SEMI-LINEAR REPRESENTATIONS: SOME EXAMPLES

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1. Introduction

Let $F$ be a field, $G$ a group acting on $F$ by field automorphisms and $k = F^G$ be the subfield of invariants of this action.

By an $F$-semi-linear $G$-representation we mean an $F$-space $V$ with a $(k$-linear) $G$-action such that $\sigma(a \cdot v) = \sigma(a) \cdot \sigma(v)$ for any $\sigma \in G$, $v \in V$ and $a \in F$. This is the same as a module over the algebra $F(G) := F \otimes \mathbb{Z}[G]$ with evident action of $F$ and the diagonal action of $G$. If $G$ acts on $F$ faithfully, then $F(G)$ is a central $k$-algebra. Semi-linear $G$-representations finite-dimensional over $F$ form an abelian tensor rigid $k$-linear category.

A typical example of a semi-linear $G$-representations over $F$ is given by the space $\Omega^1_{F/k}$ of differential 1-forms on $F$ over $k$. In a certain conjectural sense, this is the only (up to tensor algebra operations and extensions) ‘universal’ semi-linear representation. This is the main motivation of the present note.

However, this means neither that $\Omega^1_{F/k}$ is non-trivial for arbitrary extension $F/k$, cf. §3, nor that $\Omega^1_{F/k}$ is the only source of semi-linear $G$-representations, cf. §2.1.

The natural map $V^G \otimes_k F \rightarrow V$ is injective for any semi-linear $F$-representation $V$ of $G$. Hilbert Theorem 90 states that this is an isomorphism if $F$ is algebraic over $k$, $G$ acts faithfully on $F$ and the stabilizers of the elements of $V$ are open in the Krull topology.

If $F$ is not algebraic over $k$ then, a priori, there exist non-trivial (i.e., non-isomorphic to direct sums of copies of $F$) smooth (i.e. with open stabilizers in a natural generalization of the Krull topology studied in [J], p.151, Exercise 5, [H-W-H], [Sh] Ch.6, §6.3, and [I] Ch.2, Part 1, Section 1) semi-linear representation of $G$.

For example, if $F = k(t)$, where $t \in F$ is transcendental over $k$, and $G \cong \mathbb{Z}$ is generated by the automorphism $\sigma : t \mapsto t + 1$ then the one-dimensional $F$-space with the action of $\sigma$ given by multiplication by $t$ on a fixed non-zero vector defines a non-trivial $F$-semi-linear $G$-representation (as otherwise there would exist an element $f = f(t) \in F^\times = k(t)^\times$ such that $\sigma f / f = t$, in other words, $f(t + 1) = tf(t)$, which is impossible).

If $G$ is an algebraic $k$-group acting an a $k$-variety $X$ with $k(X) = F$, a part of examples comes from the fibre over the generic point of the $G$-equivariant bundles on $X$.

The main results of this note are Corollary 4.2 and Proposition 5.1, where it is shown that if $F$ is the function field of a projective space $\mathbb{P}$ over a characteristic zero field $k$ with sufficiently many roots of unity and $G$ is either the group of projective transformations, or a certain bigger subgroup in the Cremona group, then any semi-linear $G$-representations of

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degree one is an integral $F$-tensor power of $\det_F \Omega^1_{F/k}$. This bigger subgroup has such an advantage that it has no non-trivial representations of finite degree if $\dim_F P \geq 2$ (cf. Proposition 5.2), so at least this source of unexpected semi-linear representations is excluded.

There is a decreasing $\gamma$-filtration on any $\lambda$-ring. In the case of the $K_0$-ring of the category of coherent sheaves on a variety $X$ the graded pieces of the $\gamma$-filtration are canonically isomorphic to the Chow groups $CH^q(X) = H^q_{\text{Zar}}(X, K^M_q)$. In the case of the $K_0$-ring $K^0_0(F \langle G \rangle)$ of the category of semi-linear $G$-representations of finite degree one has, by definition, $gr^0_0K^0_0(F \langle G \rangle) \cong \mathbb{Z} = H^0(G, K^M_0(F))$ and, as the group $H^1(G, GL_n F)$ classifies the isomorphism classes of $F$-semi-linear $G$-representations of degree $r$, $gr^1_0K^0_0(F \langle G \rangle) = H^1(G, K^M_0(F))$.

