Excision and the Hodge filtration
in periodic cyclic homology:
the case of splitting and invertible extensions

Michael Puschnigg

Periodic cyclic homology of complex algebras was invented by Connes [Co] as a non-commutative generalization of de Rham cohomology. It is defined as the homology of the periodic cyclic bicomplex. This is a $\mathbb{Z}/2\mathbb{Z}$-graded complex of differential forms over the given algebra with grading given by the parity of forms. The degree of algebraic differential forms yields a descending filtration, called the Hodge filtration, of the periodic cyclic bicomplex by subcomplexes. The corresponding filtration on (co)homology is known as the dimension filtration [Co]. This filtration yields an important piece of extra structure on periodic cyclic (co)homology. In particular, it provides the link between the periodic and ordinary cyclic groups [Co], [FT]. The associated spectral sequence yields Connes exact sequence [Co] relating cyclic and Hochschild homology. In general it is difficult to determine the Hodge filtration on the periodic cyclic (co)homology of a given algebra.

One of the basic properties of periodic cyclic (co)homology is excision. It was established by Cuntz and Quillen [CQ3] and states that every algebra extension

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

gives rise to a natural six term exact sequence of periodic cyclic groups. The boundary map

$$\delta_s : HP_s(B) \rightarrow HP_{s-1}(I)$$

in this sequence and the excision isomorphism

$$\iota_s : HP_s(I) \xrightarrow{\sim} HP_s(A, B)$$

preserve the Hodge filtrations only under rather restrictive conditions [Wo]. In general these maps shift the dimension, i.e. the weight with respect to the Hodge filtration, of (co)homology classes.

In the present paper we continue our investigation [Pu] of the dimension shift under the excision isomorphism and the boundary map in periodic cyclic (co)homology. We evaluate the possible values of this shift in the case of splitting respectively...
invertible algebra extensions. For them we obtain considerably lower bounds than those valid for general extensions [Pu].

Here an extension is called splitting if it possesses a multiplicative linear section and invertible if it occurs as direct summand of a splitting extension. The latter extensions play an important role in operator K-theory [Ka] because the odd bivariant K-groups of $C^*$-algebras can be interpreted in terms of invertible $C^*$-algebra extensions.

For both types of extensions we exhibit rigid extension classes, i.e. classes of extensions which realize a maximal dimension shift, and for which the dimension shift is determined by the dimension of the involved homology classes alone. In fact we find

**Theorem 0.1.** Let

$$0 \rightarrow qA \rightarrow QA \rightarrow A \rightarrow 0$$

be the universal splitting extension [Cu] for $A$ and let

$$\iota_* : H^p_{\ast}(qA) \rightarrow H^p_{\ast}(QA, A)$$

be the excision isomorphism. Then

$$dim \iota_*^{-1}(\alpha) = \varphi(dim \alpha)$$

for all $\alpha \in H^p_{\ast}(QA, A)$ where

$$\varphi(4n) = \varphi(4n + 2) = 2n \quad \text{and} \quad \varphi(4n + 1) = \varphi(4n + 3) = 2n + 1$$

**Theorem 0.2.** Let

$$0 \rightarrow \epsilon A \rightarrow E_{\ast}A \rightarrow A \rightarrow 0$$

be the universal invertible extension [Ze] for $A$ and let

$$\delta_* : H^p_{\ast}(A) \rightarrow H^p_{\ast-1}(\epsilon A)$$

be the boundary map. Then

$$dim \delta_* (\beta) = \psi(dim \beta)$$

for all $\beta \in H^p_{\ast}(A)$ where

$$\psi(4n) = \psi(4n - 2) = 2n - 1 \quad \text{and} \quad \psi(4n + 1) = \psi(4n - 1) = 2n$$

There are no similar results in cohomology because the shift functions $\varphi$ and $\psi$ are not one to one.

With the help of the rigidity theorems one easily calculates the possible values for the dimension shift of the excision respectively boundary maps for arbitrary splitting respectively invertible extensions. In particular it follows that the pseudodifferential operator extension [Do]

$$0 \rightarrow \ell^{2n+1}(H) \rightarrow \mathcal{P}(M^{2n}) \xrightarrow{\sigma_m} \mathcal{C}^\infty(S^*M^{2n}) \rightarrow 0$$
associated to a smooth compact manifold of even dimension realizes the maximal dimension shift of the boundary map among invertible extensions.

Turning to cohomology one obtains from the previous results the bounds

$$\dim (\iota^*)^{-1}(\alpha) \leq 2 \dim \alpha + 2 \quad \text{and} \quad \dim \delta^*(\beta) \leq 2 \dim \beta + 2$$

for splitting respectively invertible extensions which are considerable sharper than the estimates

$$\dim (\iota^*)^{-1}(\alpha) \leq 3 \dim \alpha + 2 \quad \text{and} \quad \dim \delta^*(\beta) \leq 3 \dim \beta + 3$$

which are optimal for general algebra extensions [Pu].

The demonstration of our results relies essentially on an (unpublished) paper of Cuntz and Quillen [CQ] which builds on former work of Connes [Co2]. In [Co2] Connes showed that normalized cocycles of infinite support on the cyclic bicomplex can be identified with odd supertraces on suitable adic completions of the superalgebras $QA = A \ast A$ [Cu] respectively $EA = QA \times \mathbb{Z}/2\mathbb{Z}$ [Ze]. Cuntz and Quillen [CQ] studied homotopy relations between such supertraces and arrived at a description (of the chain homotopy type) of the whole cyclic bicomplex in terms of the odd part of the $X$-complexes of $\widehat{QA} = \lim QA/(qA)^n$ and $\widehat{EA} = \lim EA/(\epsilon A)^n$. Under this identification the Hodge filtration on periodic cyclic homology corresponds to the obvious adic filtrations on the homology of the various $X$-complexes. This is exactly what we will use for the purpose of this paper.

The boundary map associated to the universal invertible extension can be given by a rather explicit chain map [Pu], which makes it possible to obtain the lower bound $\dim \delta_\ast(\alpha) \geq \psi(\dim \alpha)$ for $\alpha \in HP_\ast(A)$. We exhibit transgression formulas which relate the class $\delta_\ast(\alpha)$ to the image of $\alpha$ under the Cuntz-Quillen isomorphism $\Psi : HP_\ast(A) \rightarrow H_\ast(X(EA, \mathcal{F}[F])^{-}).$ These transgression formulas allow to control the $\epsilon A$-adic valuation of $\Psi(\alpha).$ The compatibility of the Cuntz-Quillen isomorphism with filtrations yields then the upper bound $\dim \delta_\ast^{-1}(\alpha) \leq \psi(\dim \alpha)$ which achieves the demonstration of the rigidity theorem. A similar strategy applies in the case of splitting extensions.

I want to thank Joachim Cuntz heartily for making his joint work with Quillen accessible to me. It is the basis of the present paper. I am also indebted to him for his kind permission to include the demonstrations of the relevant results from their work.
Contents

1 Universal algebras and universal extensions 5
   1.1 Algebraic differential forms .......................... 5
   1.2 The universal splitting extensions (after Cuntz): ...... 5
   1.3 The universal invertible extensions (after Zekri): ..... 6

2 X-complexes of universal algebras
   (after Cuntz and Quillen) 7
   2.1 Cyclic chain complexes ............................. 7
      2.1.1 Complexes and mapping cones .................... 7
      2.1.2 Operators on differential forms .................. 8
      2.1.3 The cyclic bicomplex $\widetilde{CC}(A)$ .............. 9
   2.2 $X$-complexes of universal algebras .................. 11
      2.2.1 The complex $X(\widehat{QA})^-$ ................... 11
      2.2.2 The complex $X(\widehat{EA})^-$ .................... 15
   2.3 Identification of the Cuntz-Quillen maps ................ 19

3 Results 21
   3.1 Free ideal extensions ............................... 21
   3.2 Explicit boundary maps ................................ 23
   3.3 Rigidity of the universal extensions ................. 25
      3.3.1 Rigidity of the universal splitting extensions .... 25
      3.3.2 Rigidity of the universal invertible extensions ... 28
   3.4 Universal bounds for dimension shifts ................ 30
1 Universal algebras and universal extensions

1.1 Algebraic differential forms

Throughout this paper we work in the category of (not necessarily unital) complex algebras. Such an algebra will usually be denoted by $A$.

The functor which assigns to a differential graded algebra its subalgebra of homogeneous elements of degree zero possesses a left adjoint $A \rightarrow \Omega A$. Thus $\Omega A$ is universal among the differential graded algebras which coincide in degree zero with $A$. There is a canonical linear isomorphism

$$\Omega A \simeq \bigoplus_{n=0}^\infty \tilde{A} \otimes A^n$$

$$a^0 da^1 \ldots da^n \leftrightarrow \tilde{a}^0 \otimes \ldots \otimes a^n$$

The ideal of forms of strictly positive degree will be denoted by $\Omega A_+$. There exists still another natural product on $\Omega A$, the Fedosov product [CQ1], which is given in terms of the usual product by

$$\alpha \ast \alpha' = \alpha \cdot \alpha' - (-1)^{|\alpha|} d\alpha \cdot d\alpha'$$

In particular

$$Gr_{\Omega A_+}(\Omega A, \ast) \simeq \Omega A$$

as graded algebras.

1.2 The universal splitting extensions (after Cuntz):

The following results are due to Cuntz [Cu] The forgetful functor from superalgebras ($= \mathbb{Z}/2\mathbb{Z}$-graded algebras) to algebras possesses a left adjoint

$$A \rightarrow QA := A \ast A$$

assigning to an algebra $A$ the free product of two copies of $A$. The superalgebra structure arises from the $\mathbb{Z}/2\mathbb{Z}$-action which flips the two copies of $A$. Associated to the adjunction are a canonical algebra homomorphism $\theta : A \rightarrow QA$, which is an isomorphism onto the first copy of $A$ generating $QA$, and a (super)algebra epimorphism $\pi_{QA} : QA \rightarrow A$ which equals the identity on each of the two copies of $A$ generating $QA$.

