On the Expressiveness of Interaction Nets and their Non-deterministic Variants

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Marseille, 18 June 2008

*Supported by a post-doctoral fellowship of the *Fondation Sciences Mathématiques de Paris*.

Motivations

- Interaction nets are Turing-complete. This means nothing!
- There are several non-deterministic variants of interaction nets:
  - multiwire (Alexiev, Beffara-Maurel);
  - multiport (Alexiev, Khalil, Mazza);
  - multirule (Alexiev, Ehrhard-Regnier).
- How do these relate to each other?
- We want to compare computational models in terms of the *computational dynamics* they can express.
- Causality, parallelism, non-determinism must be seen as key ingredients.

Event structures

- An event structure is a triple $E = (|E|, \leq, \smile)$ such that:
  - $|E|$ is a set of *events*;
  - $\leq$ is a partial order on $|E|$, called *causal order*, such that, for all $x \in |E|$, $\lceil x \rceil = \{y \in |E| \mid y \leq x\}$ is finite;
  - $\smile$ is an anti-reflexive symmetric relation on $|E|$, called *conflict relation*, such that, for all $x, y, z \in |E|$, $x \smile y \leq z$ implies $x \smile z$.

- Let $u \subseteq |E|$. We say that u is a *configuration* iff

  causality: $x \in u$ and $y \leq x$ implies $y \in u$.
  coherence: $x, y \in u$ implies $x \mathrel{\overset{\frown}{\smile}} y$;

  The set of configurations of E is denoted by $\mathcal{D}(E)$.

- An event structure E may be *labelled* over a set $\mathcal{L}$ by a partial function $\ell : |E| \rightharpoonup \mathcal{L}$. If $u \in \mathcal{D}(E)$, we pose $|u| = \{x \in u \mid \ell(x)\downarrow\}$.

History preserving bisimulation

- Let $u, v \in \mathcal{D}(E)$ with $v = u \cup \{x\}$, $x \notin u$. We define:
  - $u \stackrel{a}{\longrightarrow} v$ iff $\ell(x) = a$, and $u \longrightarrow v$ iff $\ell(x)\uparrow$;
  - $\stackrel{a}{\Longrightarrow} = \longrightarrow^* \stackrel{a}{\longrightarrow} \longrightarrow^*$.
- A *history preserving bisimulation* between E and F is a relation

$$\mathcal{B} \subseteq \mathcal{D}(E) \times |E| \rightharpoonup |F| \times \mathcal{D}(F)$$

such that, $(\emptyset, \bot, \emptyset) \in \mathcal{B}$ and, whenever $(u, f, v) \in \mathcal{B}$, we have:

i. the restriction of f to $|u|$ is a poset isomorphism between $|u|$ and $|v|$ (hence, in particular, $|u| \subseteq \mathrm{dom}\, f$);

ii. $u \stackrel{a}{\longrightarrow} u'$ implies $v \stackrel{a}{\Longrightarrow} v'$ with $(u', f', v') \in \mathcal{B}$ and $f \subseteq f'$;

iii. $u \longrightarrow u'$ implies $v \longrightarrow^* v'$ with and $(u', f, v') \in \mathcal{B}$;

iv. $v \stackrel{a}{\longrightarrow} v'$ implies $u \stackrel{a}{\Longrightarrow} u'$ with $(u', f', v') \in \mathcal{B}$ and $f \subseteq f'$;

v. $v \longrightarrow v'$ implies $u \longrightarrow^* u'$ with and $(u', f, v') \in \mathcal{B}$.

- We write $E \simeq F$ if there exists a hp-bisimulation between E and F.

Abstract rewriting systems

- An *abstract rewriting system* (ARS) is a tuple $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \partial_0, \partial_1, \cdot[\cdot])$
  - $\mathcal{S}_0$ is a set of *objects*;
  - $\mathcal{S}_1$ is a set of *radicals*;
  - $\partial_0 : \mathcal{S}_1 \to \mathcal{S}_0$ is the *source* function;
  - $\partial_1 : \mathcal{S}_1 \to \mathcal{S}_0$ is the *target* function;
  - $\cdot[\cdot] : \mathcal{S}_1 \times \mathcal{S}_1 \rightharpoonup \mathcal{P}(\mathcal{S}_1)$ is the *residue* function;
  - $s[r]$ is defined exactly when r and s are coinitial, in which case we have, for all $t \in s[r]$, $\partial_0(t) = \partial_1(r)$. The radicals of $s[r]$ are said to be the *residues* of s under r.
- A *standard* ARS additionally satisfies
  - for all $\mu \in \mathcal{S}_0$, the set $\{r \in \mathcal{S}_1 \mid \partial_0(r) = \mu\}$ is finite;
  - for all $r \in \mathcal{S}_1$, $r[r] = \emptyset$.
- If S ia a set of coinitial radicals we pose $S[r] = \bigcup_{s \in S} s[r]$.

Reductions and equivalence

- $\rho \in \mathcal{S}_0^*$ is *well-formed* if it is non-empty and, for all $1 \leq i < |\rho|$, $\partial_0(\rho_{i+1}) = \partial_1(\rho_i)$. $\mathcal{S}^+$ is the set of well-formed words.