The natural questions are: Whether there exists a theory of Chern classes of semi-linear $G$-representations with values in $\bigoplus_{q \geq 0} H^q(G, K^M_q(F))$ so that the above equalities could be generalized to $gr^j_0K^0_0(F \langle G \rangle) = H^j(G, K^M_j(F))$ for all $j \geq 0$? Such “Chern” classes can probably be defined, using a construction of $[G]$, for semi-linear $G$-representations with “sufficiently small” $G$-action. Would this imply that the Chern character $K^0_0(F \langle G \rangle) \otimes \mathbb{Q} \xrightarrow{\text{ch}} \bigoplus_{q \geq 0} H^q(G, K^M_q(F)) \otimes \mathbb{Q}$ is, as in Riemann–Roch Theorem, a ring isomorphism? Is there a generalization of Hilbert Theorem 90, like $H^q(G, K^M_q(F)) = 0$ if $q > \text{tr.deg}(F/k)$?

As I do not know the answers to the above questions, the classical construction involving the curvature is adapted in §2.3 to our context to get the Chern classes of semi-linear $G$-representations with values in $\bigoplus_{q \geq 0} H^q(G, \Omega^F_q)$, where $\Omega^F_q$ is the exterior $F$-algebra of the Kähler differentials on $F$.

2. The curvature

Fix an absolute connection $V \xrightarrow{\nabla} \Omega^1_F \otimes_F V$ in a semi-linear finite-dimensional $F$-representation $V$ of $G$. Then $\sigma \rightarrow \sigma \nabla \sigma^{-1}$ is a 1-cocycle on $G$ with values in $\Omega^1_F \otimes_F \text{End}_F(V)$. Its class in $H^1(G, \Omega^1_F \otimes_F \text{End}_F(V))$, the curvature, is obviously independent of the choice of the connection. $G$ acts on $\Omega^1_F \otimes_F \text{End}_F(V)$ by $\sigma \rightarrow (\omega \rightarrow a_\sigma \omega a_\sigma^{-1})$, where $\omega$ is considered as a matrix in the same basis where $a_\sigma$ has been defined.

For any subfield $k'$ in $F$ preserved by $G$ one can also consider relative $k'$-connections $V \xrightarrow{\nabla} \Omega^1_{F/k'} \otimes_{F/k'} V$ on $V$ and the corresponding $k'$-curvatures in $H^1(G, \Omega^1_{F/k'} \otimes_{F/k'} \text{End}_{F/k'}(V))$. The set of isomorphism classes of semi-linear $F$-representations of $G$ of degree $r$ is canonically identified with the set $H^1(G, GL_n F)$.

2.1. A source of flat semi-linear representations. To any subfield $k'$ in $F$ invariant under the $G$-action and to any finite-dimensional semi-linear $k'$-representation $V_0$ of $G$ one associates the semi-linear $F$-representation $V_0 \otimes_{k'} F$ of $G$. On the level of isomorphism classes of semi-linear representations of $G$ of degree $r$ this operation coincides with the natural map

\begin{equation}
H^1(G, \text{GL}_n F) \rightarrow H^1(G, \text{GL}_r F).
\end{equation}
As the $k'$-connection trivial on $V_0$ commutes with the $G$-action, the $k'$-curvature of $V_0 \otimes_{k'} F$ vanishes.

The following lemma gives a sufficient condition for injectivity of the map (1).

**Lemma 2.1.** Let $k'$ be a Galois extension of $k$ in $F$. If the $G$-orbit of any element of $F - k'$ spans a $k'$-subspace in $F$ of dimension $> r^2$ then the map (1) is injective.

**Proof.** Let $(a_\sigma)$ and $(a'_\sigma)$ be two 1-cocycles representing some classes in $H^1(G, \text{GL}_r k')$. Suppose they become the same in $H^1(G, \text{GL}_r F)$, i.e., there is an element $b \in \text{GL}_r F$ such that $a_\sigma = b^{-1} a'_\sigma b$ for all $\sigma \in G$. Equivalently, $b a_\sigma = a'_\sigma b$ for all $\sigma \in G$. If $b \not\in \text{GL}_r k'$, i.e., there are some $1 \leq s, t \leq r$ such that $b_{st} \not\in k'$, then there is $\sigma \in G$ such that $\sigma b_{st} \not\in \langle b_{ij} | 1 \leq i, j \leq r \rangle_{k'}$, which contradicts $b a_\sigma = a'_\sigma b$. This means that $b \in \text{GL}_r k'$, and thus, the classes of $(a_\sigma)$ and $(a'_\sigma)$ in $H^1(G, \text{GL}_r k')$ coincide. \qed

**Remark.** The tensor multiplication over $F$ is a group law on the set $\mathcal{L}$ of isomorphism classes of $F$-semi-linear representations of degree 1 admitting presentation of type $V_0 \otimes_{k'} F$ for $k'$-representations $V_0$ of $G$, where $k'$ is the algebraic closure of $k$ in $F$. One has

$$\mathcal{L} = \ker[H^1(G, F^\times) \longrightarrow H^1(G, F^\times / k^\times)] = \text{coker}[(F^\times / k^\times)^G \longrightarrow H^1(G, k^\times)].$$

The following is a series of examples of semi-linear representations of degree 1 with vanishing $k$-curvature, whose isomorphism classes are outside of $\mathcal{L}$.