The natural $\mathbb{Z}/2\mathbb{Z}$-equivariant extension

$$0 \rightarrow qA \rightarrow QA \xrightarrow{\pi_{QA}} A \rightarrow 0$$
possesses two natural multiplicative linear sections given by $\theta$ and its conjugate $\theta^\gamma$ under the $\mathbb{Z}/2\mathbb{Z}$-action. This extension is universal in the sense that any extension $0 \to J \to A \to B \to 0$ with multiplicative linear section fits into a commutative diagram

\[
\begin{array}{ccc}
0 & \to & J \\
\uparrow & & \uparrow \\
0 & \to & qA \\
\end{array}
\quad \begin{array}{ccc}
& \to & A \\
& & \to \\
& \to & B \\
\end{array}
\quad \begin{array}{ccc}
\pi_{QA} & \to & A \\
\uparrow & & \uparrow \\
& & \to \\
\end{array}
\quad 0
\]

which is compatible with the given sections.

There is a linear isomorphism

\[
\Omega A \cong QA
\]

\[a^0 da^1 \ldots da^n \leftrightarrow pa^0 qa^1 \ldots qa^n\]

where $pa$, respectively $qa$, denotes the even, respectively odd, part of $\theta(a) \in QA$ under the canonical $\mathbb{Z}/2\mathbb{Z}$-action. With respect to this isomorphism, the multiplication on $QA$ corresponds to the Fedosov product on $\Omega A$. The epimorphism $\pi_{QA}$ corresponds to the canonical projection onto forms of degree zero and the multiplicative linear sections $\theta, \theta^\gamma$ of $\pi_{QA}$ correspond to the homomorphisms $j^\pm : A \to \Omega A$ given by $j^\pm(a) := a \pm da$. Moreover, under this isomorphism,

\[Gr_{qA}(QA) \cong \Omega A\]

as graded algebras.

1.3 The universal invertible extensions (after Zekri):

The following results are due to Zekri [Ze]. Suppose that $A$ is unital and let

\[\alpha : 0 \to B \to E \to A \to 0, \quad \alpha' : 0 \to B \to E' \to A \to 0\]

be (unital) algebra extensions of $A$ by $B$. The pullback of the extension

\[0 \to M_2(B) \to E'' = \begin{pmatrix} E & B \\ B & E' \end{pmatrix} \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = A \oplus A \to 0\]

with respect to the diagonal $\Delta : A \to A \oplus A$ is called the direct sum of the given extensions and is denoted by $\alpha \oplus \alpha'$.

An algebra extension $\alpha \in Ext_{Alg}(A, B)$ is called invertible, if there exists an extension $\alpha' \in Ext_{Alg}(A, B)$ such that $\alpha \oplus \alpha'$ is splitting, i.e. possesses a multiplicative linear section.

Let

\[\alpha_A : 0 \to \epsilon A \to EA \xrightarrow{\pi E_A} A \oplus A \to 0\]
be the extension obtained from the universal splitting extension

$$0 \rightarrow qA \rightarrow QA \rightarrow A \rightarrow 0$$

by taking crossed products with respect to $\mathbb{Z}/2\mathbb{Z}$. The algebra $EA = QA \times \mathbb{Z}/2\mathbb{Z}$
is canonically isomorphic to $A[\mathbb{F}]/(F^2 - 1)$, the algebra obtained by adjoining to $A$
an involutive element $F$ satisfying $F^2 = 1$. If $\epsilon$, $\epsilon^2 = 1$, denotes the grading operator
on $\Omega A$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading given by even and odd forms, then there
is a natural linear isomorphism

$$\Omega A[\epsilon] \simeq EA$$

$$a^0 da^1 \ldots da^n \leftrightarrow a^0[F, a^1] \ldots [F, a^n]$$

$$\epsilon a_0 da_1 \ldots da_n \leftrightarrow F a_0[F, a_1] \ldots [F, a_n]$$

Moreover, under this identification,

$$Gr_{\epsilon A}(EA) \simeq \Omega A[\epsilon]$$

as graded algebras.

Let $\alpha_0^A$, $\alpha_1^A$, be the pull backs of $\alpha_A$ along the inclusions

$i_0, i_1 : A \rightarrow A \oplus A$. The extensions $\alpha_0^A$, $\alpha_1^A$ are invertible and in fact inverse to each other under addition.

The invertible extension $\alpha_0^A$ is universal in the sense that every invertible extension

$\alpha : 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ fits into a commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & B & \rightarrow & E & \rightarrow & A & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \| \\
\alpha_A^0 : 0 & \rightarrow & \epsilon A & \rightarrow & E' A & \rightarrow & A & \rightarrow & 0
\end{array}$$

2 X-complexes of universal algebras 
(after Cuntz and Quillen)

2.1 Cyclic chain complexes

2.1.1 Complexes and mapping cones

All chain complexes considered in this paper will be $\mathbb{Z}/2\mathbb{Z}$-graded complexes of complex, adically complete, vector spaces. If $F$ is a functor from the category of abstract complex algebras to adically complete chain complexes, then we extend it to a functor on the category of adically complete algebras and continuous algebra
homomorphisms as follows. For an adically complete algebra $\tilde{A}$ obtained by completion of the abstract algebra $A$ with respect to the $I$-adic topology, $I \subset A$ an ideal, we put

$$F(\tilde{A}) = \lim_{n \to \infty} F(A/I^n)$$

This depends only on $\tilde{A}$ and not on the choices of $A$ and $I$.

For a chain complex $C$ denote by $C[1] : C[1]_n = C_{n+1}$, $\partial_{C[1]} = -\partial_C$ the shifted complex. Let $f : C \to C'$ be a chain map. The mapping cone $\text{Cone}(f)$ of $f$ is defined as the complex

$$\text{Cone}(f)_n = C_n \oplus C'_{n+1}, \quad \partial_{\text{Cone}} = \begin{pmatrix} \partial_C & 0 \\ f & -\partial_{C'} \end{pmatrix}$$

Thus a chain map $C'' \to \text{Cone}(f)$ is given by a pair $(\varphi, \psi)$ consisting of a chain map $\varphi : C'' \to C$ and a contracting homotopy $\psi : C'' \to C'[1]$ of the composition $f \circ \varphi$, i.e. $f \circ \varphi = \psi \circ \partial + \partial \circ \psi$. It should be noted that the chain homotopy class of this map to the cone will in general depend on the choice of the nullhomotopy for $f \circ \varphi$.

There is a natural extension of complexes

$$0 \to C'[1] \to \text{Cone}(f) \xrightarrow{\pi_{\text{Cone}}} C \to 0$$

which possesses a natural linear section. Any pair

$$\begin{pmatrix} g : C'' \to C, & C \xrightarrow{j} C' \\ g \uparrow & \uparrow \downarrow j \\ C'' \xrightarrow{i} D \end{pmatrix}$$

consisting of a chain map $g : C'' \to C$ and a commutative diagram with contractible $D$ determines a unique chain homotopy class $C'' \to \text{Cone}(f)$ of chain maps.

Indeed, if $h$ is a contracting homotopy of $D$, then the couple $(g, j \circ h \circ i)$ defines a chain map $\alpha_h : C'' \to \text{Cone}(f)$. This time the chain homotopy class of $\alpha_h$ does not depend on the choice of $h$.

### 2.1.2 Operators on differential forms

We follow [CQ2]. The basic operators on algebraic differential forms are the **exterior differential**

$$d : \Omega^n A \longrightarrow \Omega^{n+1} A, \quad d(da^0 \cdots da^n) = da^0 da^1 \cdots da^n$$

and the **Hochschild boundary operator**

$$b : \Omega^n A \longrightarrow \Omega^{n-1} A, \quad b(\omega da) = (-1)^{|\omega|}[\omega, a]$$
They satisfy \( b^2 = d^2 = 0 \). The Karoubi operator
\[
\kappa : \Omega^n A \rightarrow \Omega^n A, \quad \kappa(a^0da^1 \ldots da^n) = (-1)^{n-1}da^n a^0 da^1 \ldots da^{n-1}
\]
can be expressed in terms of the two basic operators as
\[
\kappa = Id - (d \circ b + b \circ d)
\]
Therefore \( \kappa \) commutes with \( b \) and \( d \). Moreover it follows that \( \kappa \) is a chain endomorphism of the Hochschild complex \((\Omega A, b)\) which is chain homotopic to the identity. The Connes operator
\[
B : \Omega^n A \rightarrow \Omega^{n+1} A, \quad B = \sum_{i=0}^{n} \kappa^i \circ d = N' \circ d, \quad (N'|_{\Omega^{n+1}} = \sum_{i=0}^{n} \kappa^i)
\]
satisfies \( B^2 = 0 \) and \( bB + Bb = 0 \) and commutes with \( \kappa \) as well. From the explicit formula for the Karoubi operator one derives the identity
\[
(\kappa^n - 1)(\kappa^{n+1} - 1) = 0
\]
on \( \Omega^n A \). There is a natural spectral decomposition
\[
\Omega^n A = \text{Ker}(\kappa - Id)^2 \oplus \bigoplus_{\zeta \neq 1} \text{Ker}(\kappa - \zeta \cdot Id) = P_{CQ} \Omega^n A \oplus (1 - P_{CQ}) \Omega^n A
\]
of the space of algebraic differential forms with respect to the Karoubi operator into the direct sum of the subspace \( P_{CQ} \Omega^n A \) of normalized forms and the complementary subspace \((1 - P_{CQ}) \Omega^n A\). The spectral projection \( P_{CQ} \) is called the Cuntz-Quillen projection. It commutes with the operators \( b \), \( d \), and \( B \).

