Non concrete models of λ-calculus

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Outline

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Syntax of the untyped λ-calculus

Terms of untyped λ-calculus

λ-terms: $M, N ::= x \mid MN \mid \lambda x.M$

Examples

$\mathbf{I} \equiv \lambda x.x$, $\mathbf{T} \equiv \lambda xy.x$, $\mathbf{F} \equiv \lambda xy.y$, $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$.

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Open and closed λ-terms

Bound and free variables

An occurrence of a x in M is:

- *bound* if it lies within the "scope" of a $\lambda x.$,
- *free* otherwise.

Examples

$FV(\lambda x.x) = \emptyset$, $FV(\lambda xy.xr(yz)) = \{r, z\}$.

Notation

Λ = set of all λ-terms.

Λ^o = set of all closed λ-terms.

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Equivalence relations on λ-terms

α-conversion

$\lambda x.M =_\alpha \lambda y.M[x := y]$ for all $y \notin FV(M)$.

Recall

λ-terms are considered up to α-conversion.

β-conversion

$(\lambda x.M)N =_\beta M[x := N]$.

η-conversion (to obtain extensionality)

$\lambda x.Mx =_\eta M$, for all $x \notin FV(M)$.

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Lambda theories

λ-theory

Any congruence on Λ containing β-conversion.

Examples of λ-theories

$\lambda\beta$ = the smallest λ-theory,
$\lambda\beta\eta$ = the smallest extensional λ-theory.

Remark

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Models of the untyped λ-calculus

1969) Scott defines the first model of λ-calculus: $\mathcal{D}_\infty$.

Standard semantics: Scott-continuous semantics

$\mathcal{D}_\infty$ lives in the Scott-continuous semantics (*CPO*'s + Scott-continuous functions).

$\exists$ *non*-standard semantics: relational framework

(BEM - CSL'07) We have isolated and studied a concrete extensional model living in a relational semantics.

Problem

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What is a model of λ-calculus?

There are (at least) two possible answers...

Categorical view point

A reflexive object of a Cartesian closed category.

Algebraic view point

A combinatory algebra satisfying the 5 axioms of Curry + the Meyer-Scott axiom.

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Categorical view point

Take a Cartesian closed category...

Reflexive object

$$\mathcal{U} = (U, A, \lambda) \quad U \underset{\lambda}{\overset{A}{\rightleftarrows}} [U \Rightarrow U]$$

such that $A \circ \lambda = Id_{[U \Rightarrow U]}$ (*extensional* if $\lambda \circ A = Id_U$).

Interpretation of M

Given a list of variables $x_1, \ldots, x_n \supseteq FV(M)$, $|M|_{\vec{x}} : U^n \to U$:

- $|x|_{\vec{x}} = \pi_x$,
- $|NP|_{\vec{x}} = ev \circ (A \circ |N|_{\vec{x}}, |P|_{\vec{x}})$,
- $|\lambda z.N|_{\vec{x}} = \lambda \circ \Lambda(|N|_{\vec{x},z})$,

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Categorical view point

Enough points condition

The model $\mathcal{U} = (U, A, \lambda)$ has enough points if:

$$1 \xrightarrow{\forall h} U \overset{f}{\underset{g}{\rightrightarrows}} U$$

implies:

$$f = g$$

Algebraic view point

A *combinatory algebra* $(C, \cdot, \mathbf{k}, \mathbf{s})$ is an applicative structure such that:

$$\mathbf{k}xy = x; \qquad \mathbf{s}xyz = xz(yz).$$

Barendregt proposed 2 classes of algebras as models of λ-calculus:

λ-algebra

A combinatory algebra $\mathcal{A}$ satisfying the 5 Curry's axioms:

$$\text{(intuitively) } M =_\beta N \Rightarrow \mathcal{A} \models M = N$$

λ-model

A λ-algebra satisfying the Meyer-Scott's axiom:

$$\forall x, y[\forall z(xz = yz) \Rightarrow \varepsilon x = \varepsilon y]$$

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Advantages and disadvantages

λ-algebras

They satisfy all provable equations on λ-calculus and form an equational class, but they are *not sound* for λ-theories:

$$\mathcal{A} \models M = N \not\Rightarrow \mathcal{A} \models \lambda x.M = \lambda x.N.$$

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How these definitions are related?