Let $G$ be a connected algebraic $k$-group acting on a smooth proper $k$-variety $X$, and $F = k(X)$ be the function field of $X$. The short exact sequence of $G$-modules

$$0 \longrightarrow F^\times / k^\times \longrightarrow \text{Div}_{\text{alg}}(X) \longrightarrow \text{Pic}^0(X) \longrightarrow 0$$

gives an exact sequence of cohomologies

$$0 \longrightarrow \text{Pic}^0(X)^G / \text{Div}_{\text{alg}}(X)^G \longrightarrow H^1(G, F^\times / k^\times) \longrightarrow H^1(G, \text{Div}_{\text{alg}}(X)).$$

Explicitly, $\delta D = (\sigma \mapsto \sigma(D) - D)$ for any $D \in \text{Div}_{\text{alg}}(X)$ with the image in $\text{Pic}^0(X)$ fixed by $G$, where we identify the elements of $F^\times / k^\times$ with their divisors.

On the other hand, for any $D \in \text{Div}_{\text{alg}}(X)$ there exists a (unique modulo $\Gamma(X, \Omega^1_{X/k})$) closed 1-form $\eta$ on $X$ with no poles except for logarithmic ones in the support of $D$ and with the residue $D$. Then, modulo $\Gamma(X, \Omega^1_{X/k})$, $d \log$ of the element $f_\sigma \in F^\times / k^\times$ with the divisor $\sigma(D) - D$ is exactly $\sigma \eta - \eta$.

Without any loss of generality, we may suppose that $k = \mathbb{C}$. Then for any 1-cycle $\gamma \in H_1(X(\mathbb{C}), \mathbb{Z})$ and any $\sigma \in G$ one has $\gamma = \sigma \gamma$, so the periods of $\sigma \eta - \eta$ are in $\mathbb{Z}(1)$. This implies that the periods of $d \log f_\sigma - (\sigma \eta - \eta)$ vanish, and thus, $d \log f_\sigma = \sigma \eta - \eta$. This means that the image of $\delta$ is contained in the kernel of $H^1(G, F^\times / k^\times) \longrightarrow H^1(G, \Omega^1_{F/k}).$

Now we let

- $k = \mathbb{C}$,
- $X = \mathbb{C}/\Lambda$ be an elliptic curve = one-dimensional complex compact torus for a $\mathbb{Z}$-lattice
  \[ \Lambda = \langle 1, \omega \rangle_{\mathbb{Z}} \supset \mathbb{Z} \text{ in } \mathbb{C}, \]
  and
- $G = X(\mathbb{C})$ be the group of translations of $X$.
Then \( \text{Div}_{\text{alg}}(X)^G = 0 \) and \( G \) acts trivially on \( \text{Pic}^G(X) \), so

\[
\text{Pic}^G(X) \overset{\delta}{\twoheadrightarrow} \ker(H^1(G, F^\times/k^\times) \longrightarrow H^1(G, \Omega^1_{F/k})).
\]

We only have to check that \( \delta(\text{Pic}^G(X)) \) is contained in the image of \( H^1(G, F^\times) \longrightarrow H^1(G, F^\times/k^\times) \).

Let \( \vartheta_\Lambda(z) \) be a \( \mathbb{Z} \)-periodic theta-function with simple zeroes in \( \Lambda \) and \( \vartheta_\Lambda(z - \omega) = e^{\alpha z + \beta \vartheta}(z) \). For each \( \bar{a} \in X = \mathbb{C}/\Lambda \) fix some lifting \( a \in \mathbb{C} \).

For each pair \( A, B \in \mathbb{R} \), set \( b_{A+B,\omega} = e^{\alpha A + B} \). Then \( f_\sigma = b_{\sigma} \vartheta_\Lambda(z) \vartheta_\Lambda(z, \omega - \omega \bar{z}) \in F^\times \) depends only on the class \( \sigma \in \mathbb{C}/\Lambda = X = G \) of \( \sigma \in \mathbb{C} \), i.e., \( f_\sigma(a) := f_\sigma \) is well-defined. It is clear that \( (f_\sigma) \) is a 1-cocycle on \( G \) with coefficients in \( F^\times \), and its projection to \( H^1(G, F^\times/k^\times) \) is \( \delta(\bar{a}) \).

Finally, the \( F \)-semi-linear representations defined by the cocycles \( (f_\sigma(a)) \) are all distinct, and they are not of type \( V_0 \otimes_k F \) if \( \bar{a} \neq 0 \).

