DUAL BILLIARDS, FAGNANO ORBITS AND REGULAR POLYGONS.

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ABSTRACT. We study the notion of Fagnano orbits for dual polygonal billiards. We used them to characterize regular polygons and we study the iteration of the developing map.

In this article we consider the dual notion of two results on polygonal billiards. We begin by describing these original results. DeTemple and Robertson [3] have shown that a closed convex polygons P is regular if and only if P contains a periodic billiard path Q similar to P. The second result is about the dynamics of the so called pedal map related to billiards in a triangle P. The three altitudes of P intersect the opposite sides (or their extensions) in three points called the feet. These three points form the vertices of a new triangle Q called the pedal triangle $P^{(1)}$ of the triangle $P^{(0)}$ (Figure 1). It is well known that for acute triangles the pedal triangle forms a period three billiard orbit often referred to as the Fagnano orbit, i.e., the polygon Q is inscribed in P and satisfies the usual law of geometric optics: "the angle of incidence equals the angle of reflection" or equivalently (this is a theorem) that the pedal triangle has least perimeter among all inscribed triangles. The name Fagnano is used since in 1775 J.F.F. Fagnano gave the first proof the variational characterization. In a sequence of elegant and entertaining articles, J. Kingston and J. Synge [5], P. Lax [6], P. Ungar [11] and J. Alexander [1] studied the dynamics of the pedal map given by iterating this process.

There is a dual notion to billiards, called dual or outer billiards. The game of dual billiards is played outside the billiard table. Suppose the table is a polygon P and that z is a point outside P and not on the

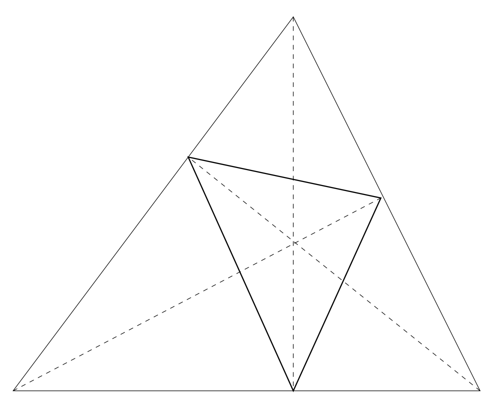

FIGURE 1. A pedal triangle.

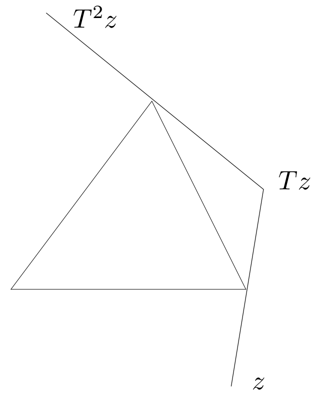

FIGURE 2. The polygonal dual billiard map.

continuation of any of P's sides. A line L is a *support line* of P if it intersects the boundary ∂P of P and P lies entirely in one of the two regions into which the line L divides the plane. There are two support lines to P through z, choose the right one as viewed from z. If z is not on the continuation of a side of P then this support line intersects P at a single point which we call the *support vertex* of P. Reflect z in the support vertex of P to obtain z's image under the dual billiard map denoted by T (see Figure 2). The point z has an infinite orbit if none of its images belongs to the continuation of a side of the polygon.

Dual billiards have been extensive studied by S. Tabachnikov (see [7],[8],[10] and the reference therein). Corresponding to the fact that the Fagnano orbit hits each side of the triangle call a periodic dual billiard orbit of period n for an n-gon *Fagnano* if the orbit reflects from the vertices in a cyclic manner. Fagnano orbits for dual billiards were introduce by Tabachnikov in [8] where he studied a certain variational property analogous to one in the pedal case studied by Gutkin [4].

Motivated by the notion of pedal triangles we introduce here dedal n-gons. Throughout the article we will identify the polygon having vertices $z_i \in \mathbb{C}$, $i = 1, \ldots, n$ ordered cyclically with the n-tuple $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. In particular a polygon for us is an oriented object. The notion of polygon includes self intersecting polygons (which we call star shaped polygons) and geometrically degenerate n-gons: sides of length 0, or an angle of π or 2π at a vertex.

