Tree Automata Make Ordinal Theory Easy

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Plan:
Motivations, background
First Order theory of $(\omega^\omega, +)$, extension to $(\omega^{\omega^i}, +)$
Monadic Second Order theory of $(\omega^2, <)$, extension to $(\omega^i, <)$
Comparisons, ordinal automata, other orderings, ...

Abstract

We give a new simple proof of the decidability of the First Order Theory of $(\omega^{\omega^i}, +)$ and the Monadic Second Order Theory of $(\omega^i, <)$, improving the complexity in both cases. Our algorithm is based on tree automata and a representation of (sets of) ordinals by (infinite) trees. We give some extensions to other countable scattered linear orderings.

Les automates d'arbres facilitent la théorie des ordinaux

On propose une nouvelle démonstration, plus simple et avec une meilleure complexité, de la décidabilité de la logique du premier ordre de $(\omega^{\omega^i}, +)$ et de la logique monadique du second ordre de $(\omega^i, <)$. Notre algorithme utilise les automates d'arbres et une représentation des (ensembles d') ordinaux par des arbres (infinis). On donnera des extensions à certains ordres linéaires dispersés.

Motivations

Automata based decision procedure are efficient and simple.

e.g. FO, MSO / $\mathbb{N}$, binary trees, . . .

Ordinal theories: original proof uses ordinal automata [Büchi65].
Complexity improved using a complicated coding by finite words [Maurin97].

"We have a proof without theorem" Wolfgang Thomas, 2002
(closure of tree automata under boolean operations and projection)

Presburger arithmetic: FO($\mathbb{N}, +$)

[Presburger1930, Skolem1931]: quantifier elimination

[Büchi1960]: using finite automata

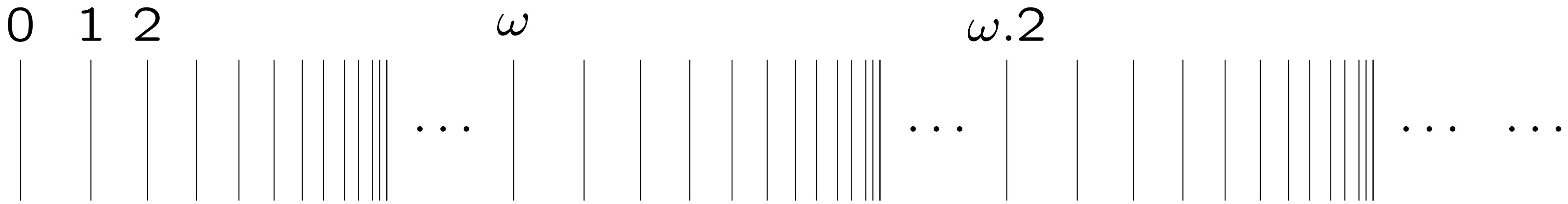
integer $x \in \mathbb{N}$ ↔ $u \in \{0, 1\}^*$ in binary
relation $x + y = z$ ↔ automaton over $\{0, 1\}^3$
models: set $X \subseteq \mathbb{N}$ ↔ automaton over $\{0, 1\}$
formula $\psi(x_1, \ldots, x_k)$ ↔ automaton over $\{0, 1\}^k$
quantification $\exists$ ↔ projection ($\rightsquigarrow$ non-determinism)
$\vee$, $\wedge$ ↔ union, product
negation ↔ complementation ($\rightsquigarrow$ costly)

Induction on the structure of the formula.
Emptiness is decidable

see [Klaedtke04]

Ordinal arithmetic

$1 + \omega = \omega \neq \omega + 1 \qquad 2.\omega = \omega \neq \omega.2$



∀ ordinal α: $\alpha = \{\beta : \beta < \alpha\}$

$$\begin{aligned} \omega^i + \omega^j &= \omega^j && \text{if } i < j \\ &= \omega^j.2 && \text{if } i = j \end{aligned}$$

Cantor Normal Form

$\forall\ 0 < \alpha < \omega^\omega$, $\exists$ unique integers $p, n_0, n_1, \ldots, n_p$, s.t.

