E[t] \in \mathcal{SN}$$

Theorem

The following are equivalent:

(i) *unions of biorthogonals are reducibility candidates,*

(ii) *for every t which is non-normal, strongly normalizing and neutral, there is a term u such that*

$$t \rightarrow_R u \qquad \textit{and} \qquad u \lesssim_{\mathcal{SN}} t$$

u is a principal reduct of t.

Comparison [Rib07b]

Lemma

$$\forall t, u \in \mathcal{SN}. \qquad t \lesssim_{\mathfrak{CR}} u \qquad \Longrightarrow \qquad u \lesssim_{\mathcal{SN}} t$$

Comparison [Rib07b]

Lemma

$$\forall t, u \in \mathcal{SN}. \qquad t \lesssim_{\mathfrak{CR}} u \qquad \Longrightarrow \qquad u \lesssim_{\mathcal{SN}} t$$

Every strong principal reduct is a principal reduct.

Lemma

$$\mathfrak{CR} \textit{ stable by union} \qquad \Longrightarrow \qquad \mathfrak{O} \subseteq \mathfrak{CR}$$

Comparison [Rib07b]

Lemma

$$\forall t, u \in \mathcal{SN}. \qquad t \lesssim_{\mathfrak{CR}} u \qquad \Longrightarrow \qquad u \lesssim_{\mathcal{SN}} t$$

Every strong principal reduct is a principal reduct.

Lemma

$\mathfrak{CR}$ *stable by union* $\quad \Longrightarrow \quad \mathcal{O} \subseteq \mathfrak{CR}$

The converse is false, consider

$$p \mapsto_{\mathcal{R}} \lambda x.c_1 \qquad p \mapsto_{\mathcal{R}} \lambda x.c_2 \qquad c_i \mapsto_{\mathcal{R}} d$$

Indeed,

$$p \not\lesssim_{\mathfrak{CR}} \lambda x.c_i \qquad \text{but} \qquad \lambda x.c_i \lesssim_{\mathcal{SN}} p$$

Back to Saturated Sets

Pure λ-calculus

- $S \subseteq \mathcal{SN}_\beta$ is a **saturated set** ($S \in \mathcal{SAT}$) iff

  $(\mathcal{SAT}1)$ if $E[\ ] \in \mathcal{SN}_\beta$ and $x \in \mathcal{V}ar$ then $E[x] \in S$,

  $(\mathcal{SAT}2_\beta)$ if $E[t[u/x]] \in S$ and $u \in \mathcal{SN}_\beta$ then $E[(\lambda x.t)u] \in S$.

Back to Saturated Sets

Pure λ-calculus

- $S \subseteq \mathcal{SN}_\beta$ is a **saturated set** ($S \in \mathcal{SAT}$) iff

  $(\mathcal{SAT}1)$ if $E[\,] \in \mathcal{SN}_\beta$ and $x \in \mathcal{V}ar$ then $E[x] \in S$,

  $(\mathcal{SAT}2_\beta)$ if $E[t[u/x]] \in S$ and $u \in \mathcal{SN}_\beta$ then $E[(\lambda x.t)u] \in S$.

Rewriting

- In addition to $(\mathcal{SAT}1)$ and $(\mathcal{SAT}2_\beta)$ we need stability by reduction (if $t \in S$ and $t \to_{\beta\mathcal{R}} u$ then $u \in S$), and that for all $E[f(t_1, \dots, t_n)]$,

$$
\begin{array}{ccccc}
 & E[f(t_1,\dots,t_n)] & & \Longrightarrow & E[f(t_1,\dots,t_n)] \in S \\
\swarrow_{\beta\mathcal{R}} & & \searrow^{\beta\mathcal{R}} & & \\
u_1 \in S & \dots & u_n \in S & &
\end{array}
$$

Back to Saturated Sets

Pure λ-calculus

- $S \subseteq \mathcal{SN}_\beta$ is a **saturated set** ($S \in \mathcal{SAT}$) iff

  $(\mathcal{SAT}1)$ if $E[\ ] \in \mathcal{SN}_\beta$ and $x \in \mathcal{V}ar$ then $E[x] \in S$,

  $(\mathcal{SAT}2_\beta)$ if $E[t[u/x]] \in S$ and $u \in \mathcal{SN}_\beta$ then $E[(\lambda x.t)u] \in S$.