2.1.3 The cyclic bicomplex \( \widehat{CC}(A) \)

We recall some well known facts about cyclic complexes taken from [Co],[FT], and [CQ2]. Let \( \widehat{\Omega} A = \prod \Omega^n A \) be the \( \Omega A \)-adic completion of \( \Omega A \). It is naturally \( \mathbb{Z}/2\mathbb{Z} \)-graded by the decomposition into forms of even and odd degree.

The periodic cyclic bicomplex of an algebra \( A \) is
\[
\widehat{CC}(A) = (\widehat{\Omega} A, b + B)
\]
It is a \( \mathbb{Z}/2\mathbb{Z} \)-graded chain complex of adically complete vector spaces and its homology
\[
HP_n(A) = H_n(\widehat{CC}(A))
\]
is called the periodic cyclic homology of \( A \). The cohomology of the dual complex is the periodic cyclic cohomology of \( A \). In the definition of the cyclic bicomplex it is essential to pass to the completion: the dense subcomplex \( CC(A) := (\Omega A, b + B) \) is contractible.

The periodic cyclic bicomplex carries a natural decreasing filtration
\[
\text{Fil}_{\text{Hodge}}^n \widehat{CC}(A) = (b\Omega^n A \oplus \prod_{k \geq n} \Omega^k A, b + B)
\]

9
given by the subcomplexes generated by algebraic differential forms of degree at least $n$. It is called the \textbf{Hodge filtration}. The Hodge filtration induces a corresponding filtration of periodic cyclic (co)homology.

The weight of a cyclic (co)homology class $\alpha \in HP(A)$ with respect to the Hodge filtration is called its \textbf{dimension} $\dim(\alpha) = \dim \{ k \in \mathbb{N}, \alpha \notin Ker(HP_k \to HC_k) \}$. Thus $\dim(\alpha) = n$ if $\alpha \in Fil_{Hodge}^n HP(A)$, but $\alpha \notin Fil_{Hodge}^{n+1} HP(A)$ whereas $\dim(\alpha) = \infty$ if $\alpha \in Fil_{Hodge}^m HP(A)$ for all $m \in \mathbb{N}$.

The homology

$$HC_*(A) = H_*(\widetilde{CC}(A)/Fil_{Hodge}^{*+1}\widetilde{CC}(A))$$

of the quotient complexes is called the \textbf{cyclic homology} of $A$. The corresponding cohomology of the dual complexes is the cyclic cohomology of $A$. (Note that cyclic (co)homology is $\mathbb{Z}$-graded while the underlying complexes are only $\mathbb{Z}/2\mathbb{Z}$-graded.)

Thus for $\alpha \in HP_*(A)$ one has

$$\dim(\alpha) = \min \{ k \in \mathbb{N}, \alpha \notin Ker(HP_k \to HC_k) \}$$

while for $\tilde{\alpha} \in HP^*(A)$

$$\dim(\tilde{\alpha}) = \min \{ l \in \mathbb{N}, \tilde{\alpha} \in Im(\text{HC}_l \to HP_l) \}$$

Among the quotient complexes $\widetilde{CC}(A)/Fil_{Hodge}^k\widetilde{CC}(A)$ the $X$-complex [Qu]

$$X_*(A) = \widetilde{CC}_*(A)/Fil_{Hodge}^2\widetilde{CC}_*(A)$$

plays a particularly important role. It is given by

$$X_0 A = A, \quad X_1 A = \Omega^1 A = \Omega^1 A/\Omega^1 A, A$$

and its differentials equal $d_0 = d$ and $d_1 = b$.

We denote by $\pi_{\widetilde{CC}} : \widetilde{CC}(A) \to X(A)$ the natural quotient map. It is a chain homotopy equivalence if $A$ is \textbf{quasifree} [CQ1].

Let $S \subset A$ be a subalgebra. The quotient complex

$$X(A, S) : \quad X_0(A, S) = A/[A, S] \quad X_1(A, S) = (\Omega^1 A/AdSA)$$

of $X(A)$ is called the \textbf{relative $X$-complex} of the pair $(A, S)$ [CQ]. If every derivation of $A$ with values in an $A$-bimodule becomes inner under restriction to $S$, then the quotient map $\pi_{rel} : X(A) \to X(A, S)$ is a chain homotopy equivalence [CQ].

A \textbf{quasifree presentation} of an algebra $A$ is a homomorphism $R \to A$ from an (adicantly complete) quasifree algebra $R$ [CQ1] to $A$ which induces a chain homotopy equivalence on periodic cyclic bicomplexes. Every algebra possesses quasifree presentations.

The Cuntz-Quillen projection defines a natural decomposition of $\widetilde{CC}(A)$ into the direct sum of the normalized subcomplex $PCQ \widetilde{CC}(A)$ and the subcomplex

$$(1 - P_{CQ}) \widetilde{CC}(A)$$

which is contractible. There is a similar decomposition for the subcomplexes $Fil_{Hodge}^n \widetilde{CC}(A)$. The Karoubi operator and thus the Cuntz Quillen projection equal the identity on the quotient complex $X(A)$. 

10
2.2 X-complexes of universal algebras

From the very beginning beginning of the theory [Co] cyclic cocycles were interpreted as generalized traces. In [Co2] Connes identified normalized cocycles of infinite support on the periodic cyclic bicomplex of \( A \) with odd supertraces on the universal algebras \( \overline{QA} \) and \( \overline{EA} \). Cuntz and Quillen [CQ] derived homotopy formulas for such supertraces. They obtained thus a complete description (up to chain homotopy) of the cyclic bicomplex in terms of suitable \( X \)-complexes of universal algebras. Their results play a crucial role in this paper and we recall the aspects which are relevant for us. I am indebted to Joachim Cuntz for his kind suggestion to include complete demonstrations of the results used from their work.

2.2.1 The complex \( X(\overline{QA})^- \)

Let \( \overline{QA} \) be the \( qA \)-adic completion of the universal super-algebra \( QA = A \ast A^7 \) over \( A \) (1.2) and let \( \theta : A \rightarrow \overline{QA} \) be the canonical inclusion. Consider the chain map

\[
X(\overline{QA})^- : X(A) \overset{X(\overline{QA})^-}{\rightarrow} X(\overline{QA}) \rightarrow X(\overline{QA})^- 
\]

which is given by the composition of \( X(\overline{QA})^- \) and the canonical projection of \( X(\overline{QA})^- \) onto its odd part with respect to the natural \( \mathbb{Z}/2\mathbb{Z} \)-action. Let \( \text{Cone}(X(\overline{QA})^-) \) be the mapping cone of \( X(\overline{QA})^- \) and recall that it is isomorphic to \( X(A) \oplus X(\overline{QA})[1]^- \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space.

In the first step this space is identified with a space of algebraic differential forms.

**Proposition 2.1.** [CQ] There exists a natural linear isomorphism of \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces

\[
(\overline{\Omega}^{ev} A \oplus \Omega^1 A_o) \oplus (\overline{\Omega}^{odd} A \oplus \Omega^1 A_1) \cong X(A) \oplus X(\overline{QA})[1]^- 
\]

which is given by the formulas

\[
a^0 \quad \leftrightarrow \quad (a^0, 0) \\
a^0 da^1 \ldots da^{2n} \quad \leftrightarrow \quad (0, \theta(a^0)qa^1 \ldots qa^{2n-1} d\theta(a^{2n})^-), \quad n \geq 1 \\
\overline{a^0 da^1} \quad \leftrightarrow \quad (0, \theta(a^0)d\theta(a^1)^-) \\
\overline{a^0 da^1} \quad \leftrightarrow \quad (\overline{a^0 da^1}, pa^0 qa^1) \\
a^0 da^1 \ldots da^{2n+1} \quad \leftrightarrow \quad (0, pa^0 qa^1 \ldots qa^{2n+1}), \quad n \geq 1 \\
\overline{a^0 da^1} \quad \leftrightarrow \quad (\overline{a^0 da^1}, 0)
\]
**Proof:** The algebra $QA = A \ast A^\gamma$ is a free product which implies that

$$\Omega^1 QA \simeq QA(dA)QA \oplus QA(dA^\gamma)QA$$

as $QA$-bimodules. From this one deduces easily that the canonical map

$$QA(dA) \oplus QA(dA^\gamma) \rightarrow \Omega^1(QA)/[\Omega^1(QA), QA] = \Omega^1(QA)_{\mathbb{H}}$$

is onto. An elementary calculation shows that

$$(QA(dA) + QA(dA^\gamma)) \cap [\Omega^1QA, QA] = [QA(dA), A] + [QA(dA^\gamma), A^\gamma]$$

Altogether one obtains a $\mathbb{Z}/2\mathbb{Z}$-equivariant linear isomorphism

$$QA(dA)/[QA(dA), A] \oplus QA(dA^\gamma)/[QA(dA^\gamma), A^\gamma] \xrightarrow{\sim} \Omega^1QA/[\Omega^1QA, QA] = \Omega^1QA_{\mathbb{H}}$$

where the action on the left hand side flips the two factors. Taking coinvariants and projecting to the first factor yields then an isomorphism

$$QA(dA)/[QA(dA), A] \xrightarrow{\sim} \Omega^1QA_{\mathbb{H}}^-$$

We are going to describe the commutator subspace $[QA(dA), A]$ in detail. There is a linear identification of $QA(dA)$ with the space of differential forms of strictly positive degree given by

$$\Omega A_+ \simeq QA(dA)$$

$$a^0da^1 \ldots da^n \rightarrow \theta(a^0)qa^1 \ldots qa^{n-1}d\theta(a^n) = \theta q^{n-1}d\theta(a^0 \otimes \ldots \otimes a^n)$$