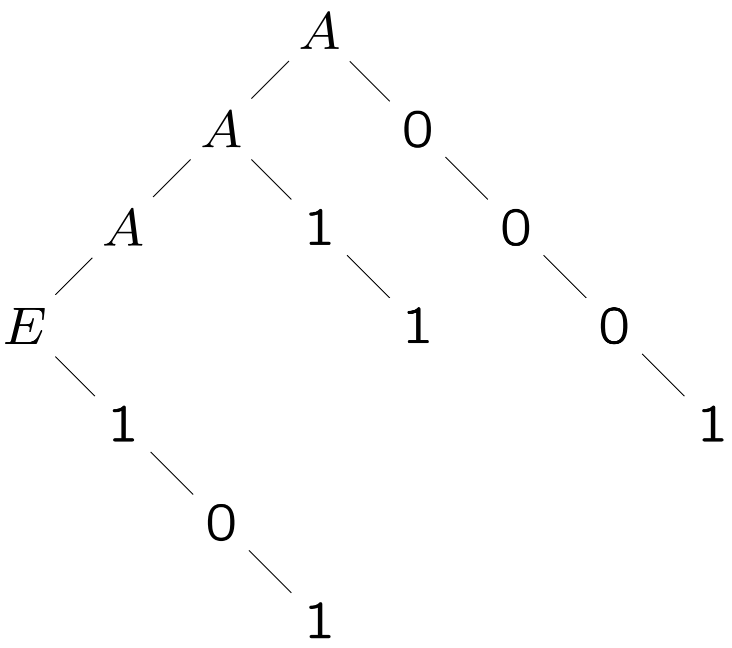
$$n_p > 0 \text{ and } \alpha = \omega^p n_p + \omega^{p-1} n_{p-1} + \ldots + \omega^1 n_1 + n_0$$

Given $\alpha = \omega^p n_p + \ldots + \omega^1 n_1 + n_0$
and $\alpha' = \omega^{p'} n'_{p'} + \ldots + \omega^1 n'_1 + n'_0$ in CNF,

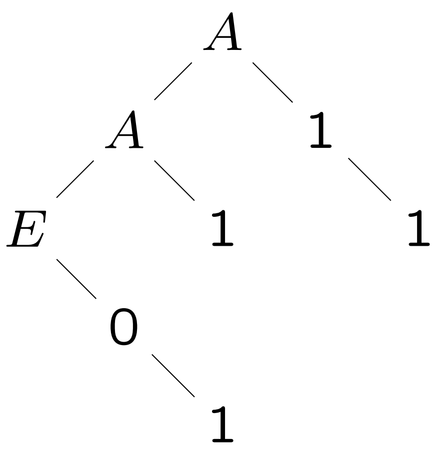
$$\alpha + \alpha' = \omega^p n_p + \ldots + \omega^{p'}(n_{p'} + n'_{p'}) + \ldots + \omega^1 n'_1 + n'_0$$

Representation by finite trees

$\alpha = \omega^3.5 + \omega.3 + 8$



$\alpha' = \omega^2.2 + \omega.1 + 3$



$\alpha + \alpha' = \omega^3.5 + \omega^2.2 + \omega.1 + 3$

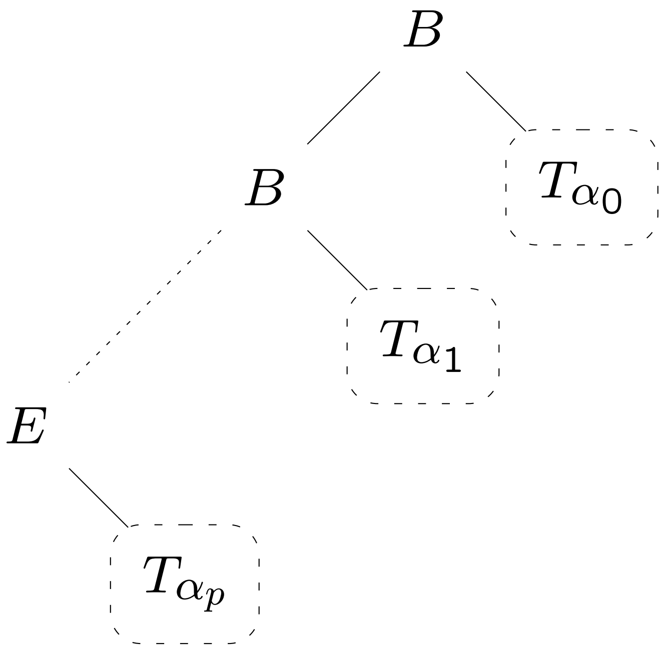
addition $\leftrightarrow$ tree-automaton over $\{E, A, 0, 1\}^3$
boolean operations $\leftrightarrow$ closure properties
$\exists$ quantification $\leftrightarrow$ projection