Karubi envelope: λ-models $\to$ ccc

Every λ-model gives rise to a reflexive object with enough points.

Koymans' construction: ccc $\to$ λ-algebras/models

- Every reflexive object $\mathcal{U} = (U, A, \lambda)$ gives rise to a λ-algebra with carrier set $C(1, U)$,
- if $\mathcal{U}$ has enough points, then we obtain a λ-model.

Remark: Koymans had as *aim* to provide λ-algebras!

Common belief:

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Ideas behind the construction

Every reflexive object $\mathcal{U} = (U, A, \lambda)$ can be presented as a λ-model. Idea:

$$|M| : U^{Var} \to U \qquad \textit{(finitary morphism)}$$

only depending on a finite number of variables ($FV(M)$).

Free extension of λ-algebras

$\mathcal{A}[x_1, \dots, x_n] \models M = N \Rightarrow \mathcal{A}[x_1, \dots, x_n] \models \lambda x.M = \lambda x.N$ for all λ-terms M, N with $FV(M, N) \subseteq x_1, \dots, x_n$.

Correlation with Cartesian closed categories

$\mathcal{A}[x_1, \dots, x_n] \cong C(U^n, U)$ where $\mathcal{U} \mapsto \mathcal{A}$ (Koymans).

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Bucciarelli, Ehrhard, Manzonetto's construction

Combinatory algebra associated with $\mathcal{U} = (U, A, \lambda)$

Take $\mathcal{C}_{\mathcal{U}} = (C_f(U^{Var}, U), \bullet, |K|, |S|)$, where $a \bullet b = ev \circ (A \circ a, b)$.

BEM - CSL'07

The combinatory algebra $\mathcal{C}_{\mathcal{U}}$ is a λ-model.

Hence, also the reflexive objects without enough points can be turned into λ-models.

- There exists a *unique* definition of model of λ-calculus -

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Why ccc's without enough points?

Categories without enough points

In denotational semantics: they naturally appear when morphisms carry some "intensional" information.

- Sequential algorithms [Berry, Curien],
- strategies in various categories of games.

Simpler examples:

Relational frameworks

Cartesian closed categories with:

- Objects: sets,
- Morphisms: relations.

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Relational semantics of λ-calculus

From linear logic to λ-calculus...

Ideas underlying Girard'88

Rel $+ \mathcal{M}_f(-)$ generates (via the Kleiseli construction) a ccc **MRel**.

MRel

- Objects: sets,
- Morphisms: **MRel**$(A, B) = \mathcal{P}(\mathcal{M}_f(A) \times B)$ (relations between $\mathcal{M}_f(A)$ and B).

MRel constitutes a "relational semantics" of λ-calculus.

MRel is "strongly" intensional

No object ($\neq 1$) of **MRel** has enough points.

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Building a relational model of λ-calculus

$\mathcal{D} = (D, A, \lambda)$ in MRel

Sort of completion process:

- let $D_0 = \emptyset$,
- let $D_{n+1} = \mathcal{M}_f(D_n)^{(\omega)}$.

and take the union $D = \cup_{n\in\mathbb{N}} D_n$

Extensional model

- $(\sigma_1, \sigma_2, \sigma_3, \cdots) \in D$,
- $(\sigma_1, (\sigma_2, \sigma_3, \cdots)) \in \mathcal{M}_f(D) \times D = [D \Rightarrow D]$.

Hence, there is a trivial isomorphism $D \cong [D \Rightarrow D]$.

$\mathcal{D}$ does not have enough points

The intensionality of **MRel** *don't* affect the extensionality of $\mathcal{D}$.

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The intensionality of **MRel** *don't* affect the extensionality of $\mathcal{D}$.

What is the λ-theory induced by $\mathcal{D}$?

BEM's construction provides a λ-model $\mathcal{C}_\mathcal{D}$ associated with $\mathcal{D}$:

$$|M|_\rho \in \mathbf{MRel}(D^{Var}, D)$$

What equalities are induced?

$\mathcal{C}_\mathcal{D}$ can be "well stratified" using the stratification

$$D = \bigcup_{n \in \mathbb{N}} D_n.$$

Adapting a technique of [Hyland, Wadsworth], we prove $Th(\mathcal{C}_\mathcal{D}) = \mathcal{H}^*(= Th(\mathcal{D}))$.