An n-gon Q is called a *dedal* n-gon of the n-gon P if reflecting the vertex w_i of Q in the vertex z_i of P yields the vertex w_{i+1}. There is no requirement that the sides of Q touch P only at a vertex. From the definition it is clear that if the Fagnano orbit of an n-gon P exists then it is a dedal n-gon. A non-degenerate polygon can have a degenerate dedal polygon and vice-versa. Examples are shown in Figures 3 and 4. The degeneracy not shown can not occur: i.e. it is impossible for two consecutive vertices of P to coincide if Q is non-degenerate. We will not dwell on this aspect.

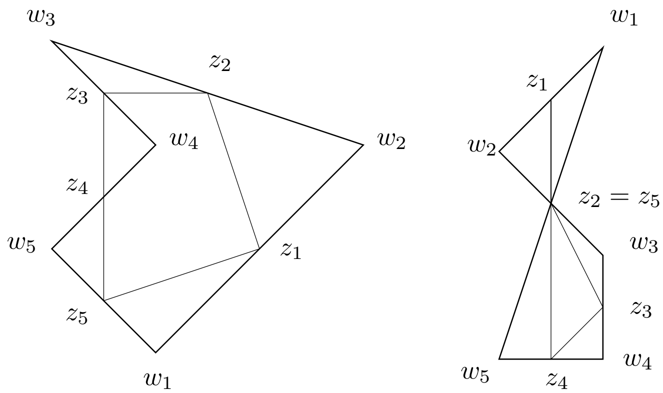

FIGURE 3. Non-degenerate dedal pentagon corresponding to a degenerate pentagons, a) angle π and b) 2π.

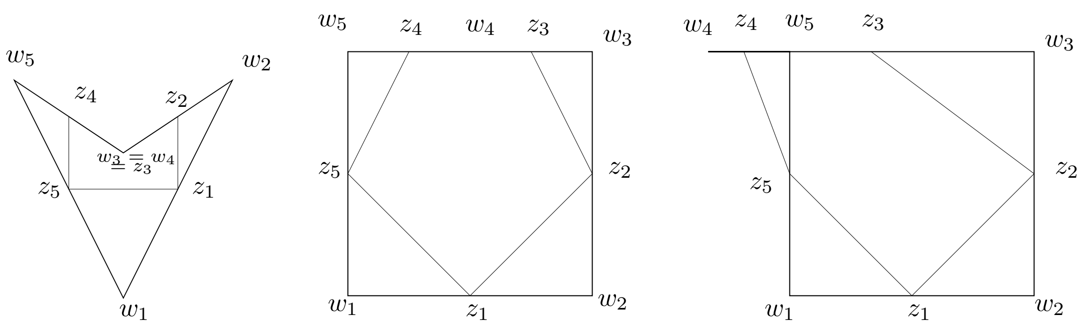

FIGURE 4. Degenerate dedal polygons: loss of a vertex, angle π, angle 2π.

In this article we study the dedal n-gon Q of an n-gon P. Our main results are the following. If n is odd then its dedal n-gon exists and is unique. For n odd we give a necessary and sufficient condition for the existence of dedal n-gons and describe the space of all dedal n-gons of P. Then we go on to characterize regular and affinely regular n-gons by similarity to their dedal n-gons. Finally we show that the iteration of the dedal map $\mu : Q \to P$ is $2n$ periodic. The proofs of all our results boil down to some linear algebra of the dedal map.

After writing this article one of the anonymous referees pointed out that iteration of the dedal map has already been studied by Berlekamp, Gilbert and Sinden in 1965 [2]. They answer a question they attribute to G.R. MacLane, namely they prove that for a.e. polygon Q there exists an $M \geq 1$ such that $\mu^M(Q)$ is convex. Note that the image of a convex polygon is convex, thus this implies that for a.e. Q there exists an M such that $\mu^m(Q)$ is convex for all $m \geq M$.