Using it one finds the relations

$$[\theta q^{2n-1}d\theta, \theta] = \theta q^{2n-1}d\theta \circ b - 2\theta q^{2n}d\theta, \ n \geq 1,$$

$$[\theta q^{2n}d\theta, \theta] = -\theta q^{2n}d\theta \circ b, \ n \geq 0,$$

in $[QA(dA), A]$. In fact, if one denotes

$$\partial_i(a^0, \ldots, a^n) = (a^0, \ldots, a^i a^{i+1}, \ldots, a^n), \ 0 \leq i \leq n-1,$$

$$\partial_n(a^0, \ldots, a^n) = (a^n a^0, \ldots, a^{n-1})$$

the simplicial Hochschild face operators, then

$$[\theta q^{2n-1}d\theta, \theta] = \theta q^{2n-1}d(\theta^2) - \theta q^{2n-1}\theta d\theta - \theta^2 q^{2n-1}d\theta$$

$$= \theta q^{2n-1}d\theta \circ (\partial_{2n} - \partial_{2n+1}) - \theta q^{2n-1}(p + q)d\theta$$

$$= \theta q^{2n-1}d\theta \circ \left( \sum_{i=1}^{2n+1} (-1)^i \partial_i \right) + \theta pq^{2n-1}d\theta - \theta q^{2n}d\theta$$

$$= \theta q^{2n-1}d\theta \circ \left( \sum_{i=1}^{2n+1} (-1)^i \partial_i \right) + \theta (\theta - q) q^{2n-1}d\theta - \theta q^{2n}d\theta$$

$$= \theta q^{2n-1}d\theta \circ \left( \sum_{i=1}^{2n+1} (-1)^i \partial_i \right) + \theta (\theta - q) q^{2n-1}d\theta - \theta q^{2n}d\theta$$
\[
\begin{align*}
&= \partial q^{2n-1} d\theta \circ \left( \sum_{i=0}^{2n+1} (-1)^i \partial_i \right) - 2\partial q^{2n} d\theta \\
&= \partial q^{2n-1} d\theta \circ b - 2\partial q^{2n} d\theta
\end{align*}
\]

and a similar calculation shows the other relation above. We have thus shown that the composition

\[
\Omega^{ev} A_+ \oplus \Omega^1 A_0 \longrightarrow QA(dA) \longrightarrow \Omega^1 QA_0^-
\]
is an isomorphism. The proposition follows easily from this, (1.2) and (2.1.1). □

This linear identification allows to interpret the differentials of the complex \( \text{Cone}(X(\hat{\theta})^-) \) in terms of standard operators on differential forms.

**Proposition 2.2.** [CQ] Under the linear isomorphism

\[
(\tilde{\Omega}^{ev} A \oplus \Omega^1 A_0) \oplus (\tilde{\Omega}^{odd} A \oplus \Omega^1 A_0) \simeq X(A) \oplus X(QA)[1]^{-1}
\]
of (2.1) the differentials of the complex \( \text{Cone}(X(\hat{\theta})^-) \) correspond to the following operators on differential forms:

\[
\partial^e|_{\tilde{\Omega}^e} = d \oplus 0, \quad \partial^e|_{\tilde{\Omega}^{e \geq 0}} = (b - 2d) \oplus 0, \quad \partial^e|_{\tilde{\Omega}^1} = -d \circ b \oplus 0,
\]

\[
\partial^{odd}|_{\tilde{\Omega}^{odd}} = (b + B) \oplus 0, \quad \partial^{odd}|_{\tilde{\Omega}^{odd \geq 1}} = (B - b \circ N') \oplus 0, \quad \partial^e|_{\tilde{\Omega}^1} = b \oplus Id
\]

**Proof:** Straightforward in view of (2.1). □

The harmonic decomposition of Cuntz and Quillen [CQ2] yields finally a complete description of the homological properties of the complex under consideration.

**Theorem 2.3.** [CQ] Let \( \hat{\theta} : A \to QA \) be the canonical inclusion (1.2) and denote by \( \text{Cone}(X(\hat{\theta})^-) \) the cone of the chain map

\[
X(\hat{\theta})^- : X(A) \xrightarrow{X(\hat{\theta})} X(QA) \xrightarrow{\pi} X(QA)^-
\]

which is given by the composition of \( X(\hat{\theta}) \) and the canonical projection of \( X(QA) \) onto its odd part with respect to the natural \( \mathbb{Z}/2\mathbb{Z} \)-action. Let the natural identification (2.1) of this complex with a space of algebraic differential forms be understood.

i) The differentials of the complex \( \text{Cone}(X(\hat{\theta})^-) \) commute with the Karoubi operator \( \kappa \) and with the Cuntz-Quillen projection \( P_{CQ} \) (2.1.2). Thus there is a natural decomposition of complexes

\[
\text{Cone}(X(\hat{\theta})^-) \simeq P_{CQ} \text{Cone}(X(\hat{\theta})^-) \oplus (1 - P_{CQ}) \text{Cone}(X(\hat{\theta})^-)
\]

13
ii) The linear map

\[ \Psi : P_{\text{CQ}} \widehat{\text{CC}}(A) \rightarrow P_{\text{CQ}} \text{Cone}(X(\bar{\theta})^-) \]

of normalized subcomplexes, which is given on differential forms of degree
\[2n - 1, 2n, n \geq 1,\]
by multiplication with \( \lambda_n = (-1)^{n-1}2^{2n-1} \frac{n!}{(2n)!} \) and by the identity in degree zero, is an inclusion of chain complexes with natural linear section. The cokernel of \( \Psi \) is naturally contractible.

iii) The complementary subcomplex \((1 - P_{\text{CQ}}) \text{Cone}(X(\bar{\theta})^-)\) is naturally contractible.

**Proof:** Assertion i) is clear from (2.1.2) and (2.2). It is also clear from (2.1) that \( \Psi \) is a linear inclusion with cokernel equal to \( \Omega^1 \Lambda_i \) in both even and odd degree. A natural linear section of \( \Psi \) is given by suitable multiples of the obvious projection of \( P_{\text{CQ}} \text{Cone}(X(\bar{\theta})^-) \) onto \( P_{\text{CQ}} \widehat{\Omega} A \). From the identity \( \kappa |_{\Omega^n} = 1 - \frac{bR}{n(n+1)}, \) which holds on the space of normalized differential forms \([CQ1]\), and which implies the identities \( \kappa \circ b = b \circ \kappa = b \) and \( \kappa \circ d = d \circ \kappa = d \), one deduces that the differentials of \( P_{\text{CQ}} \text{Cone}(X(\bar{\theta})^-) \) are given by \( \partial \mid_{\Omega^n} = B, \partial \mid_{\Omega^{2n}} = b - \frac{bR}{n+1}, n \geq 1, \)
and \( \partial \mid_{\Omega} = b + B, \partial \mid_{\Omega^{2n+1}} = B - (n + \frac{1}{2})b, n \geq 1. \) It follows immediately that \( \Psi \)

is a chain map. The differentials on its cokernel are given by zero respectively the identity which shows that the latter complex is naturally contractible. To verify iii) note that on \( \text{Im}(1 - P_{\text{CQ}}) \) the operator \( 1 - \kappa = db + bd \) is invertible. The identity \( \partial \circ (d - \frac{1}{2}b) = (d - \frac{1}{2}b) \circ \partial = db + bd \) shows then that
\( \partial = b - 2d : (1 - P_{\text{CQ}}) \text{Cone}(X(\bar{\theta})^-)_0 \rightarrow (1 - P_{\text{CQ}}) \text{Cone}(X(\bar{\theta})^-)_1 \)
is bijective. \( \square \)

The following corollaries collect what will be needed in the sequel. The notations are those of (2.3).

**Corollary 2.4.** [CQ] There exists a natural chain homotopy equivalence

\[ \Psi : \widehat{\text{CC}}(A) \xrightarrow{\sim} \text{Cone}(X(\bar{\theta})^-) \]

such that

\[ \pi_{\text{Cone}} \circ \Psi = \pi_{\widehat{\text{CC}}} \]

and such that \( \Psi \) as well as a suitable chain homotopy inverse of \( \Psi \) preserve the degree of algebraic differential forms.

In particular, the restriction of \( \Psi \) to the second step of the Hodge filtration defines a natural quasiisomorphism

\[ \text{Fil}^2_{\text{Hodge}} \widehat{\text{CC}}(A) \xrightarrow{\text{qi}} X(QA)^- [1] \]

In fact the latter map is given on normalized differential forms of fixed degree by a multiple of the isomorphism (2.1).
**Proof:** The chain map $\Psi$ has already be defined on the normalized subcomplex $P_{CQ}\widehat{CC}(A)$ in the previous theorem. Extend it to a chain map on the whole cyclic bicomplex by demanding that it vanishes on the complementary subcomplex $(1 - P_{CQ})\widehat{CC}(A)$. It follows then immediately from the previous theorem that $\Psi$ is a chain homotopy equivalence which preserves the degree of differential forms and which possesses a chain homotopy inverse which also does so. The identity $\pi_{Cone} \circ \Psi = \pi_{\widehat{CC}}$ is clear from the definitions. Note that the restriction of $\Psi$ to the subcomplex $Fil^2_{\text{Hodge}}\widehat{CC}(A)$ yields only a natural quasiisomorphism but no natural chain homotopy equivalence because $Fil^2_{\text{Hodge}}\widehat{CC}(A)$ has no natural complement in $\widehat{CC}(A)$. \hfill $\square$

**Corollary 2.5.** [CQ] The chain complex $\text{Cone}(X(\theta)^-): X(A) \to X(QA)^-$ is naturally contractible.

**Proof:** In view of the previous theorem the assertion amounts to the contractibility of the complex $CC(A)$. \hfill $\square$

### 2.2.2 The complex $X(\widehat{EA})^-$

Let $A$ be unital and let $\widehat{EA}$ be the $\epsilon A$-adic completion of the algebra $EA = QA \times \mathbb{Z}/2\mathbb{Z} \simeq A[F]/(F^2 - 1)$ obtained from $A$ by adjoining an involutive element $F$ (1.3). The algebra $EA$ carries a canonical $\mathbb{Z}/2\mathbb{Z}$-action which can be described either as the dual action on $QA \times \mathbb{Z}/2\mathbb{Z}$ or as the nontrivial action on $A[F]$ which fixes $A$ pointwise. We denote by $X(EA, F)$ the relative $X$-complex of the pair $(\widehat{EA}, \mathcal{C}[F])$ and let finally $X(EA, F)^-$ be its odd part with respect to the $\mathbb{Z}/2\mathbb{Z}$-action.