2.2. Another description of the curvature. Let \( V \) be a semi-linear finite-dimensional \( F \)-representation of \( G \). Denote by \( I \) is the kernel of the multiplication map:

\[
0 \longrightarrow I \longrightarrow F \otimes_k F \overset{x}{\longrightarrow} F \longrightarrow 0.
\]

The map \((F \otimes_k F)/I^2 \longrightarrow \Omega^1_{F/k}\) given by \( x \otimes y \mapsto -xy \) induces a canonical isomorphism \( I/I^2 \overset{\sim}{\longrightarrow} \Omega^1_{F/k} \), so there is a short exact sequence

\[
0 \longrightarrow \Omega^1_{F/k} \longrightarrow (F \otimes_k F)/I^2 \overset{x}{\longrightarrow} F \longrightarrow 0.
\]

There are two \( F \)-space structures on \((F \otimes_k F)/I^2\). However, they coincide in the restriction to \( I/I^2 \). Applying to (2) the (composition of two) exact functor(s) \( \text{Hom}_F(V, - \otimes_F V) \), we get an extension

\[
0 \longrightarrow \Omega^1_{F/k} \otimes_F \text{End}_F(V) \longrightarrow V^\vee \otimes_{F \otimes k} \big((F \otimes_k F)/I^2\big) \otimes_{k \otimes k} V \overset{x}{\longrightarrow} \text{End}_F(V) \longrightarrow 0,
\]

where \( V^\vee = \text{Hom}_F(V, F) \) is the dual.

Then the curvature of \( V \) corresponds to the extension

\[
0 \longrightarrow \Omega^1_{F/k} \otimes_F \text{End}_F(V) \longrightarrow x^{-1}(F) \longrightarrow F \longrightarrow 0
\]

in the category of \( F \)-semi-linear \( G \)-representations, where \( x^{-1}(F) \) is the preimage of the scalars in \( \text{End}_F(V) \) in \( V^\vee \otimes_{F \otimes k} \big((F \otimes_k F)/I^2\big) \otimes_{k \otimes k} V \).

2.3. Chern classes with values in \( \bigoplus_{q=0}^n H^q(G, \Omega^q_{F/k}) \). In the standard way, invariant polynomials on the spaces of matrices corresponding to the elementary symmetric functions, evaluated on the curvature give the Chern classes of semi-linear \( F \)-representations of \( G \) with values in the \( k \)-algebra \( \bigoplus_{q=0}^n H^q(G, \Omega^q_{F/k}) \).

3. Examples of extensions \( F/k \) with trivial \( \Omega^1_{F/k} \)

If \( F \) is algebraic over \( k \) then \( \Omega^1_{F/k} = 0 \), so we assume that this is not the case.
1. Let $\mathcal{A}$ be an absolutely irreducible algebraic group scheme over $k$ and $G_0 \subseteq \mathcal{A}(k)$ a subgroup Zariski dense in $\mathcal{A}$. Then $F^{G_0} = k$, where we set $F = k(\mathcal{A})$. On the other hand, the tangent bundle of $\mathcal{A}$ is trivial, so $(\Omega^1_{F/k})^{G_0} \cong \text{Hom}_k(T_1\mathcal{A}, k)$, thus one has $(\Omega^1_{F/k})^{G_0} \otimes_k F \cong \Omega^1_{F/k}$, i.e., $\Omega^1_{F/k}$ is a trivial $F$-semi-linear $G_0$-representation.

Let $G_0 \subseteq G \subseteq \text{Aut}_k(\mathcal{A})$ be a subgroup containing $G_0$ as a normal subgroup. Assume that the $G/G_0$-action on $T_1\mathcal{A}$ is scalar, and an index $u$ subgroup of $G/G_0$ acts trivially on $T_1\mathcal{A}$.

Then $\left(\Omega^1_{F/k}\right)^{G_0} \otimes_k F \cong \left(\Omega^1_{F/k}\right)^{G_0}$ for any $j \geq 0$, i.e., $\left(\Omega^1_{F/k}\right)^{G_0}$ is a trivial $F$-semi-linear $G$-representation for any $j \geq 0$.

2. Let $n \geq 2$, $F = k(x_1, \ldots, x_n)$ and $G \subseteq \text{GL}_n k$ acts by linear substitutions of $x_1, \ldots, x_n$ so that $F^{G_0} = k$. Then $\Omega^1_{F/k} \cong F \otimes_k V_0$, where $V_0$ is the restriction of the standard $k$-representation of $\text{GL}_n k$ to $G$, so $\Omega^1_{F/k} \cong F \otimes_k \bigwedge^k V_0$. In particular, if $G = SL_n k$ then $\Omega^1_{F/k} \cong F$.

4. Semi-linear representations of $PGL$ of degree one

Fix an $n$-dimensional projective $k$-space $\mathbb{P}$ and some coordinates $x_1, \ldots, x_n$ on $\mathbb{P}$.