Induction on the formula. Emptiness decidable
Add dummy symbols #

Extension to ω^{ω^i}

Cantor Normal Form $\rightsquigarrow$ Any $\beta < \omega^{\omega^2}$ is uniquely written as

$$\omega^{\omega.p}\alpha_p + \ldots + \omega^{\omega.2}\alpha_2 + \omega^{\omega}\alpha_1 + \alpha_0 \ , \text{ where } p < \omega, \alpha_i < \omega^{\omega}, \alpha_p > 0 \ .$$



and so on by induction . . .

Complexity

Let $\text{Tower}(0, n) = n$ and $\text{Tower}(k+1, n) = 2^{\text{Tower}(k,n)}$.

Thm: $\forall i < \omega \; \exists c_i$ such that $\text{FO}(\omega^{\omega^i}, +)$ is decidable in time $\mathcal{O}(\text{Tower}(n, c_i))$, where n is the length of the formula.

Best known result:
via reduction to Weak Monadic Second Order logic $(\omega^\omega, <)$,
in turn decidable in time $\mathcal{O}(\text{Tower}(6n, c'))$ [Maurin96].

Non-elementary lower bound via WMSO

Application: Similar technique for $FO(\mathbb{N}, \cdot)$, via prime number decomposition,

$$x = \prod p_i^{x_i}, \qquad x \cdot y = \prod p_i^{x_i + y_i}$$

Monadic Second Order $(\mathbb{N}, <)$ [Büchi62]

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	$\cdots$
$X \subseteq \mathbb{N}$	●		●	●			●		●	●	$\cdots$