Suppose $Q(w_1, \ldots, w_n)$ is the dedal polygon of $P(z_1, \ldots, z_n)$. The definition implies that

$$z_i = (w_i + w_{i+1})/2 \quad \text{(subscripts read modulo } n\text{)} \tag{1}$$

(see Figures 3, 4 and 5). Throughout this article the subscripts will be taken modulo n without explicit mention. The linear transformation $\mu : \mathbb{C}^n \to \mathbb{C}^n$ given by $\mu(w_1, \ldots, w_n) = (z_1, \ldots, z_n)$ is called the *developing map.*

The characteristic polynomial of μ is

$$(1 - 2x)^n - (-1)^n.$$

Its eigenvalues are $(1+q^i)/2$ for $i = 0, 1, \ldots, n-1$ where $q := \exp(2\pi i/n)$. All the eigenvalues differ from zero except for the $n/2$nd eigenvalue when n is even. The ith eigenvector is $X_i := (1, q^i, q^{2i}, \ldots, q^{(n-1)i})$. A simple calculation (see [2],[8]) shows that the vectors X_i form a basis of our space of polygons. If i divides n then X_i is a polygon with n/i sides. None the less for the sake of clarity (avoiding stating special cases) we will think of this as an n-gon which is traced i times, for example if $n = 6$ then $X_2 = (1, q^2, q^4, q^6, q^8, q^{10}) = (1, q^2, q^4, 1, q^2, q^4)$ traces the triangle $(1, q^2, q^4)$ twice. An exception to this rule is the case n even and $i = n/2$. In this case $X_{n/2}$ is a segment which we do not consider as a polygon.

The following proposition clarifies the existence of dedal polygons.

Proposition 1. *a) Suppose that $n \geq 3$ is odd. Then for any n-gon P there is a unique dedal n-gon Q.*
b) If $n \geq 3$ is even dedal n-gons exist if and only if the vertices of P satisfy $z_1 - z_2 + z_3 - \cdots - z_n = 0$. If this equation is satisfied then there is a unique dedal n-gon Q_0 in the space $X_{n/2}^{\perp} := \{\vec{z} \in \mathbb{C}^n : \ \vec{z} \cdot X_{n/2} = 0\}$. The space $D := \{Q_0 + sX_{n/2} : \ s \in \mathbb{C}\}$ consists of the dedal n-gons of P. In particular for each $i \in \{1, \ldots, n\}$ every point $w \in \mathbb{C}$ is the ith vertex of a unique dedal n-gon $Q_i(w)$.

We remark that the condition $z_1 - z_2 + z_3 - \cdots - z_n = 0$ means that the center of mass of the even vertices coincides with the center of mass of the odd vertices.

Proof of Proposition 1. If $n \geq 3$ is odd then the map μ is invertible with

$$w_i = z_i - z_{i+1} + z_{i+2} - \cdots + z_{i-1}, \tag{2}$$

and thus the dedal polygon exists and is unique.

On the other hand if $n \geq 3$ is even then the map μ is not invertible. The kernel is one-complex dimensional and is generated by the vector $X_{n/2} := (1, -1, 1, -1, \ldots, 1, -1)$. Thus dedal polygons exist if and only if the vector $\vec{z} = (z_1, z_2, \ldots, z_n)$ is orthogonal to the vector $X_{n/2}$, i.e., it satisfies $X_{n/2} \cdot \vec{z} = 0$, or equivalently

$$z_1 - z_2 + z_3 - \cdots - z_n = 0. \tag{3}$$

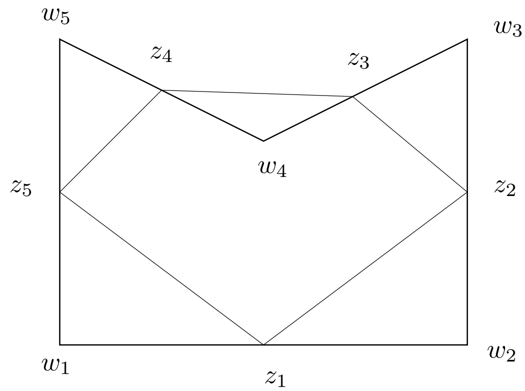

FIGURE 5. A pentagon without a Fagnano orbit, the unique dedal pentagon is pictured.