Let $G = \text{Aut}(\mathbb{P}) \cong \text{PGL}_{n+1} k$ be the group of automorphisms of $\mathbb{P}$ and $F = k(\mathbb{P})$ be the function field of $\mathbb{P}$.

Let $A = (A_{ij})_{1 \leq i, j \leq n+1} \in \text{PGL}_{n+1} k$ act on $F$ by $x_j \mapsto \frac{A_{j1} x_1 + \cdots + A_{jn} x_n + A_{j,n+1}}{A_{n+1,1} x_1 + \cdots + A_{n+1,n} x_n + A_{n+1,n+1}}$.

The aim of this section is to show that in characteristic zero any $F$-semi-linear $G$-representations of degree one is a ‘rational $F$-tensor power’ of the space $\Omega^1_{F/k}$ of differential forms on $F$.

**Proposition 4.1.** For any characteristic zero field $k$ the group $H^1(G, F^x/k^x)$ is infinite cyclic.

**Proof.** Let $\Lambda \cong \mathbb{Z}^n$ be the standard lattice in the subgroup $\mathbb{Z}^n \cong U_0 \subset G$ of upper triangular unipotent matrices with the only non-zero entries in the $(n+1)$-st column (except the diagonal). First, by induction on $n$, we check the vanishing of $H^1(\Lambda_{\mathbb{Q}}, F^x/k^x)$.

As the terms $E_2^{s,0} = H^s(\Lambda_{\mathbb{Q}}/\Lambda, H^0(\Lambda, F^x/k^x))$ of the Hochschild–Serre spectral sequence for the subgroup $\Lambda \subset \Lambda_{\mathbb{Q}}$ vanish, we get $H^1(\Lambda_{\mathbb{Q}}, F^x/k^x) = E_2^{0,1} = H^1(\Lambda, F^x/k^x).$ Let $\lambda_0 = (0, \ldots, 0, 1) \in \Lambda$ and $\Lambda' = \mathbb{Z}^{n-1} \times \{0\} \subset \Lambda$. For any 1-cocycle $(f_\lambda) \in H^1(\Lambda, F^x/k^x)$ and any $\mu \in \Lambda_{\mathbb{Q}}$ there is an element $g_\mu \in k(x_1, \ldots, x_n)^x$ such that $f_{\lambda_0}(x + \mu)/f_{\lambda_0}(x) = g_\mu(x + \lambda_0)/g_\mu(x)$. Multiplying $f_\lambda$ with rational functions of type $h(x + \lambda)/h(x)$ (which does not change the cohomology class), we may suppose that there are no pairs of irreducible components of the support of the divisor of $f_{\lambda_0}$ that differ by a translation by an integer multiple of $\lambda_0$.

Then, for any $\mu \neq 0$ sufficiently small there are no pairs of irreducible components of the support of the divisor of $f_{\lambda_0}(x + \mu)/f_{\lambda_0}(x)$ that differ by a translation by an integer multiple of $\lambda_0$, and therefore, $f_{\lambda_0}(x + \mu)/f_{\lambda_0}(x) = g_\mu(x + \lambda_0)/g_\mu(x)$ if and only if $f_{\lambda_0}(x + \mu)/f_{\lambda_0}(x)$ is constant, which means that $f_{\lambda_0}(x)$ is constant itself.

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Then for any \( \lambda \in \Lambda \) the cocycle condition gives
\[
f_{\lambda+\lambda_0}(x) = f_{\lambda}(x) = f_{\lambda}(x + \lambda_0),
\]
and thus, \( f_{\lambda}(x) \in k(x_1, \ldots, x_{n-1})^\times / k^\times \). By the induction assumption, there is some \( g \in k(x_1, \ldots, x_{n-1})^\times / k^\times \) such that \( f_{\lambda}(x) = g(x + \lambda)/g(x) \) for all \( \lambda \in \Lambda \), and thus, \( f_{\lambda}(x) = g(x + \lambda)/g(x) \) for all \( \lambda \in \Lambda_Q \), i.e., \( H^1(\Lambda_Q, F^\times / k^\times) = 0 \).

For a normal subgroup \( A \triangleleft B \) and a \( B \)-module \( W \) such that \( H^0(A, W) = H^1(A, W) = 0 \) the Hochschild–Serre spectral sequence gives:
\[
E_2^{p, q} = H^p(B/A, H^q(A, W)) = 0 \quad \text{and} \quad E_2^{1, 1} = H^1(B/A, H^1(A, W)) = 0,
\]
so \( H^1(B, W) = 0 \).