set $X \subseteq \mathbb{N}$ ↔ infinite word $v \in \{0,1\}^\omega$
$x \in \mathbb{N}$ ↔ $\{x\} \subseteq \mathbb{N}$ ↔ 0^x10^ω
$x \in X$ ↔ Büchi automaton over $\{0,1\}^2$
$x < y$ ↔ Büchi automaton over $\{0,1\}^2$
formula $\psi(x_1, \ldots, x_k)$ ↔ Büchi automaton over $\{0,1\}^k$
quantification $\exists$ ↔ projection ($\rightsquigarrow$ non-determinism)
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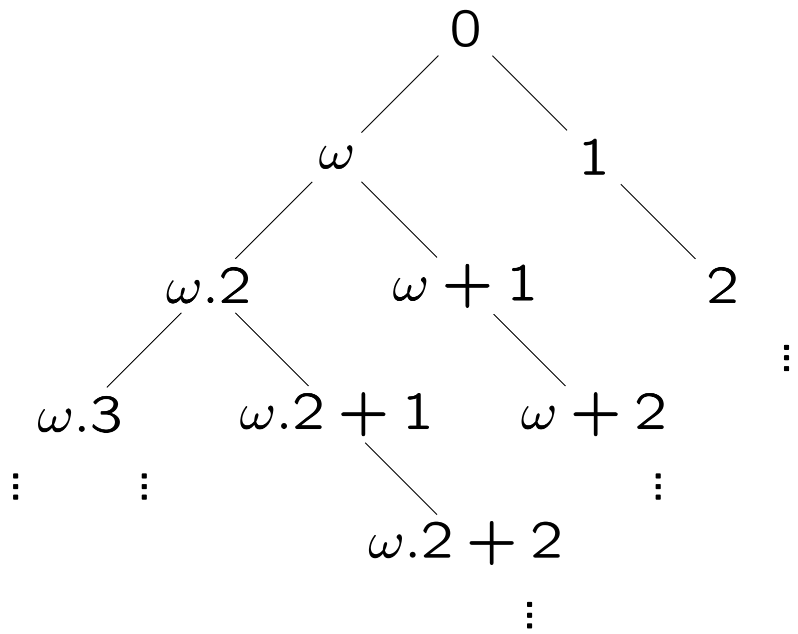
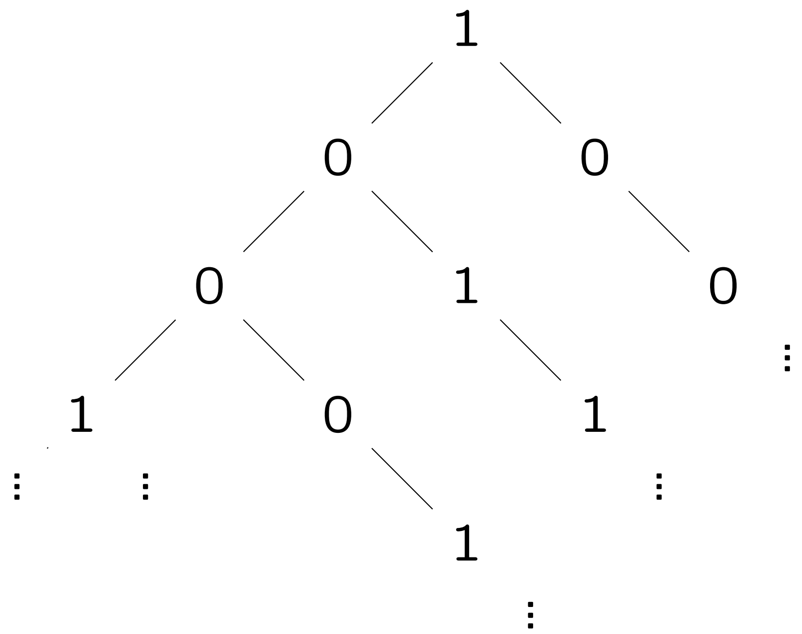
Non-elementary lower bound

Representation of any $X \subseteq \omega^2$ by an infinite tree

positions



labels $\in \{0, 1\}$



$\{0, \omega + 1, \omega + 2, \omega.2 + 2, \omega.3\}$

Use Rabin tree-automaton (or Muller or parity)
cannot be determinized yet complemented

Example of automata

Test X is $\{x\}$: non-deterministicaly search the unique node labeled 1

Test $x < y$: either on the same branch, or on different branches . . .

Other cases are easy and similar to $(\mathbb{N}, <)$

Extension to ω^i: as in the case of FO$(\omega^{\omega^i}, +)$ with e.g.

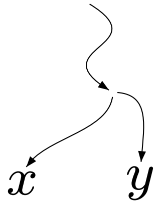
$$X \subseteq \omega^3 \text{ is written } \bigcup_{j<\omega} \left(\omega^2.j + X_j\right) \text{ where } \forall j : X_j \subseteq \omega^2$$

Thm: $\forall i < \omega\ \exists d_i$ such that MSO$(\omega^i, <)$ is decidable in time $\mathcal{O}(\text{Tower}(n, d_i))$, where n is the length of the formula.

Lower bound in Tower(n, c) even for $i = 1$

[Maurin96] WeakMSO $\mathcal{O}(\text{Tower}(6n, d'))$

MSO interpretation

Ordering among the nodes : $x < y$ iff 

[Rabin69] $(\mathbb{Q}, <)$ interpreted in the full binary tree. $\rightsquigarrow$ MSO$(\mathbb{Q}, <)$
Any countable linear ordering $(\sigma, <)$ interpreted in $(\mathbb{Q}, <)$ $\rightsquigarrow$ MSO$(\sigma, <)$

$(\omega^2, <) \leftrightsquigarrow b^*ab^*$
$(\omega^3, <) \leftrightsquigarrow b^*ab^*ab^*$
$(\omega^i, <) \leftrightsquigarrow b^*(ab^*)^i$

The graph of $(\omega^i, <)$ is prefix recognizable, *i.e.* at level 1 in the Caucal Hierarchy [Caucal02]. Open problem: higher ordinals at higher levels of the hierarchy?