Alternatively to see this note that

$$z_1 + z_3 + \cdots + z_{n-1} = \tfrac{1}{2}\left((w_1 + w_2) + (w_3 + w_4) + \cdots + (w_{n-1} + w_n)\right)$$
$$= \tfrac{1}{2}\left((w_2 + w_3) + (w_4 + w_5) + \cdots (w_n + w_1)\right) = z_2 + z_4 + \cdots z_n.$$

The uniqueness of Q_0 follows since the map μ is invertible on the space $X_{n/2}^{\perp}$. The statement on the set D follows immediately since $X_{n/2}$ is the kernel of μ. Let $Q_0 := (w_1^0, \ldots, w_n^0)$. For each i any point $w \in \mathbb{C}$ can be uniquely expressed as $w_i^0 + s$ for some $s \in \mathbb{C}$. □

Note that the dedal polygon Q of a convex polygon P is a Fagnano orbit if and only if Q is convex. In particular Fagnano orbits always exist for triangles, but not for polygons with more sides. An example of a pentagon without a Fagnano orbit is given in Figure 5. Although not every polygon has a Fagnano periodic orbit, it does have a periodic orbit that is nearly as simple: it is a simple periodic orbit of the second iteration of the dual billiard map [9].

The map μ preserves the center of mass, thus we can assume that the center of mass is at the origin: i.e., that $w_1 + w_2 + \cdots + w_n = 0$. This reduces the complex dimension by 1. The eigenvector $X_0 = (1, 1, \ldots, 1)$ corresponds to polygons which degenerate to a point. The reduced space is orthogonal to this eigenvector.

X_1 and X_{n-1} are the usual regular n-gon in counter clockwise and clockwise orientation (Figure 6a). If i and n are relatively prime and $i \notin \{1, n-1\}$ then X_i is star shaped and we also call it regular (Figures 6b and c). Finally if i divides n and $i \neq n/2$ then X_i is naturally a regular n/i-gon which, as mentioned above, we will think of as (a multiple cover of) a regular n-gon.

Two (unoriented) polygons are called *similar* if all corresponding angles are equal and all distances are increased (or decreased) in the same ratio. Since we study oriented ordered polygons we will call two polygons $P = \sum b_i X_i$ and $Q = \sum a_i X_i$ *eigen-similar* if there exists a

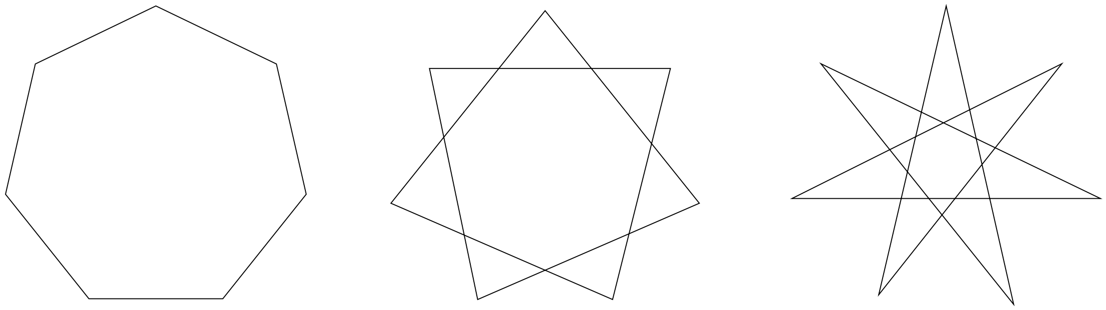

FIGURE 6. Up to orientation there are three regular 7-gons.

non-zero complex constant such that $b_i = const\, a_i$ for all i. Note that if P and Q are eigen-similar then they are similar. On the other hand if P and Q are similar then P is eigen-similar to a cyclic permutation $Q^{(k)} := (w_k, w_{k+1}, w_{k+2}, \dots, w_{k-1})$ of Q or a cyclic permutation $\bar{Q}^{(k)} := (w_k, w_{k-1}, w_{k-2}, \dots, w_{k+1})$ of Q with the opposite orientation.

In analog to DeTemple and Robertson's result we have

Theorem 2. *Fix $n \geq 3$. An n-gon P is regular if and only if it has a dedal polygon Q which is eigen-similar to P.*

Note that if n is odd then Q is the unique dedal polygon of P while if n is even then Q is the unique dedal polygon $Q_0 \in X_{n/2}^{\perp}$.