Taking \( B = U_0 \supset A = \Lambda_Q \), and \( W = F^\times / k^\times \), we get
\[
H^1(U_0, F^\times / k^\times) = 0.
\]

Taking \( B = U \), the subgroup of upper triangular matrices in \( \text{PGL}_{n+1} k \), \( A = U_0 \) and \( W = F^\times / k^\times \), we get
\[
H^1(U, F^\times / k^\times) = 0.
\]

As the subgroup \( U_\triangleleft \) of lower triangular matrices in \( \text{PGL}_{n+1} k \) is conjugated to the subgroup \( U \) of upper triangular matrices in \( \text{PGL}_{n+1} k \), one has
\[
H^1(U_\triangleleft, F^\times / k^\times) = 0,
\]
any element of \( H^1(\text{PGL}_{n+1} k, F^\times / k^\times) \) can be presented by a cocycle of type \( U \rightarrow 1 \),
\[
(A_{ij})_{1 \leq i, j \leq n+1} \rightarrow f \left( \frac{A_{11} x_1}{x_1 + \cdots + A_{1,n+1} x_{n+1}}, \ldots, \frac{A_{n+1,1} x_1 + \cdots + A_{n+1,n+1} x_{n+1}}{x_1 + \cdots + A_{n+1,n+1} x_{n+1}} \right)
\]
for some \( f \in k(x_1, \ldots, x_{n})^\times / k^\times \), where \( A_{ij} = 0 \) if \( i < j \). The diagonal matrices are simultaneously lower and upper triangular, so \( f(x) = x^{m_1} \ldots x^{m_n} \) for some \( m_1, \ldots, m_n \in \mathbb{Z} \), which gives us that the above 1-cocycle on the lower triangular matrix
\[
(A_{ij})_{1 \leq i, j \leq n+1},
\]
where \( A_{ij} = 0 \) if \( i < j \), is \( (A_{n+1,1} x_1 + \cdots + A_{n+1,n+1} x_{n+1})^{-m} \).

As the subgroups of lower and upper triangular matrices generate \( \text{PGL}_{n+1} k \),
\[
H^1(G, F^\times / k^\times) = \left\{ (A_{ij})_{1 \leq i, j \leq n+1} \rightarrow (A_{n+1,1} x_1 + \cdots + A_{n+1,n+1} x_{n+1})^{-m} \mid m \in \mathbb{Z} \right\} \cong \mathbb{Z}.
\]

**Corollary 4.2.** Let \( k \) be a field of characteristic zero. Then the natural homomorphism
\[
H^1(G, F^\times) \rightarrow H^1(G, F^\times / k^\times)
\]
is an injection of infinite cyclic groups.

Its cokernel is canonically isomorphic to the image of \( \mathbb{Z} \rightarrow \text{End}(k^\times) \otimes \mathbb{Z}/(n+1)\mathbb{Z} \).

The class \(^1\) of \( \Omega^1_{F/k} \) generates a subgroup of index \( n+1 \) in \( H^1(G, F^\times / k^\times) \).

In particular, the number of \((n+1)\)-st roots of unity in \( k \) divides the index of \( H^1(G, F^\times) \) in \( H^1(G, F^\times / k^\times) \) (thus, \( \Omega^1_{F/k} \) generates \( H^1(G, F^\times) \) if \( k \) contains all \((n+1)\)-st roots of unity).

**Example.** If \( k = \mathbb{R} \) then the index is 1 for even \( n \), and it is 2 for odd \( n \).

**Proof.** As \( G \) coincides with its commutant, \( \text{Hom}(G, k^\times) = 0 \), so the short exact sequence
\[
1 \rightarrow k^\times \rightarrow F^\times \rightarrow F^\times / k^\times \rightarrow 1
\]
gives an embedding \( H^1(G, F^\times) \hookrightarrow H^1(G, F^\times / k^\times) \). Suppose that the 1-cocycle
\[
A = (A_{ij})_{1 \leq i, j \leq n+1} \rightarrow (A_{n+1,1} x_1 + \cdots + A_{n+1,n+1} x_{n+1})^{-m}
\]
\(^1\) of the cocycle \( (\sigma \mapsto \sigma \omega / \omega) \in H^1(G, F^\times) \) for any non-zero \( n \)-form \( \omega \in \Omega^n_{F/k} \).
on \( G \) with values in \( F^\times /k^\times \) can be lifted to a 1-cocycle \( A = (A_{ij})_{1 \leq i,j \leq n+1} \mapsto \Phi(A) \cdot (A_{n+1,i} x_1 + \cdots + A_{n+1,n} x_n + A_{n+1,n+1})^{-m} \) on \( G \) (considered as a 1-cocycle on \( GL(V) \) for an \((n + 1)\)-dimensional \( k \)-vector space \( V \)) with values in \( F^\times \). Then \( \Phi : GL(V) \mapsto k^\times \) is a homomorphism, and thus, \( \Phi \) factors through the determinant: \( \Phi(A) = \phi(\det A) \) for a homomorphism \( k^\times \mapsto k^\times \). The cocycle on \( GL(V) \) defined by \( \Phi \) descends to a cocycle on \( G \) if and only if \( \Phi \) is homogeneous of degree \( m \), so \( \phi(\lambda)^{n+1} = \lambda^m \). This implies that \( m \), considered as element of \( \text{End}(k^\times) \supset \mathbb{Z} \), should be divisible by \( n + 1 \).