The relation between the absolute $X$-complex and the relative $X$-complex with respect to a separable subalgebra is clarified in

**Lemma 2.6.** [CQ] If $S \subset R$ is a separable subalgebra of the algebra $R$ [CQ1], i.e. any bimodule derivation on $S$ is inner, then the natural projection $X(R) \to X(R, S)$ (2.1.3) is a chain homotopy equivalence. This applies in particular to the inclusion $\mathcal{C}[F] \subset \widehat{EA}$ (1.3). Consequently the canonical projection

$$\pi_{rel} : X(\widehat{EA})^- \to X(\widehat{EA}, F)^-$$

is a chain homotopy equivalence (2.1.3).

**Proof:** Let $S \subset R$ be a subalgebra such that the restriction of any derivation of $R$ (with values in an $R$-bimodule) to $S$ is inner. Suppose that the restriction of the universal derivation $d : R \to \Omega^1 R$ to $S$ is of the form $ds = [s, Y]$ with $Y \in RdSR$, the $R$-subbimodule of $\Omega^1 R$ spanned by $dS$. This is the case for example if $S$ is separable [CQ1]. Define an operator $h_s : X_s(R) \to X_{s+1}(R)$ of degree one by
\( h_0(r) = r^{\bar{Y}} \), \( h_1 = 0 \) and put \( 1-p = h \circ \partial + \partial \circ h \). Then \( p \) is a deformation retraction onto a subcomplex of \( X(R) \) which is complementary to \( \text{Ker}(p) = [R, S] \oplus RdSR \).

In fact \( \text{Ker}(p) = Im(1-p) \subset b(h(X_0)) + h(b(X_1)) \subset [R, S] \oplus (RdSR)_{\mathbb{K}} \) and on this subcomplex \( (1-p)([r, s]) = b((r, s)Y)_Y) = b((r[s, Y])_Y) = b((rds)_\mathbb{K}) = (r, s) \) and \( (1-p)((rds)_\mathbb{K}) = h((r[s, Y])_Y) = (rds)_\mathbb{K} \). Therefore the image \( Im(p) \) is isomorphic to the relative \( X \)-complex \( X(R, S) \) and the first assertion is proved.

For the second note just that \( \mathbb{C}[[F]] \cong \mathbb{C} \oplus \mathbb{C} \) is separable and that the canonical projection \( \pi_{rel} : \widehat{X(EA)} \to X(EA, F) \) is \( \mathbb{Z}/2\mathbb{Z} \)-equivariant.

\[ \square \]

It is the relative \( X \)-complex \( \widehat{X(EA, F)} \) that can be identified with a space of algebraic differential forms.

**Proposition 2.7.** ([CQ]) There exists a natural linear isomorphism of \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces

\[ \widehat{\Omega}A \cong \widehat{X(EA, F)}^- \]

which is given by the formulas

\[ a^0 da^1 \ldots da^{2n} \leftrightarrow F a^0[F, a^1] \ldots [F, a^{2n}] \]

\[ a^0 da^1 \ldots da^{2n+1} \leftrightarrow a^0 F[F, a^1] \ldots [F, a^{2n}] da^{2n+1} \]

**Proof:** Denote by \( \eta : A \to EA \) the canonical inclusion. Recall that there is a linear isomorphism (1.3)

\[ \Omega A[c] \cong EA \]

\[ a^0 da^1 \ldots da^n \leftrightarrow a^0[F, a^1] \ldots [F, a^n] \]

\[ \epsilon a_0 da_1 \ldots da_n \leftrightarrow Fa_0[F, a_1] \ldots [F, a_n] \]

With respect to this identification the odd parts of the spaces \( [F, \Omega A] \) respectively \( [F, \epsilon \Omega A] \) are spanned by the elements

\[ [F, \eta][F, \eta]^{2n} = [F, \eta]^{2n+1} \]

respectively

\[ [F, F\eta][F, \eta]^{2n+1} = \eta[F, \eta]^{2n+1} + F\eta[F, \eta]^{2n+1} = 2\eta[F, \eta]^{2n+1} - F[F, \eta]^{2n+2} \]

From this one deduces easily that

\[ X_0(EA, F)^- = \Omega A[c]^-[F, \Omega A[c]^-] \cong \Omega^{ev} A \]

\[ Fa^0[F, a^1] \ldots [F, a^{2n}] \leftrightarrow a^0 da^1 \ldots da^{2n} \]

Taking \( \epsilon A \)-adic completions yields the first assertion.
Consider the $E_A$-bimodule $\Omega^1(E_A, \mathcal{C}[F])$ of relative differentials. Its commutator quotient is given by

$$\Omega^1(E_A, \mathcal{C}[F])_1 \simeq E_A(dA)/[E_A(dA), A]$$

The odd part of the commutator subspace $[E_A(dA), A]$ is spanned by the elements

$$[\eta[F, \eta]^{2n-1} d\eta, \eta] = \eta[F, \eta]^{2n-1} d\eta \circ b$$

and

$$\left[\eta F[F, \eta]^{2n} d\eta, \eta\right]$$

$$= -\eta F[F, \eta]^{2n} d\eta \circ \sum_{i=1}^{2n+2} (-1)^i \partial_i - \eta F\eta[F, \eta]^{2n} d\eta$$

so that

$$\left\{ X_1(E_A, F) \right\}^\perp = \Omega^1(E_A, \mathcal{C}[F])_1 \simeq \Omega_{\text{odd}} A$$

$$d^0 F[F, a^1] \ldots [F, a^{2n}] da^{2n+1} \iff a^0 da^1 \ldots da^{2n+1}$$

Passing to $\epsilon A$-adic completion one obtains the second assertion. □

Again this identification allows to interpret the differentials of the complex $X(\widehat{E_A}, F)$ in terms of the standard operators on algebraic differential forms.

**Proposition 2.8.** [CQ] Under the linear isomorphism

$$\widehat{\Omega} A \simeq X(\widehat{E_A}, F)$$

of (2.7) the differentials of $X(\widehat{E_A}, F)\perp$ correspond to the following operators on differential forms:

$$\partial^w = B + 2b \circ N', \quad \partial^{\text{odd}} = b + \frac{1}{2} d$$

The harmonic decomposition yields as before a complete description of the homological properties of the considered complex.

**Theorem 2.9.** [CQ] Let $A$ be unital and let $\widehat{E_A}$ be the $\epsilon A$-adic completion of the algebra $E_A = QA \times \mathbb{Z}/2\mathbb{Z} \simeq A[F]/(F^2 - 1)$ obtained from $A$ by adjoining an involutionary element $F$ (1.3). Denote by $X(\widehat{E_A}, F)$ the relative $X$-complex of the pair $(\widehat{E_A}, \mathcal{C}[F])$ and let $X(\widehat{E_A}, F)\perp$ be its odd part with respect to the canonical $\mathbb{Z}/2\mathbb{Z}$-action (2.1.3). Let the natural identification (2.7) of this complex with a space of algebraic differential forms be understood.

i) The differentials of the complex $X(\widehat{E_A}, F)\perp$ commute with the Karoubi operator $\kappa$ and thus also with the Cuntz-Quillen projection $P_{CQ}$ (2.1.2). Thus there is a natural decomposition of complexes

$$X(\widehat{E_A}, F)\perp \simeq P_{CQ} X(\widehat{E_A}, F)\perp \oplus (1 - P_{CQ}) X(\widehat{E_A}, F)\perp$$
ii) The linear map
\[ \chi : P_{CQ} \hat{CC}(A) \longrightarrow P_{CQ} X(\hat{E}A, F)^{-} \]
of normalized subcomplexes, which is given on differential forms of degree
\(2n, 2n + 1, n \geq 0,\) by multiplication with \(\mu_n = \frac{1}{4^n n!}\) is an isomorphism of
chain complexes.

iii) The complementary subcomplex \((1 - P_{CQ}) X(\hat{E}A, F)^{-}\) is naturally contractible.

**Proof:** Assertion i) is obvious from (2.1.2) and (2.8). The identities \(\kappa b = b\kappa = b\)
and \(\kappa d = d\kappa = d\) of operators on the space of normalized differential forms shows
that the differentials on the normalized complex \(P_{CQ} X(\hat{E}A, F)^{-}\) coincide with
\(\partial' = B + 4nb\) on \(\Omega^n, n \geq 0,\) and with \(\partial' = b + \frac{4d}{4^n}\) on \(\Omega^{2n-1}, n \geq 1.\) It is clear from
this description that the linear isomorphism \(\chi\) introduced above is a chain map. On
the complementary subcomplex \((1 - P_{CQ}) X(\hat{E}A, F)^{-}\) the operator \(1 - \kappa = bd + db\) is
invertible. The identity \(\partial^{\text{odd}} \circ (b + 2d) = (b + 2d) \circ \partial^{\text{odd}} = bd + db\) shows therefore that
the differential \(\partial^{\text{odd}} : (1 - P_{CQ}) X_1(\hat{E}A, F)^{-} \rightarrow (1 - P_{CQ}) X_0(\hat{E}A, F)^{-}\) is invertible,
whence the result. \(\Box\)

The following corollaries sum up the facts that will be needed in the sequel. The
notations are those of (2.9).