Proof. Suppose P is regular, i.e., $P = const\, X_j$ where $j \neq n/2$ if n is even. If n is odd let Q be the unique dedal polygon of P. If n is even then since P is regular it satisfies (3) and thus has dedal polygons and we choose $Q = Q_0 \in X_{n/2}^{\perp}$. In both cases let $Q = \sum a_i X_i$. Since $P = \mu(Q) = \sum \frac{1+q^i}{2} a_i X_i$ we have $a_i = 0$ for $i \neq j$ and thus $Q = a_j X_j$ is eigen-similar to P.

Conversely suppose that $Q = \sum a_i X_i$ is a dedal polygon of $P = \sum b_i X_i$ and that P and Q are eigen-similar, i.e., $a_i = const\, b_i$. Since $P = \mu(Q)$ we have $\frac{1+q^i}{2} a_i = b_i$ for $i = 1, \dots, n$. Combining this with the eigen-similarity yields $\frac{1+q^i}{2} = const$ for each i such that $a_i \neq 0$. It follows $q^i = q^j$ if a_i and $a_j \neq 0$. Since $q^i \neq q^j$ if $i \neq j$ it follows that only a single a_j is non-null and thus $P = b_j X_j$ and $Q = a_j X_j$ are regular. Note that if n is even then $i \neq n/2$ since in this case $(1+q^i)/2 = 0$ and thus $b_i = a_i = 0$. □

Of course we would like to have the analog of Theorem 2 with the regular notion of similarity. For each j (except $j = n/2$ when n is even) the subspace generated by X_j and X_{n-j} is the space of *affine-regular n-gons* (if $j = 1$ in the usual sense, if j and n are relatively prime then affine-regular n-stars, and if j divides n then a multiply covered affine-regular n-gon).

Theorem 3. *a) Suppose $n \geq 3$ is odd. An n-gon P is affinely regular if and only if it has a dedal polygon Q which is similar to P.*
b) If $n \geq 4$ is even then an n-gon P appears in the following list of affinely regular polygons if and only if it has a dedal polygon Q which is similar to P.
List: i) regular n-gons, ii) affinely regular n-gons $P = b_j X_j + b_{n-j} X_{n-j}$ such that there exits $k \in \{1, 2, \dots, \frac{n}{2} - 1, \frac{n}{2} + 1, \dots, n\}$ such that n divides $j(2k-1)$, iii) affinely regular n-gons $P = b_j X_j + b_{n-j} X_{n-j}$ such that there exists $k \in \{1, \dots, n\}$ with $\frac{b_j}{b_{n-j}} = \pm q^{j(k+3/2)}$.

All triangles are affinely regular. Berlekamp et. al. noticed that every triangle is similar to its dedal triangle [2]. Of course one would like to know if there are other special properties of the polygons in the list.

Proof. Consider the representation $(a_1, \dots, a_{n-1})$ of the polygon $Q^{(1)} = Q$. It is a simple exercise in linear algebra to compute the other representations, the representation of $Q^{(k)}$ is $(a_1 q^k, a_2 q^{2k}, \dots, a_{n-1} q^{(n-1)k})$ and that of $\bar{Q}^{(k)}$ is $(a_{n-1} q^{(k+1)}, a_{n-2} q^{2(k+1)}, \dots, a_1 q^{(n-1)(k+1)})$. That P is similar to Q is equivalent to the existence of a $k \in \{1, \dots, n-1\}$ and a nonzero complex constant such that $P = \mathit{const}\, Q^{(k)}$ or $P = \mathit{const} \bar{Q}^{(k)}$. The first equation means that $b_i = \mathit{const}\, a_i q^{ik}$ for $i = 1, \dots, n-1$ and the second that $b_i = \mathit{const}\, a_{n-i} q^{i(k+1)}$ for $i = 1, \dots, n-1$.

The structure of the proof is as follows. We group the even and odd cases and start by proving that the various classes of affinely regular polygons are similar to their dedal polygons. Then we turn to the converse.

Suppose that n is odd and $P = b_j X_j + b_{n-j} X_{n-j}$ is affinely regular. Let $Q = Q^{(1)} = \sum a_i X_i$ be the unique dedal polygon of P. Since $P = \mu(Q) = \sum \frac{1+q^i}{2} a_i X_i$ we have $a_i = 0$ for $i \notin \{j, n-j\}$. We claim that $P = \mathit{const}\, Q^{((n+1)/2)}$. To see this note that

$$1 + q^j = q^j(1 + q^{-j}) = q^{j(n+1)}(1 + q^{-j}) = q^{j(n+1)/2} q^{j(n+1)/2}(1 + q^{-j})$$

or

$$\frac{1 + q^j}{2q^{j(n+1)/2}} = \frac{1 + q^{-j}}{2q^{-j(n+1)/2}}.$$

Thus, choosing $c := \frac{1+q^j}{2q^{j(n+1)/2}}$ yields $P = cQ^{(n+1)/2}$.