As any endomorphism of \( k^\times \) induces an endomorphism of the subgroup of \((n + 1)\)-st roots of unity, if \( k \) contains \( t \) out of \( n + 1 \) roots of unity of order \( n + 1 \), then \( n + 1 \) divides \( m \) as element of \( \mathbb{Z} / t \mathbb{Z} \), which simply means that \( m \equiv 0 \pmod{t} \).

5. SEMI-LINEAR REPRESENTATIONS OF DEGREE ONE OF A SUBGROUP OF THE CREMONA GROUP

As in the previous section, we fix an \( n \)-dimensional projective \( k \)-space \( \mathbb{P} \) and some coordinates \( x_1, \ldots, x_n \) on \( \mathbb{P} \).

Let \( P = \text{Aut}(\mathbb{P}) \cong \text{PGL}_{n+1} k \) be the group of automorphisms of \( \mathbb{P} \) (denoted by \( G \) in §4), and \( F = k(\mathbb{P}) \) be the function field of \( \mathbb{P} \).

Let \( G \) be the subgroup of the Cremona group \( \text{Cr}_n(k) = \text{Aut}(F/k) \) generated by \( P \) and by the involution \( \sigma \) such that \( \sigma x_1 = x_1^{-1} \) and \( \sigma x_j = x_j \) for all \( 2 \leq j \leq n \).

The aim of this section is to show that in characteristic zero any \( F \)-semi-linear \( G \)-representations of degree one is an integral \( F \)-tensor power of the space \( \Omega^1_{F/k} \) of differential forms on \( F \).

**Proposition 5.1.** Let \( k \) be a field of characteristic zero.

Then the isomorphism class of \( \Omega^1_{F/k} \) generates the group \( H^1(G, F^\times) \).

**Proof.** Let \( (a_\tau) \) be a 1-cocycle on \( G \). We may suppose that (in notation of the proof of Corollary 4.2) the restriction of \( (a_\tau) \) to \( P \) coincides with \( A \mapsto \phi(\det A) \cdot (A_{n+1,1} x_1 + \cdots + A_{n+1,n} x_n + A_{n+1,n+1})^{-m} \) for a homomorphism \( k^\times \mapsto k^\times \).

Let \( T \subseteq P \) be the maximal torus subgroup such that \( \tau x_j / x_j =: \lambda_j(\tau) \in k^\times \) for any \( \tau \in T \) and any \( 1 \leq j \leq n \). Then \( \sigma \) normalizes \( T \) and for any \( \tau \in T \) one has \( \lambda_j(\sigma \tau \sigma^{-1}) = \lambda_j(\tau)^{1-2n,j} \), and \( a_\tau = \phi(\lambda_1(\tau) \cdots \lambda_n(\tau)) \). As \( a_{\sigma \tau \sigma^{-1}} = a_\sigma \cdot a_\tau \cdot \sigma \tau \cdot a_\sigma^{-1} \cdot a_\tau^{-1} \), this implies that \( \sigma \tau \sigma^{-1} a_\sigma \cdot a_\sigma^{-1} = \phi(\lambda_1(\tau)^2) \).

This means, in particular, that \( a_\tau \) does not depend on the variables \( x_2, \ldots, x_n \), i.e., \( a_\sigma \in k(x_1)^\times \), and \( a_\tau(\lambda_{1}^{-1}x_1) = \phi(\lambda_1^2) \cdot a_\sigma(x_1) \) for any \( \lambda_1 \in k^\times \).

It is now clear that \( a_\tau(x_1) \) is homogeneous, say of degree \( s \in \mathbb{Z} \), so \( \phi(\lambda_1^s) = \lambda^s \). Evaluating both sides at \(-1\), we see that \( s \) is even. Recall from the proof of Corollary 4.2 that \( \phi(\lambda)^{n+1} = \lambda^m \), so \( \lambda^{-s(n+1)} = \phi(\lambda^2)^{n+1} = \lambda^{2m} \), and thus, \( m = \frac{-s(n+1)}{2} \) is divisible by \( n + 1 \). Then \( a_\tau(x_1) = c \cdot x_1^{-2m/(n+1)} \) for some \( c \in k^\times \), so \( (a_\tau) \) is the product on an integer power of the class of \( \Omega^1_{F/k} \) and a homomorphism \( G \mapsto k^\times \) trivial on \( P \).
Let $\iota_0$ be the involution in $P$ sending $(x_1, \ldots, x_n)$ to $(1/x_1, x_2/x_1, \ldots, x_n/x_1)$, $s_1 = \iota_0 \sigma \iota_0 : (x_1, \ldots, x_n) \mapsto (1/x_1, x_2/x_1^2, \ldots, x_n/x_1^2)$ and $s_0 : (x_1, \ldots, x_n) \mapsto (x_1^{-1}, \ldots, x_n^{-1})$.