**Corollary 2.10.** [CQ] There exists a natural chain homotopy equivalence
\[ \chi : \hat{CC}(A) \simto X(\hat{E}A, F)^{-} \]
such that
\[ X(\pi_{\hat{E}A}) \circ \chi = (i_{0*} - i_{1*}) \circ \pi_{\hat{CC}} \]
and such that \(\chi\) as well as a suitable chain homotopy inverse of it preserve the degree
of algebraic differential forms.

In fact \(\chi\) is given on normalized differential forms of fixed degree by a multiple of
the isomorphism (2.5).

**Proof:** Extend the operator \(\chi\) introduced in the previous theorem to a chain map
\( \chi : \hat{CC}(A) \longrightarrow X(\hat{E}A, F)^{-} \) by putting \(\chi = 0\) on \((1 - P_{CQ}) X(\hat{E}A, F)^{-}\). Then \(\chi\)
is a chain homotopy equivalence which satisfies all the conditions listed above. In
fact the complementary subcomplexes \((1 - P_{CQ}) \hat{CC}(A)\) and \((1 - P_{CQ}) X(\hat{E}A, F)^{-}\)
are contractible through chain homotopies which preserve Hodge filtrations. A look
at the explicit formulas in (2.9) shows finally that \(X(\pi_{\hat{E}A}) \circ \chi = (i_{0*} - i_{1*}) \circ \pi_{\hat{CC}}, \Box\)

**Corollary 2.11.** [CQ] The complex \(X(\hat{E}A)^{-}\) is naturally contractible.

**Proof:** Again this amounts in view of the previous theorem to the contractibility
of the complex \(CC(A).\) \(\Box\)
2.3 Identification of the Cuntz-Quillen maps

In this section we will express the Cuntz-Quillen maps in terms of familiar operators on cyclic complexes. This yields on the one hand a better understanding of these chain maps and provides us on the other hand with much more flexibility in dealing with them.

**Theorem 2.12.** The Cuntz-Quillen map

\[ \Psi: \widehat{CC}(A) \rightarrow \text{Cone}(X(\bar{\theta}) - X(\bar{\theta}^\gamma)) \]

of (2.2) represents the chain homotopy class of morphisms defined (2.1.1) by the chain map

\[ \pi_{\widehat{CC}}: \widehat{CC}(A) \rightarrow X(A) \]

and the commutative diagram

\[
\begin{array}{ccc}
\widehat{CC}(A) & \xrightarrow{\bar{\delta} - \bar{\delta}} & \widehat{CC}(QA^-) \\
\downarrow & & \downarrow \\
X(A) & \xrightarrow{\bar{\delta} - \bar{\delta}} & X(QA^-)
\end{array}
\]

**Proof:**

Let \( \pi_{QA}: QA \rightarrow A \) be the natural \( \mathbb{Z}/2\mathbb{Z} \)-equivariant epimorphism. By Goodwillie’s theorem [Go] the corresponding equivariant chain map \( \widehat{CC}(QA) \rightarrow \widehat{CC}(A) \) is a chain homotopy equivalence. As \( \widehat{CC}(A)^- = 0 \), it follows that \( \widehat{CC}(QA)^- \) is contractible.

The chain map \( \pi_{\widehat{CC}} \) and the diagram above define therefore by (2.1.1) a chain homotopy class of chain maps from the cyclic bicomplex to the mapping cone in question.

Let \( R \rightarrow A \) be a quasifree presentation of \( A \) (2.1.3). By the naturality of the Cuntz-Quillen map there exists a commutative diagram

\[
\begin{array}{ccc}
\widehat{CC}(R) & \xrightarrow{\Psi} & \text{Cone}(X(\bar{\theta}_R) - X(\bar{\theta}^\gamma_R)) \\
\downarrow & & \downarrow \\
\widehat{CC}(A) & \xrightarrow{\Psi} & \text{Cone}(X(\bar{\theta}_A) - X(\bar{\theta}^\gamma_A))
\end{array}
\]

The theorem of Cuntz and Quillen (2.2) implies that the horizontal arrows are chain homotopy equivalences. As the left vertical arrow is a chain homotopy equivalence by construction, it follows that all arrows in the diagram are so. It suffices therefore to prove the assertion in the case that the underlying algebra is quasifree. For a quasifree algebra \( R \) the free product \( QR = R \ast R \) and its adic completion \( \widehat{QR} \) are quasifree again [CQ1]. Therefore the natural projection \( \pi_{\widehat{CC}}: \widehat{CC}(QR) \rightarrow X(QR) \) is a chain homotopy equivalence. As this map is equivariant we may pass to the odd subspaces and deduce that \( X(QR)^- \) is contractible. From the natural extension
with linear section \( 0 \to X(\hat{Q}R)[1]^- \to \text{Cone}(X(\hat{\theta}) - X(\hat{\theta}')) \xrightarrow{\pi_{\text{Cone}}} X(R) \to 0 \) we get therefore that \( \pi_{\text{Cone}} : \text{Cone}(X(\hat{\theta}) - X(\hat{\theta}')) \to X(R) \) is a chain homotopy equivalence for quasifree \( R \).

Let now \( \psi : \hat{CC}(R) \to \text{Cone}(X(\hat{\theta}) - X(\hat{\theta}')) \) be a chain map associated (2.1.3) to the pair of data given above. Then by construction \( \pi_{\text{Cone}} \circ \psi = \pi_{\hat{CC}} \). But \( \pi_{\text{Cone}} \circ \Psi = \pi_{\hat{CC}} \) by (2.2). Thus \( \Psi \) and \( \psi \) are chain homotopic because \( \pi_{\text{Cone}} \) is a chain homotopy equivalence. This completes the proof of the theorem. \( \square \)

**Theorem 2.13.** Let \( i_0, i_1 : A \to A \oplus A \) be the canonical inclusions and denote by \( \pi_{\text{CE}} : \hat{EA} \to E\!A/eA = A \oplus A \) the canonical epimorphism (1.3). The Cuntz-Quillen map

\[
\chi : \hat{CC}(A) \to X(\hat{E}A, F)^-
\]

of (2.6) represents the chain homotopy class of morphisms defined by the diagram of chain maps

\[
\begin{array}{ccc}
\hat{CC}(A) & \xrightarrow{i_0 - i_1} & \hat{CC}(A \oplus A)^- \\
\downarrow{\pi_{\text{CE}}} & & \downarrow{\pi_{\text{CE}}} \\
\hat{CC}(\hat{EA}) & \to & X(\hat{EA})^- \\
\end{array}
\]

**Proof:** The projection \( (\pi_{\text{CE}})_* \) is an equivariant chain homotopy equivalence by Goodwillie’s theorem. Thus the diagram above determines a well defined morphism in the chain homotopy category of chain complexes.

Let \( R \to A \) be a quasifree presentation of \( A \). Theorem (2.6) of Cuntz-Quillen implies that \( X(\hat{ER}, F)^- \to X(\hat{EA}, F)^- \) is a chain homotopy equivalence. The naturality of \( \chi \) and of all the morphisms in the diagram above allows thus to reduce the verification of the theorem to the case of quasifree algebras.

If \( R \) is quasifree, then so are the algebras \( R \oplus R, ER = R[F]/(F^2 - 1) \), and the completion \( \hat{E}A \) [CQ1]. Therefore \( \pi_{\text{CE}} : \hat{ER} \to R \oplus R \) as well as the identity are quasifree presentations of \( R \oplus R \). It follows [CQ2] that the projection \( X(\pi_{\text{CE}}) : X(\hat{EA}) \to X(R \oplus R) \) as well as its restrictions to the even respectively odd subcomplexes are chain homotopy equivalences. Denoting by \( \pi_{\text{CE}}^l : \hat{ER} \to R \) the composition of \( \pi_{\text{CE}} \) and the projection of \( R \oplus R \) onto the first factor, we conclude that \( X(\pi_{\text{CE}}^l) : X(\hat{ER})^- \to X(R) \) is a chain homotopy equivalence. Moreover \( X(\pi_{\text{CE}}^l) \) factors through the relative \( X \)-complex \( X(\hat{ER}, F) \) so that by lemma (2.4) we obtain finally a chain homotopy equivalence \( X(\pi_{\text{CE}}^l) : X(\hat{ER}, F)^- \to X(R) \).

It suffices to verify that \( X(\pi_{\text{CE}}^l) \circ \chi \) represents the homotopy class obtained by composing the morphism described by the diagram with \( X(\pi_{\text{CE}}^l) \). Now a little diagram chase shows that the latter morphism coincides with the chain homotopy class of the canonical projection \( \pi_{\hat{CC}} : \hat{CC}(R) \to X(R) \). But by theorem (2.6) \( X(\pi_{\text{CE}}^l) \circ \chi = \pi_{\hat{CC}} \), which completes thus the proof of the theorem. \( \square \)
3 Results

Let

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

be an extension of algebras with linear section. The excision theorem of Cuntz and Quillen [CQ3] asserts that the natural inclusion

$$\iota : \widehat{CC}(I) \rightarrow \widehat{CC}(A, B)$$

of the cyclic bicomplex of the ideal $I$ into the relative cyclic bicomplex of the pair $(A, B)$ is a chain homotopy equivalence. Consequently there exists a boundary map

$$\delta : \widehat{CC}(B) \rightarrow \widehat{CC}(I)[1]$$

which is a chain map of degree one that is well defined up to chain homotopy and gives rise to a natural six-term exact sequence relating the periodic cyclic (co)homology groups associated to the given extension.