Weak MSO and FO

Weak MSO: same syntax as MSO, but sets are interpreted as finite

$\forall \alpha$: WMSO$(\alpha, <)$ linearly reducible to FO$(\omega^\alpha, +)$ $\rightsquigarrow$ lower bound

Any ordinal β is uniquely written as

$$2^{\gamma_{n-1}} + \ldots + 2^{\gamma_0} \text{ where } (\gamma_{n-1} > \gamma_{n-2} > \ldots > \gamma_0)$$

Examples: $2^\omega = \omega$, $2^{\omega.i+j} = 2^{\omega.i}.2^j = \omega^i.2^j$, $2^{\omega^2} = (2^\omega)^\omega = \omega^\omega$.

[Buechi65] theories WMSO$(\alpha, <)$ and FO$(2^\alpha, +, E)$ are equireducible in linear time. $\rightsquigarrow$ WMSO$(\omega^{i+1}, <)$ and FO$(\omega^{\omega^i}, +, E)$

Any ordinal β can yet be written in a unique way in the form

$$\alpha = \gamma.\omega^\omega + \omega^p n_p + \omega^{p-1} n_{p-1} + \ldots + \omega^1 n_1 + n_0 \quad \text{where} \quad n_p > 0$$

$WMSO(\alpha, <)$ depends only on ($(\gamma > 0)$, $n_p, \ldots, n_0$)

Ordinal Automata

$(Q, \Sigma, \Delta, I, F)$
states Q (finite), alphabet Σ (finite),
transitions $\Delta \subseteq Q \times \Sigma \times Q \cup \mathscr{P}(Q) \times Q$,
$I \subseteq Q$ initial, $F \subseteq Q$ final states.

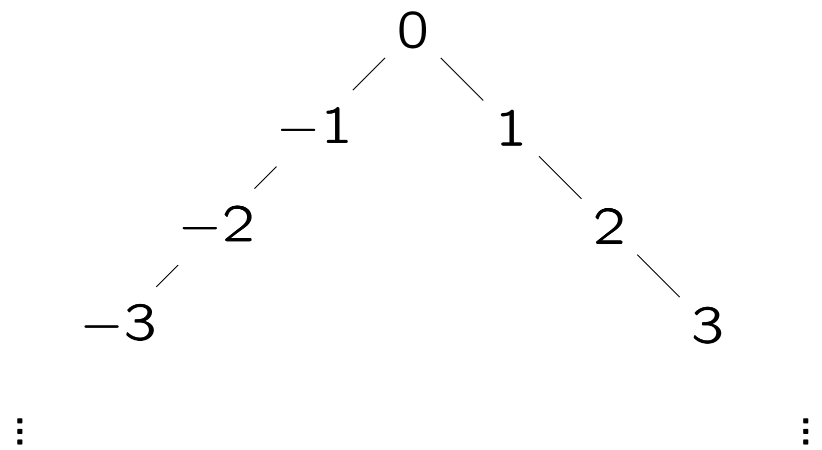
Ordinal automaton $\leftrightsquigarrow$ Tree automaton

Automata on (countable scattered) linear orderings:
transitions $\Delta \subseteq Q \times \Sigma \times Q \cup \mathscr{P}(Q) \times Q \cup Q \times \mathscr{P}(Q)$

Word-automaton $\rightsquigarrow$ tree-automaton
complementation of word-automaton: new proof of [CartonRispal04]

Extension to other linear orderings

MSO($\mathbb{Z}, <$)



Consider $-\omega + \omega \quad (-\omega).\omega \quad \ldots$

Open problems: linear orders of infinite rank, ordinals higher than ω^ω.

[Delhommé04] $\omega^{(\omega^\omega)}$ is not tree-automatic