Rewriting

- In addition to $(\mathcal{SAT}1)$ and $(\mathcal{SAT}2_\beta)$ we need stability by reduction (if $t \in S$ and $t \to_{\beta\mathcal{R}} u$ then $u \in S$), and that for all $E[f(t_1, \dots, t_n)]$,

$$E[f(t_1, \dots, t_n)] \xrightarrow{\beta\mathcal{R}} u_1 \in S \quad \dots \quad E[f(t_1, \dots, t_n)] \xrightarrow{\beta\mathcal{R}} u_n \in S \qquad \Longrightarrow \qquad E[f(t_1, \dots, t_n)] \in S$$

If $\to_R$ has principal reducts, then we only need

- $S \in \mathcal{SAT}_R$ iff $\mathcal{V}ar^{\perp\!\!\perp} \subseteq S \subseteq \mathcal{SN}$ and

  $(\mathcal{SAT})$ if $E[u] \in S$ and u is a principal reduct of t then $E[t] \in S$.

Outline

Constructor Rewriting

- **Constructors** are symbols c of type

$$\frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash \mathsf{c}(t_1, \dots, t_n) : \mathsf{B}}$$

Constructor Rewriting

- **Constructors** are symbols c of type

$$\frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash \mathsf{c}(t_1, \dots, t_n) : \mathsf{B}}$$

- **Constructor patterns**

$$p \quad ::= \quad x \quad | \quad \mathsf{c}(p_1, \dots, p_n)$$

Constructor Rewriting

- **Constructors** are symbols c of type

$$\frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash \mathsf{c}(t_1, \dots, t_n) : \mathsf{B}}$$

- **Constructor patterns**

$$p \quad ::= \quad x \quad | \quad \mathsf{c}(p_1, \dots, p_n)$$

- **Rewrite rules**

$$\mathsf{f}(p_1, \dots, p_n) \quad \mapsto_{\mathcal{R}} \quad r$$

where $p_1, \dots, p_n$ are constructor patterns.

Values

- **Destructors**
  For each n-ary c with $n \geq 1$ and each $i \in \{1, \ldots, n\}$,

Values

- **Destructors**
  For each n-ary c with $n \geq 1$ and each $i \in \{1, \ldots, n\}$,

$$\mathsf{d}_{\mathsf{c},i}(\mathsf{c}(x_1, \ldots, x_n)) \quad \mapsto_{\mathcal{D}} \quad x_i$$

Values

- **Destructors**
  For each n-ary c with $n \geq 1$ and each $i \in \{1, \ldots, n\}$,

$$\mathsf{d}_{\mathsf{c},i}(\mathsf{c}(x_1, \ldots, x_n)) \quad \mapsto_{\mathcal{D}} \quad x_i$$

  For each nullary c,

$$\mathsf{d}_{\mathsf{c}}\,\mathsf{c} \quad \mapsto_{\mathcal{D}} \quad \mho$$

Values

- **Destructors**
  For each n-ary c with $n \geq 1$ and each $i \in \{1, \dots, n\}$,

$$\mathsf{d}_{\mathsf{c},i}(\mathsf{c}(x_1, \dots, x_n)) \quad \mapsto_{\mathcal{D}} \quad x_i$$

  For each nullary c,

$$\mathsf{d}_{\mathsf{c}}\,\mathsf{c} \quad \mapsto_{\mathcal{D}} \quad \mho$$

- **Elimination contexts**

$$E[\,] \in \mathcal{E}_{\Rightarrow\mathcal{C}} \quad ::= \quad [\,] \quad | \quad E[\,]\,t \quad | \quad \mathsf{d}(E[\,])$$

Values

- **Destructors**
  For each n-ary c with $n \geq 1$ and each $i \in \{1, \dots, n\}$,

$$\mathsf{d}_{\mathsf{c},i}(\mathsf{c}(x_1, \dots, x_n)) \quad \mapsto_{\mathcal{D}} \quad x_i$$

  For each nullary c,

$$\mathsf{d}_\mathsf{c}\,\mathsf{c} \quad \mapsto_{\mathcal{D}} \quad \mho$$

- **Elimination contexts**

$$E[\,] \in \mathcal{E}_{\Rightarrow\mathcal{C}} \quad ::= \quad [\,] \quad | \quad E[\,]\,t \quad | \quad \mathsf{d}(E[\,])$$

- **Values**

$$\lambda x.t \qquad\qquad \mathsf{c}(t_1, \dots, t_n)$$

External Redexes (CCERSs) [KOO01]

Let $\mathcal{R}$ be a constructor rewrite system.