Our goal will be to understand the dimension shift, i.e. the behavior of the Hodge filtration, under the excision isomorphism

$$\iota_*^{-1} : HP_*(A, B) \xrightarrow{\cong} HP_*(I)$$

and the boundary map

$$\delta_* : HP_*(B) \rightarrow HP_{*-1}(I)$$

For arbitrary extensions this problem has been solved in [Pu], Section 4. We consider here the case of splitting and of invertible extensions for which we obtain considerably more precise results.

3.1 Free ideal extensions

We recall a few notions and facts from [Pu]. An extension

$$0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$$

of algebras with linear section is called a free ideal extension if $J$ is free as $A$-left (or right)-module. In this case there is a canonical linear injection

$$\sigma : Gr_J(A) \hookrightarrow A$$

which preserves $J$-adic filtrations and extends to a continuous linear isomorphism

$$\widetilde{\sigma} : \widehat{Gr_J(A)} \xrightarrow{\cong} \widehat{A}$$
A free ideal extension is said to be multiplicatively split up to order \(d\) if

\[
\omega_\sigma \left( Gr_J(A)^i_+ \otimes Gr_J(A)^j_+ \right) \subset J^{i+j+d}
\]

for all \(i, j \geq 0\), where \(\omega_\sigma (a, a') = \sigma(aa') - \sigma(a)\sigma(a')\) denotes the curvature of \(\sigma\) in the sense of Quillen [Qu] and \(Gr_J(A)^i_+ \subset Gr_J(A)\) is the ideal of elements of strictly positive degree [Pu], (3.1). This condition means essentially that \(\sigma\) is multiplicative modulo \(J^d\).

Let \(0 \to I \to A \to B \to 0\) be an extension of algebras. The Hodge- and the \(\tilde{I}\)-adic filtration

generate a filtration of \(\widehat{CC}(\tilde{A})\) which we call the canonical one. It defines the topology on \(\widehat{CC}(A)\). By definition

\[
\operatorname{Fil}_n^{\operatorname{can}}(\widehat{CC}(\tilde{A})) = \sum_{i+j = n} \operatorname{Fil}_i^{\operatorname{Hodge}} \operatorname{Fil}_j^{\tilde{I}} \widehat{CC}(\tilde{A})
\]

We can now prove the following refinement of Goodwillie’s theorem [Go].

**Proposition 3.1.** Let \(0 \to J \to A \to B \to 0\) be a free ideal extension which splits multiplicatively up to order two. Then there exists a contracting homotopy for the complex \(\widehat{CC}(A, B)\) which decreases canonical filtrations by at most one.

**Proof:** Let \(f_t : \widehat{Gr}_J(A) \to \widehat{Gr}_J(A)\) be the formal one parameter family of endomorphisms given on homogeneous elements by \(x \to t^\deg(x) x\). Then \(f_1\) equals the identity whereas \(f_0\) is the canonical projection onto \(B = \widehat{Gr}_J(A)_0\). The Cartan homotopy formula for periodic cyclic homology [Go] provides a chain homotopy \(h_N\) between \((f_0)_*\) and \((f_1)_* = \operatorname{Id}\). It preserves the \(J\)-adic filtration and decreases the Hodge filtration by at most one. Define now an operator \(h' : \widehat{CC}_s(\tilde{A}, B) \to \widehat{CC}_{s+1}(\tilde{A}, B)\) by

\[
h' := \tilde{\sigma} \circ h_N \circ (\tilde{\sigma})^{-1}
\]

where \(\tilde{\sigma}, (\tilde{\sigma})^{-1}\) are the linear maps of cyclic complexes induced by the linear isomorphisms \(\tilde{\sigma}, (\tilde{\sigma})^{-1}\). Put \(\varphi := \operatorname{Id} - (h' \circ \partial + \partial \circ h')\). The chain map \(\varphi\) is chain homotopic to the identity and decreases Hodge filtrations by two but increases at the same time \(J\)-adic filtrations by at least two. Thus \(\varphi\) preserves the canonical filtration and increases the \(J\)-adic filtration strictly. It follows that

\[
h := h' \circ \sum_{n=0}^{\infty} \varphi^n
\]

is well defined and provides a contracting homotopy operator for \(\widehat{CC}_s(\tilde{A}, B)\) which preserves \(\tilde{J}\)-adic filtrations and decreases the canonical filtration by at most one. \(\square\)

Free ideal extensions play an important role in [Pu] because one can construct for them an explicit chain homotopy inverse for the canonical inclusion

\[
i : \widehat{CC}(J) \hookrightarrow \widehat{CC}(A, B)
\]

of the cyclic bicomplex of the ideal \(J\) into the relative cyclic bicomplex of the pair \((A, B)\). One obtains for example

**Proposition 3.2.** [Pu] Let

\[
0 \to J \to A \to B \to 0
\]

22
be a free ideal extension which splits multiplicatively up to order \( d \). Then there exists a chain map
\[
\iota^{-1} : \widehat{CC}(A, B) \to \widehat{CC}(J)
\]
which is a chain homotopy inverse of the natural inclusion
\[
\iota : \widehat{CC}(J) \to \widehat{CC}(A, B)
\]
and satisfies
\[
\iota^{-1}(\text{Fil}^m_{\text{Hodge}} \text{Fil}^m_j \widehat{CC}(A, B)) \subset \text{Fil}^{m-2k}_{\text{Hodge}} \widehat{CC}(J)
\]
for \( k \geq \frac{m}{d+2} \).

### 3.2 Explicit boundary maps

The reason for studying free ideal extensions in our context is that the universal splitting respectively invertible extensions are of this type. This allows to exhibit chain maps which realize the excision isomorphisms for them in a controlled way [Pu]. These explicit maps lead to an upper bound for the corresponding dimension shifts. Note that this bound is much smaller than the one obtained for arbitrary free ideal extensions in [Pu]. We will show later on by a completely different argument that the bounds thus obtained are actually sharp.

**Lemma 3.3.** The universal splitting extensions (1.2)
\[
0 \quad \rightarrow \quad qA \quad \rightarrow \quad QA \quad \rightarrow \quad A \quad \rightarrow \quad 0
\]
are free ideal extensions for every algebra \( A \). The Zekri extensions (1.3)
\[
0 \quad \rightarrow \quad eA \quad \rightarrow \quad EA \quad \rightarrow \quad A \oplus A \quad \rightarrow \quad 0
\]
from which the universal invertible extensions are obtained as pullback, are free ideal extensions for every unital algebra \( A \). Both types of extensions split multiplicatively up to order two.

**Proof:** An ideal \( I \subseteq A \) is free as \( A \)-left-module if there exists a complex vector space \( V \) such that \( I \cong A \otimes \mathbb{Q} V \) as \( A \)-left-modules. With the structure of the universal algebras in mind it is clear that the vector space \( V_{qA} = \{qa, a \in A\} \) may serve as a basis of \( qA \) over \( QA \) and that \( V_{eA} = \{[F,a], a \in A\} \) may serve as a basis of \( eA \) over \( EA \). The extensions under study are thus free ideal extensions.

The canonical linear maps
\[
\sigma_{qA} : \text{Gr}_{qA}(QA) \cong \Omega A \rightarrow (\Omega A, \ast) \cong QA
\]
respectively
\[
\sigma_{eA} : \text{Gr}_{eA}(EA) \cong \Omega A \times \mathbb{Z}/2\mathbb{Z} \rightarrow (\Omega A \times \mathbb{Z}/2\mathbb{Z}, \ast) \cong EA
\]
coincide with the identity on differential forms. The equality
\[
\omega \ast \omega' - \omega \cdot \omega' = \pm \, d\omega \cdot d\omega'
\]
shows therefore that both types of extensions split multiplicatively up to order two. \( \square \)
Proposition 3.4. Let 
\[ 0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0 \]
be a splitting extension of algebras, i.e. an extension which possesses a multiplicative linear section. Let 
\[ \iota : \widehat{CC}(I) \longrightarrow \widehat{CC}(A, B) \]
be the canonical inclusion and let 
\[ \iota_* : HP_*(I) \longrightarrow HP_*(A, B) \]
be the induced map on homology, which is an isomorphism by excision \([CQ3]\). Then 
\[ \iota_*^{-1} : Fil^{ln}_{\text{Hodge}} HP_0(A, B) \longrightarrow Fil^{2n}_{\text{Hodge}} HP_0(I) \]
and 
\[ \iota_*^{-1} : Fil^{ln+1}_{\text{Hodge}} HP_1(A, B) \longrightarrow Fil^{2n+1}_{\text{Hodge}} HP_1(I) \]
for all \( n \in \mathbb{N} \).

Proof: Let \( 0 \rightarrow I \rightarrow A \xrightarrow{f} B \rightarrow 0 \) be an extension and let \( g : B \rightarrow A \) be an algebra homomorphism which splits \( f \). There exists a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & I \\
\uparrow & & \uparrow u \\
0 & \longrightarrow & qA \\
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow & & \uparrow \\
QA & \xrightarrow{f} & A \\
\end{array}
\longrightarrow 0
\]

where \( u = Id \ast (g \circ f) \). The induced map \( u_* : HP_*(QA, A) \rightarrow HP_*(A, B) \) is an epimorphism and \( \theta_* - \theta_*^{g} : HP_*(A) \rightarrow HP_*(QA) \) defines a section of \( u_* \) which preserves Hodge filtrations. By the naturality of the boundary map in periodic cyclic homology it suffices therefore to study the universal splitting extensions (1.2)

\[ 0 \longrightarrow qA \longrightarrow QA \longrightarrow A \longrightarrow 0 \]

Proposition (3.2), which we may apply due to (3.3), immediately yields the desired result. \( \square \)

Proposition 3.5. Let 
\[ 0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0 \]
be an invertible extension of algebras with linear section and let 
\[ \delta_* : HP_*(B) \longrightarrow HP_{*-1}(I) \]
be the associated boundary map on periodic cyclic homology \([CQ3]\). Then
\[ \delta_* : Fil^{ln-2}_{\text{Hodge}} HP_0(A, B) \longrightarrow Fil^{2n-1}_{\text{Hodge}} HP_1(I) \]
and
\[ \delta_* : Fil^{ln-1}_{\text{Hodge}} HP_1(A, B) \longrightarrow Fil^{2n}_{\text{Hodge}} HP_0(I) \]
for all \( n \in \mathbb{N} \).