The case n is even and P of class (ii) is similar. Consider the dedal polygon $Q = Q_0 \in X_{n/2}$. Again $P = \mu(Q)$ implies that $a_i = 0$ if $i \notin \{j, n-j\}$. We claim that $P = \mathit{const} Q^{(k)}$. To see this note that

$$1 + q^j = q^j(1 + q^{-j}) = q^{j+j(2k-1)}(1 + q^{-j}) = q^{kj} q^{kj}(1 + q^{-j})$$

or

$$\frac{1 + q^j}{2q^{kj}} = \frac{1 + q^{-j}}{2q^{-kj}}.$$

Thus, choosing $c := \frac{1+q^j}{2q^{kj}}$ yields $P = cQ^{(k)}$.

The case n even and P regular has already been treated in Theorem 2 thus it rest to treat the case (iii) of n even. We suppose P is of this form and want to show that P is similar to $\bar{Q}^{(k)}$, i.e., that $b_i = const a_{n-i}q^{i(k+1)}$ for $i \in \{j, n-j\}$. As before $P = \mu(Q)$ implies that $a_i = 0$ if $i \notin \{j, n-j\}$. Combining the two relations, $b_i = \frac{1+q^i}{2}a_i$ and $\frac{b_j}{b_{n-j}} = \pm q^{j(k+3/2)}$, yields

$$b_j = \pm q^{j(k+3/2)} b_{n-j} = \pm q^{j(k+3/2)} \frac{1+q^{-j}}{2} a_{n-j} = \pm q^{j(k+1)} \frac{q^{j/2}+q^{-j/2}}{2} a_{n-j}$$

and

$$b_{n-j} = \pm q^{-j(k+3/2)} b_j = \pm q^{-j(k+3/2)} \frac{1+q^{j}}{2} a_j = \pm q^{-j(k+1)} \frac{q^{j/2}+q^{-j/2}}{2} a_j.$$

Choosing $c := \pm(q^{j/2}+q^{-j/2})/2 \in \mathbb{R}$ yields $P = c\bar{Q}^{(k)}$.

We turn to the converse. We will first prove that for n even or odd if $Q = \sum a_i X_i$ is similar to $P = \sum b_i X_i$ then P is affinely regular.

We first treat that case when $P = const\, Q^{(k)}$ for some k, i.e., $b_i = const\, a_i q^{ik}$. Since $P = \mu(Q)$ we have $\frac{1+q^i}{2}a_i = b_i$ for $i = 1, \ldots, n-1$. Note that if n is even then $(1+q^{n/2})/2 = 0$ and thus $b_{n/2} = \frac{1+q^{n/2}}{2}a_{n/2} = 0$ and $a_{n/2} = const\, b_{n/2}$ is zero as well. For each i such that $a_i \neq 0$, combining the two relations between a_i and b_i yields, $q^{-ik}\frac{1+q^i}{2} = const$. If only a single a_j is non null, then Q is regular and thus by Theorem 2 P is regular as well. Suppose a_i and a_j are non null, then the previous equation implies that

$$(1+q^i)/(1+q^j) = q^{k(i-j)}. \tag{4}$$

Taking absolute values yields$|(1+q^i)| = |(1+q^j)|$, which implies $i = \pm j$. Thus P is affinely regular.

In the case n is even we need to conclude more. Note that $(1+q^{-j})/(1+q^j) = q^{-j}$. Thus taking $i = -j$ in (4) implies $1 = q^{j(1-2k)}$. Thus $j(2k-1)$ is a multiple of n, i.e., we are in case (ii) of the list.

Finally suppose that $P = \mu(Q)$ and Q are similar but have the opposite orientation, i.e., $P = const\, \bar{Q}^{(k)}$ or equivalently

$$b_i = const\, a_{n-i} q^{i(k+1)}. \tag{5}$$

with n even or odd.