The element $s_0$ belongs to $G$, since it is the product of the elements $\iota_{ij} \sigma \iota_{ij}$ for all $1 \leq j \leq n$, where $\iota_{ij}$ are involutions in $P$ such that $\iota_{ij} x_s = x_s$ for $s \notin \{i, j\}$ and $\iota_{ij} x_j = x_j$. Let $g_0$ be the element in $P$ sending $(x_1, \ldots, x_n)$ to $(\frac{x_1}{x_1^{-1}}, \frac{x_2}{x_1^{-1}}, \ldots, \frac{x_n}{x_1^{-1}})$. Then one has the following well-known identity in $G$: $s_1 = g_0 s_0 g_0$. Then for the homomorphism $c$ as above one has $c(\sigma) = c(s_1) = c(g_0)^3 c(s_0)^2 = c(s_0)^2 = c(\sigma)^2$. As $\sigma^2 = 1$, this implies that $c(\sigma) = 1$. \qed

\textbf{Remark.} If $k$ is algebraically closed and $n = 2$ then by M.Noether theorem $G = C_{2}(k)$.

\textbf{Proposition 5.2.} Let $k$ be an algebraically closed field of characteristic zero, $A$ a noetherian algebraic group scheme over a ring $R$ and $n \geq 2$. Then $\text{Hom}(G, A(R)) = \{1\}$.

\textit{Proof.} It was shown at the end of the proof of Proposition 5.1 that there are no proper normal subgroups of $G$ containing $P$. As $P$ is simple (generated by any non-trivial conjugacy class), there are no proper normal subgroups of $G$ containing a non-trivial element of $P$.

Let the elements of $C_{2}(k)$ act identically on $k(x_3, \ldots, x_n)$. This gives an embedding of $C_{2}(k)$ into $C_n(k)$. By M.Noether theorem, $C_{2}(k)$ is generated by $\sigma$ and $P_2 := P \cap C_{2}(k)$.

Denote by $H \cong k(x_2) \ltimes k^\times$ the subgroup of $C_{2}(k) \subseteq G$ consisting of elements $\tau = (q(x_2), b)$ such that $\tau x_1 = x_1 + q(x_2)$ and $\tau x_2 = b \cdot x_2$ for some $q(x_2) \in k(x_2)$ and $b \in k^\times$.

Let $\rho : G \rightarrow A(R)$ be a homomorphism. We are going to show that $\ker \rho \cap H \cap P_2 \neq \{1\}$. As $\ker \rho$ is a normal subgroup in $G$, this will imply that $\ker \rho = G$.

For any $N \geq 3$ and any primitive $N$-th root of unity $\zeta_N^1$ the centralizer of $(0, \zeta_N^1) \in H$ is $k(x_2^{N^1}) \ltimes k^\times$.

Suppose that $\ker \rho \cap H = \{1\}$. Then $H \cong A(R)$, and thus, the centralizer of $(0, \zeta_N^1)$ in $H$ is the intersection of $H$ with the centralizer of $(0, \zeta_N^1)$ in $A(R)$. The centralizer of an element of $A(R)$ is the group of $R$-points of a closed subgroup in $A(R)$, so any descending sequence of centralizers should stabilize. This is not the case for the sequence $(k(x_2^{N^1}) \ltimes k^\times)_N \geq 1$.

Let $(q_1(x_2), b) \in \ker \rho \cap H \neq \{1\}$. If $b \neq 1$ then

$$(x_2, 1)(q_1(x_2), b)(x_2, 1)^{-1} = (x_2 + q_1(x_2), b)(-x_2, 1)(-q_1(b^{-1}x_2), b^{-1}) = ((1-b)x_2 + q_1(x_2), b)(-q_1(b^{-1}x_2), b^{-1}) = ((1-b)x_2, 1) \in \ker \rho \cap H \neq \{1\},$$

so there is $(q(x_2), 1) \in \ker \rho \cap H \neq \{1\}$.

It easy to see using prime decomposition of $q$ (or by M.Noether theorem) that $(x_1, x_2) \cong (q(x_2)x_1, x_2)$ is an element of $C_{2}(k) \subseteq G$, so $\alpha(q(x_2), 1) \alpha^{-1} \in \ker \rho \cap H$.

But $(x_1, x_2) \cong (x_1, x_2)$ is a non-trivial element of $P_2$, so $\ker \rho = G$. \qed

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