Maximal consistent sensible λ-theory

$$\mathcal{H}^* = \{(M, N) : \forall C[-](C[M] \text{ is solvable } \iff C[N] \text{ is solvable})\}.$$

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Well stratifiable categorical models

Cpo-enriched ccc

A ccc is cpo-enriched if:

- every homset is a cpo: $(C(A,B), \sqsubseteq_{(A,B)}, \perp_{(A,B)})$,
- composition is continuous,
- pairing and currying are monotonic,
- $\perp \circ f = \perp$ and $ev \circ \langle \perp, f \rangle = \perp$.

Example

- Scott's continuous semantics,
- stable semantics,
- strongly stable semantics,
- the category **MRel**.

Well stratifiable categorical models

Cpo-enriched ccc

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Well stratifiable categorical models

Let $\mathcal{U}$ be a categorical model in a cpo-enriched ccc.

Projections $\exists p_K : U \to U$

- increasing sequence $(p_k)_{k\in\mathbb{N}}$ of projections,
- $\sqcup_{k\in\mathbb{N}} p_k = Id_U$.

Well stratifiable models

For all $a \in C(U^{Var}, U)$ we have:

- $\bot \bullet a = \bot$,
- $\lambda \circ \Lambda(\bot) = \bot$,
- $a_{k+1} \bullet b = (a \bullet b_k)_k$,
- $a_0 \bullet b = (a_0 \bullet b)_0$, where $a_k =_{\text{def}} a \circ p_k$.

The theory of well stratifiable models

Theorem 1

The theory of every well stratifiable $\perp$-model is $\mathcal{H}^*$.

Theorem 2

D is a well stratifiable $\perp$-model.

Corollary

$$Th(\mathcal{D}) = Th(\mathcal{C}_{\mathcal{D}}) = \mathcal{H}^*.$$

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Properties of $\mathcal{C}_{\mathcal{D}}$

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- $\mathcal{C}_{\mathcal{D}}$ is a (simpler) analogue of $\mathcal{D}_\infty$ in the relational semantics,
- it has a rich algebraic structure,
- its algebraic properties make it suitable for dealing with non-deterministic λ-calculi.

Outline

A non deterministic extension of λ-calculus

$$M, N ::= x \mid \lambda x.M \mid MN \mid M + N \mid M||N$$

Non-deterministic choice

$M + N$ reduces non-deterministically either to M or to N.
May convergence: $(M + N) \Downarrow$ if $(M \Downarrow) \vee (N \Downarrow)$.

Parallel composition

$M||N$ reduces to $M'||N'$ if M reduces to M' and N reduces to N'.
Must convergence: $(M||N) \Downarrow$ if $(M \Downarrow) \wedge (N \Downarrow)$.

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The semantics of "non-deterministic choice"

We can interpret the non-deterministic choice in $\mathcal{C}_{\mathcal{D}}$:

$$|M + N| = |M| \cup |N|$$

Semilinear applicative structure: $\mathcal{C}_{\mathcal{D}}$ + "$\cup$"

- $\cup$ is: idempotent, commutative, associative,
- $(x \cup y) \bullet z = (x \bullet z) \cup (y \bullet z)$.

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$$|(M||N)| = |M| * |N|$$

Definition of $*$

- $\sigma * \tau = (\sigma_1 \uplus \tau_1, \sigma_1 \uplus \tau_1, \ldots)$ for $\sigma, \tau \in D$
- $a * b = \{(m_1 \uplus m_2, \sigma * \tau) : (m_1, \sigma) \in a \quad (m_2, \sigma) \in b\}$ for $a, b \in \mathbf{MRel}(D^{Var}, D)$.

Properties: $\mathcal{C}_D$ + "*"

- $*$ is: commutative and associative,
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- $*$ is NOT idempotent: $|M| \neq |(M||M)|$ "resource sensible" model.

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Conclusions

- Complete equivalence between categorial models and λ-models,
- concrete categorical model $\mathcal{D}$ living in the relational semantics of λ-calculus,
- the theory of all well stratifiable models is $\mathcal{H}^*$,
- algebraic properties of the λ-model associated with $\mathcal{D}$,
- $\mathcal{D}$ is a model of the λ-calculus extended with: non-deterministic choice + parallel composition.

Future works

We would like to:

- analyze *with hindsight* the relations among the various notions of model of λ-calculus;
- study results of *full abstraction* for must/may semantics in $\mathcal{D}$.