External Redexes (CCERSs) [KOO01]

Let $\mathcal{R}$ be a constructor rewrite system.

- A subterm (position) u occurs in a **redex argument** of a term t if either

External Redexes (CCERSs) [KOO01]

Let $\mathcal{R}$ be a constructor rewrite system.

- A subterm (position) u occurs in a **redex argument** of a term t if either
  t has a subterm $(\lambda x.t_1)t_2$ and u occurs in some t_i or

External Redexes (CCERSs) [KOO01]

Let $\mathcal{R}$ be a constructor rewrite system.

- A subterm (position) u occurs in a **redex argument** of a term t if either
  t has a subterm $(\lambda x.t_1)t_2$ and u occurs in some t_i or
  t has a subterm $\mathtt{f}(l_1\sigma, \dots, l_n\sigma)$ and u occurs in some $l_i\sigma$
  where

$$\mathtt{f}(l_1, \dots, l_n) \quad \mapsto_{\mathcal{R}} \quad r$$

External Redexes (CCERSs) [KOO01]

Let $\mathcal{R}$ be a constructor rewrite system.

- A subterm (position) u occurs in a **redex argument** of a term t if either
  t has a subterm $(\lambda x.t_1)t_2$ and u occurs in some t_i or
  t has a subterm $\mathtt{f}(l_1\sigma, \dots, l_n\sigma)$ and u occurs in some $l_i\sigma$ where
  $$\mathtt{f}(l_1, \dots, l_n) \quad \mapsto_{\mathcal{R}} \quad r$$
- A redex (position) u is **external** in a term t
  if no residual of u occurs in a redex argument of a reduct of t.

External Redexes (CCERSs) [KOO01]

Let $\mathcal{R}$ be a constructor rewrite system.

- A subterm (position) u occurs in a **redex argument** of a term t if either
  t has a subterm $(\lambda x.t_1)t_2$ and u occurs in some t_i or
  t has a subterm $\mathtt{f}(l_1\sigma, \ldots, l_n\sigma)$ and u occurs in some $l_i\sigma$
  where

$$\mathtt{f}(l_1, \ldots, l_n) \quad \mapsto_{\mathcal{R}} \quad r$$

- A redex (position) u is **external** in a term t
  if no residual of u occurs in a redex argument of a reduct of t.
- If $t \rightarrow_{\beta\mathcal{R}} u$ by contracting an external redex of t,
  then u is an **external reduct** of t.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an **orthogonal** constructor rewrite system.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an **orthogonal** constructor rewrite system.

Lemma

*Let t be a **neutral** term and u be an **external** reduct of t.*

If t reduces to a value v then u reduces to v.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an **orthogonal** constructor rewrite system.

Lemma

Let t be a neutral term and u be an external reduct of t.

If t reduces to a value v then u reduces to v.

- If t is **neutral** and u is an **external** reduct of t then $t \lesssim_{\mathfrak{CR}} u$.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an orthogonal constructor rewrite system.

Lemma

Let t be a neutral term and u be an external reduct of t.

If t reduces to a value v then u reduces to v.

- If t is neutral and u is an external reduct of t then $t \lesssim_{\mathcal{CR}} u$.
- **Theorem [KOO01]**
  If $\mathcal{R}$ is **orthogonal**
  then every reducible term has an **external** redex.

External Reducts are Strong Principal [Rib08]

Let $\mathcal{R}$ be an orthogonal constructor rewrite system.

Lemma

Let t be a neutral term and u be an external reduct of t.

If t reduces to a value v then u reduces to v.

- If t is neutral and u is an external reduct of t then $t \lesssim_{\mathfrak{CR}} u$.
- **Theorem [KOO01]**
  If $\mathcal{R}$ is orthogonal
  then every reducible term has an external redex.

Corollary

If $\mathcal{R}$ is an ***orthogonal constructor*** *rewrite system*
then $\mathfrak{CR}$ is stable by union.

Outline

Conclusion

We have studied different reducibility families,
and compared them wrt. stability by union.

Conclusion

We have studied different reducibility families,
and compared them wrt. stability by union.