This implies that $\frac{1+q^{-i}}{2}b_i = const\, b_{n-i}q^{i(k+1)}$. If n is even and $i = n/2$ then this equation implies that $b_{n/2} = 0$. For all other cases it implies that b_i and b_{n-i} are simultaneous zero or non-zero. For any i such that they are non-zero we have

$$\frac{b_i}{b_{n-i}} = \frac{2\, const\, q^{i(k+1)}}{1+q^{-i}} = \frac{1+q^i}{2\, const\, q^{-i(k+1)}}. \tag{6}$$

This implies that

$$4\, const^2 = (1+q^i)\cdot(1+q^{n-i}) = 2 + q^i + q^{n-i} := f(i). \tag{7}$$

It is easy to see that $f(i) = f(j)$ if and only if $j = i$ or $j = n - i$. Since the left hand side of this equation does not depend on i there there is exactly one pair (b_j, b_{n-j}) of non-zero coefficients or equivalently P is affinely regular.

Suppose now that n is even. Equations (6) and (7) imply that for each j there exists $k \in \{1, 2 \dots, n\}$ such that

$$\frac{b_j}{b_{n-j}} = \pm \frac{\sqrt{f(j)} q^{j(k+1)}}{1 + q^{-j}} = \pm \sqrt{\frac{1 + q^j}{1 + q^{-j}}} q^{j(k+1)} = \pm q^{j(k+3/2)}.$$

□

As noticed by Tabachnikov, [8], since $|\lambda_1| = |\lambda_{n-1}| > |\lambda_i|$ for $i = 2, \dots, n-2$ the subspace generated by the eigenvectors X_1 and X_{n-1}, i.e., the affinely regular n-gons, is an attractor for the developing map. Its basin of attraction is all polygons except for those which lie in the subspace spanned by X_i with $i = 2, \dots, n-2$.

Theorem 4. *For any affinely regular n-gon, the developing map is $2n$ periodic.*

Proof. Suppose that Q is affinely regular, i.e., $Q = aX_1 + bX_2$. Then $\mu^{2n}(Q) = (\frac{1+q}{2})^{2n} aX_1 + (\frac{1+q^{n-1}}{2})^{2n} bX_2$. Now $(1+q)^n = \sum_{k=0}^n \binom{n}{k} q^k$. If n is odd then $(1+q)^n = \sum_{k=)}^{\frac{n-1}{2}} \binom{n}{k}(q^k + q^{-k})$ while if n is even then $(1+q)^n = \sum_{k=)}^{\frac{n}{2}-1} \binom{n}{k}(q^k + q^{-k}) + \binom{n}{\frac{n}{2}} q^{\frac{n}{2}}$.

In both cases the sums of the roots of unity in the parentheses are real and the first half are positive the second half are negative. Since the coefficients are strictly increasing the sum is a negative real number and thus $((1+q)/|1+q|)^n = -1$ and $((1+q)/|1+q|)^{2n} = 1$. The same holds for q^{n-1}. Thus $\mu^{2n}(Q)$ and Q are similar. □

If $n \geq 3$ is odd we call the inverse map $f := \mu^{-1}$ the dedal map. Since $|\lambda_{\frac{n-1}{2}}| = |\lambda_{\frac{n+1}{2}}| < |\lambda_i|$ for the other i the subspace generated by the eigenvectors $X_{\frac{n-1}{2}}$ and $X_{\frac{n+1}{2}}$, i.e., the *affinely regular n-stars*, is an attractor for the dedal map. Its basin of attraction is all polygons except for those which lie in the subspace spanned by X_i with $i \neq \frac{n-1}{2}, \frac{n+1}{2}$.

Theorem 5. *If n is odd then the dedal map is $2n$ periodic on the space of affinely regular n-stars.*

The proof of this theorem is essentially the same of that of Theorem 4 with q replaced by $q^{\frac{n-1}{2}}$.

We remark that the n-gons not in the basin of attraction (for both the developing map and the dedal map) have themselves an 2 dimensional attractor, the set of n-gons with the next largest modulus of the eigenvalue, on which again the motion is $2n$ periodic. This set has a co-dimension 4 basin of attraction, and this decomposition can be iterated to yield co-dimension $6, 8, \ldots$ basins of attraction. This process stops at the repeller of the one map which is the attractor of the other.

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