- A *reduction* is an element of the set

$$\mathcal{S}^* = \mathcal{S}^+ \cup \{id_\mu \mid \mu \in \mathcal{S}_0\}.$$

- Source and target functions obviously extend to reductions. For what concerns residues, we define $S[\rho]$ by induction on the length of ρ:
  - $S[id_\mu] = S$;
  - let $\rho = \rho' r$; we put $S[\rho] = S[\rho'][r]$.

- $\rho \leftrightharpoons \sigma$ iff ρ, σ coinitial, cofinal, and, for all coinitial t, $t[\rho] = t[\sigma]$.

Normal abstract rewriting systems and homotopy

- An ARS is *normal* if, given any three coinitial radicals r, s, t, we have

  affinity: $\sharp s[r] \leq 1$;
  symmetry: $\sharp s[r] = \sharp r[s]$;
  if $\sharp s[r] = \sharp r[s] = 1$, we say that r and s are *orthogonal*, and we write $r \perp s$; in that case, the only element of $s[r]$ and $r[s]$ is denoted by s^r and r^s, respectively;
  permutation: if $r \perp s$, then $rs^r \leftrightharpoons sr^s$.

- We write $\rho \sim_0 \sigma$ iff there exist $r \perp s$ such that $\rho = \tau r s^r \tau'$ and $\sigma = \tau s r^s \tau'$.

- *Homotopy*: $\sim = \sim_0^* \subseteq \leftrightharpoons$.

Normal ARS's and event structures

- We put $\lesssim\ = \sim\leq\sim$, where $\leq$ is the word prefix relation.
- If x, y are homotopy classes, we define

$$
\begin{array}{lll}
x \leq y & \text{iff} & \forall \rho \in x, \sigma \in y.\, \rho \lesssim \sigma, \\
x \smile y & \text{iff} & \forall \rho \in x, \sigma \in y, \tau \in \mathcal{S}^*.\, \rho \lesssim \tau \text{ implies } \sigma \not\lesssim \tau.
\end{array}
$$

Proposition 1. *Let $\mathcal{S}$ be a (standard) normal ARS. Then, $(\mathcal{S}^*/\sim, \leq, \smile)$ is an event structure.*

Essential homotopy classes

- A non-identity reduction $\rho = \rho' r$ is *essential* iff, for all $\sigma \sim \rho$, $\sigma = \sigma' r$ with $\sigma' \sim \rho'$.

- ρ essential and $\sigma \sim \rho$ implies σ essential.

- A homotopy class is *essential* if it contains an essential reduction. In that case, there exists r s.t. $\rho \in x$ implies $\rho = \rho' r$; we put $\overline{x} = r$ (the *eigenradical* of x).

Event structure associated to a normal ARS

- A labelled ARS (LARS) is an ARS with a labelling $\ell : \mathcal{S}_1 \rightharpoonup \mathcal{L}$.

- The labelled event structure associated with a normal LARS $\mathcal{S}$, denoted by $\mathrm{Ev}(\mathcal{S})$, is the event structure defined above restricted to essential classes, and with label function $\ell^*(x) = \ell(\overline{x})$ if $\ell(\overline{x})\downarrow$, or undefined otherwise.

- The event structure associated to $\mu \in \mathcal{S}_0$, denoted by $\mathrm{Ev}(\mu)$, is the substructure of $\mathrm{Ev}(\mathcal{S})$ restricted to the classes x such that $\rho \in x$ implies $\partial_0(\rho) = \mu$.

Translations

- A *translation* from a normal LARS $\mathcal{S}$ to a normal LARS $\mathcal{T}$ is a relation $\mathcal{R} \subseteq \mathcal{S}_0 \times \mathcal{T}_0$ such that:

  1. for all $\mu \in \mathcal{S}_0$ there exists $\nu \in \mathcal{T}_0$ such that $(\mu, \nu) \in \mathcal{R}$;
  2. $(\mu, \nu) \in \mathcal{R}$ implies that, $\forall r \in \mathcal{S}_1$ s.t. $\partial_0(r) = \mu$, there exists $\rho \in \mathcal{T}^*$ s.t. $\partial_0(\rho) = \nu$ and $(\partial_1(r), \partial_1(\rho)) \in \mathcal{R}$;
  3. $(\mu, \nu) \in \mathcal{R}$ implies $\mathrm{Ev}(\mu) \simeq \mathrm{Ev}(\nu)$.

Results

- If one restricts to finite dynamics, interaction nets are "causally complete".
- Separation results:
  - there is no "natural" translation from the π-calculus to multirule interaction nets (hence in differential interaction nets);
  - multiport and multiwire interaction nets are stricly more expressive than multirule interaction nets (cf. Alexiev);
  - there is no finite system of multiport combinators not introducing divergence (cf. Khalil);
  - more to come. . .

Further work

- The definition of translations should be improved.
- The current notion of translation is perhaps too strict.
- Event structures are OK, but would it be possible to work directly with tools from algebraic topology? ("Real" homotopy, etc.).