- Some rewrite systems do not admit reducibility families stable by union.

Conclusion

We have studied different reducibility families,
and compared them wrt. stability by union.

- Some rewrite systems do not admit reducibility families stable by union.
- Sufficient conditions to have a reducibility family stable by union.

Conclusion

We have studied different reducibility families,
and compared them wrt. stability by union.

- Some rewrite systems do not admit reducibility families stable by union.
- Sufficient conditions to have a reducibility family stable by union.
- Investigation of the structure of Girard's Candidates.

Conclusion

We have studied different reducibility families,
and compared them wrt. stability by union.

- Some rewrite systems do not admit reducibility families stable by union.
- Sufficient conditions to have a reducibility family stable by union.
- Investigation of the structure of Girard's Candidates.
- For the combination of λ-calculus with orthogonal constructor rewriting, Girard's Candidates are stable by union.

Conclusion

We have studied different reducibility families,
and compared them wrt. stability by union.

- Some rewrite systems do not admit reducibility families stable by union.
- Sufficient conditions to have a reducibility family stable by union.
- Investigation of the structure of Girard's Candidates.
- For the combination of λ-calculus with orthogonal constructor rewriting, Girard's Candidates are stable by union.
- In [Rib07b], we studied a type system with $(\sqcup \mathrm{E})$ such that for simple rewrite systems $\mathcal{R}$, the following are equivalent:
  (i) terms typable using $(\sqcup \mathrm{E})$ are Strongly Normalizing,
  (ii) the interpretation $(\!|\_|\!) : \mathcal{T} \to \mathcal{P}^\star(\mathcal{SN})^{\perp\!\!\perp\perp\!\!\perp}$ is adequate.

Thank you for your attention !

http://www.loria.fr/~riba/

- A. Abel – “Semi-Continuous Sized Types and Termination”, *Proceedings of CSL’06*, LNCS, vol. 4207, Springer, 2006, p. 72–88.
- F. Blanqui et C. Riba – “Combining Typing and Size Constraints for Checking the Termination of Higher-Order Conditional Rewrite Systems”, *Proceedings of LPAR’06*, LNAI, vol. 4246, 2006.
- V. Danos et J.-L. Krivine – “Disjunctive Tautologies as Synchronisation Schemes”, *Proceedings of CSL’00*, LNCS, vol. 1862, 2000, p. 292–301.
- J.-Y. Girard – “Interprétation Fonctionnelle et Élimination des Coupures de l’Arithmétique d’Ordre Supérieur”, Thèse, Université Paris 7, 1972.
- — , “Linear Logic”, *Theoretical Computer Science* **50** (1987), p. 1–102.

- Z. KHASIDASHVILI, M. OGAWA et V. VAN OOSTROM – "Perpetuality ans Uniform Normalization in Orthogonal Rewrite Systems", *Information and Computation* **164** (2001), no. 1, p. 118–152.
- P.-A. MELLIÈS et J. VOUILLON – "Recursive Polymorphic Types and Parametricity in an Operational Framework", *Proceedings of LiCS'05*, IEEE Computer Society, 2005, p. 82–91.
- M. PARIGOT – "Proofs of Strong Normalization for Second Order Classical Natural Deduction", *Journal of Symbolic Logic* **62** (1997), no. 4, p. 1461–1479.
- A. M. PITTS – "Parametric Polymorphism and Operational Equivalence", *Mathematical Structures in Computer Science* **10** (2000), p. 321–359.
- C. RIBA – "On the Stability by Union of Reducibility Candidates", *Proceedings of FoSSaCS'07*, LNCS, vol. 4423, 2007.

— , “Strong Normalization as Safe Interaction”, *Proceedings of LiCS’07*, IEEE Computer Society, 2007, p. 13–22.

— , “Unions of Reducibility Candidates for Orthogonal Constructor Rewriting”, Available on HAL-INRIA `http://hal.inria.fr`, 2008.

W. W. TAIT – “A Realizability Interpretation of the Theory of Species”, *Logic Colloquium* (R. PARIKH, éd.), LNCS, vol. 453, 1975, p. 240–251.

M. TATSUTA – “Simple Saturated Sets for Disjunction and Second-Order Existential Quantification”, *Proceedings of TLCA’07*, LNCS, vol. 4583, Springer, 2007, p. 366–380.