24
Proof: Every invertible extension fits into a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \parallel & & \parallel & & \parallel \\
0 & \longrightarrow & \epsilon B & \longrightarrow & E_1 B & \longrightarrow & B & \longrightarrow & 0 \\
\parallel & & \downarrow & & \downarrow & & \downarrow_{i_0} & & \downarrow \\
0 & \longrightarrow & \epsilon B & \longrightarrow & EB & \longrightarrow & B \oplus B & \longrightarrow & 0
\end{array}
\]

By the naturality of the boundary map in periodic cyclic homology it suffices therefore to study the extensions (1.3)

\[
0 \longrightarrow \epsilon A \longrightarrow EA \longrightarrow A \oplus A \longrightarrow 0
\]

Lemma (3.3) guarantees again that we may apply proposition (3.2) which gives the desired result. \(\square\)

3.3 Rigidity of the universal extensions

The possibility of giving exact bounds on the dimension shift relies on the existence of rigid classes of extensions. The dimension shift of the excision isomorphisms and the boundary maps only depends within such a class on the dimension of the involved homology classes and can thus be given by an explicit numerical formula. For the known rigid classes of extensions one actually observes a maximal dimension shift. In [Pu] we exhibited a rigid class which allowed to determine the maximal dimension shift for general algebra extensions. It consists of extensions \(0 \to J \to R \to S \to 0\) with \(R\) quasifree and with injective boundary map \(\delta_* : HP_*(S) \to HP_{*-1}(J)\). Here we present two new rigid classes: the universal splitting extensions of Cuntz (1.2), and the universal invertible extensions of Zekri (1.3). The following two theorems are the main results of this paper.

3.3.1 Rigidity of the universal splitting extensions

The periodic cyclic (co)homology of the universal algebras \(QA\) and \(qA\) (1.2) is well known [Cu]. There is a natural \(\mathbb{Z}/2\mathbb{Z}\)-equivariant chain homotopy equivalence

\[
\widehat{CC}(QA) \sim \widehat{CC}(A) \oplus \widehat{CC}(A)
\]

where \(\mathbb{Z}/2\mathbb{Z}\) acts on the right hand side by flipping the factors. Thus one obtains from the excision theorem [CQ3] the following natural diagram of chain homotopy equivalences

\[
\begin{array}{cccccc}
\widehat{CC}(qA) & \longrightarrow & \widehat{CC}(QA, A) & \theta_+ \sim \theta_+^2 & \widehat{CC}(A) \\
\uparrow & & \uparrow & & \parallel & & \parallel \\
\widehat{CC}(qA)^- & \longrightarrow & \widehat{CC}(QA)^- & \theta_+ \sim \theta_+^2 & \widehat{CC}(A)
\end{array}
\]

The isomorphism \(\theta_* - \theta_*^2 : HP_*(A) \sim HPP_*(QA)^-\) preserves Hodge filtrations (and thus dimensions) due to the fact that a chain homotopy inverse of it is given by the chain map induced by the algebra homomorphism \(id \cdot 0 : QA \to A\).
Theorem 3.6. Let $A$ be an algebra and let

$$0 \rightarrow qA \rightarrow QA \rightarrow A \rightarrow 0$$

be the associated universal splitting extension.

Then every class $\alpha \in HP(A)$ satisfies

$$\text{dim}(\iota_*^{-1}(\theta - \theta')(\alpha)) = \varphi(\text{dim}(\alpha))$$

where $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is given by

$$\varphi(4n) = \varphi(4n + 2) = 2n \quad \text{and} \quad \varphi(4n + 1) = \varphi(4n + 3) = 2n + 1$$

for $n \geq 0$ and by $\varphi(\infty) = \infty$.

Proof:

Let $\alpha \in HP_0(A)$ be a homology class of dimension $4n$ or $4n + 2$ for some $n \geq 0$ and let $y \in \text{Fil}^{2n}_{\text{Hodge}}\overline{CC}(A)$ be a cycle which represents it. By proposition (3.4) one has $\text{dim}(\iota_*^{-1}(\theta - \theta')(\alpha)) \geq 2n$. We want to show that this estimate is sharp.

Otherwise there exists a cycle $z \in \text{Fil}^{2n+2}_{\text{Hodge}}\overline{CC}(qA) \subset \text{Fil}^2_{\text{Hodge}}\overline{CC}(qA)$ which represents the class $\iota_*^{-1}(\theta - \theta')(\alpha)$. We may assume that $z \in \text{Fil}^{2n+2}_{\text{Hodge}}\overline{CC}(qA)^{-}$ and identify $z$ with its image in $\overline{CC}(qA)^{-} \subset \overline{CC}(qA, A)$.

The idea is to construct out of $z$ a cycle $z' \in \text{Cone}(X(\overline{\theta}) - X(\overline{\theta'}))$ of high weight with respect to the Hodge filtration, which represents the image of $\alpha$ under the Cuntz-Quillen map $\Psi : HP(A) \rightarrow H(\text{Cone}(X(\overline{\theta}) - X(\overline{\theta'})))$. The results of Cuntz and Quillen lead then to a contradiction with our assumption.

We use (2.8) to exhibit a chain map which is chain homotopic to $\Psi$ and behaves in a controlled way with respect to the Hodge filtration. Fix a contracting homotopy $h$ of $\overline{CC}(qA, A)$ as constructed in (3.1). It is compatible with the canonical $\mathbb{Z}/2\mathbb{Z}$-action and provides thus also a contracting homotopy of $\overline{CC}(qA)^{-}$. The pair

$$(\pi_{\overline{CC}}, \pi_{\overline{CC}} \circ h \circ (\overline{\theta} - \overline{\theta}'))$$

defines (2.1.1) then a chain map

$$\psi : \overline{CC}(A) \rightarrow \text{Cone}(X(\overline{\theta}) - X(\overline{\theta'})) : X(A) \rightarrow X(qA)^{-}$$

which represents the chain homotopy class of the Cuntz-Quillen homomorphism $\Psi$ (2.2). Denote by $j : QA \rightarrow \overline{QA}$ the canonical inclusion.

We claim that

$$z' = (0, \pi_{\overline{CC}} \circ h \circ j_*(z)) \in \text{Cone}(X(\overline{\theta}) - X(\overline{\theta'}))_1$$
is a cycle in $Fil^{4n+3}_{\text{Hodge}} X_1(\hat{Q}A)^- \subset Fil^{4n+3}_{\text{Hodge}} \text{Cone}(X(\hat{\theta}) - X(\hat{\theta}^r))$ which is homologous to $\psi(y)$.

First we determine the homology class of $z'$. By assumption there exists $v \in \hat{CC}_1(QA)^-$ such that $\partial v = (\theta_j - \theta_j^r)(y) - z$. It follows that $(\pi_{\hat{CC}})^2(y, v)$ is a cycle in $\text{Cone}(X(\theta) - X(\theta^r) : X(A) \to X(QA)^-)$.

According to (2.3) the latter complex is contractible. Let $h'$ be a contracting homotopy of it. The chain $w = ((Id, j)_* \circ h') \circ (\pi_{\hat{CC}})^2 - \pi_{\hat{CC}} \circ h \circ j \circ \pi_2)(y, v) \in \text{Cone}(X(\theta) - X(\theta^r))_1$ satisfies $\partial(w) = \psi(y) - z'$ which verifies our first claim.

We calculate now the weight of $z'$ with respect to the Hodge filtration. For every cycle $z \in Fil^{2n+2}_{\text{Hodge}} \hat{CC}_0(QA)^-$ one has (3.0)

$$j_*(z) \in Fil^{2n+2}_{\text{Hodge}} Fil^{2n+2}_{qA} \hat{CC}_0(QA)^- \subset Fil^{4n+4}_{\text{can}} \hat{CC}_0(QA)^-$$

so that by construction (3.1)

$$(h \circ j_*)(z) \in Fil^{4n+3}_{\text{can}} \hat{CC}_1(QA)^-$$

Note that $Fil^{4n+2}_{qA} \hat{CC}(QA)^- = Fil^{4n+3}_{qA} \hat{CC}(QA)^-$ because the $qA$-adic valuation of elements which are odd with respect to the $\mathbb{Z}/2\mathbb{Z}$-action has itself to be odd. We find thus

$$Fil^{4n+3}_{\text{can}} \hat{CC}_1(QA)^- \subset Fil^{4n+3}_{qA} \hat{CC}_1(QA)^- + Fil^{4}_{\text{Hodge}} \hat{CC}_1(QA)^-$$

which finally implies that

$$(\pi_{\hat{CC}} \circ h \circ j_*)(z) \in Fil^{4n+3}_{qA} X_1(\hat{Q}A)^-$$

The latter space is spanned by elements of the form $pq^kdp$ and $pq^{k-1}dq$ for $k \geq 4n + 3$ odd (1.2). Recall that in $\Omega^1(\hat{Q}A)^-$ the identity

$$2(\theta q^m d\theta)^-(a^0 \otimes \ldots \otimes a^{2n+1}) = -(\theta q^{2n-1} d\theta)^- \circ b(a^0 \otimes \ldots \otimes a^{2n+1})$$

holds, where $b$ is the Hochschild boundary operator (2.1.2). The elements $pq^kdp = (\theta q^k d\theta)^- - (q^{k+1} d\theta)^-$ and $pq^{k-1}dq = (\theta q^{k-1} d\theta)^- - (q^k d\theta)^-$ for $k \geq 4n + 3$ odd correspond under the Cuntz-Quillen isomorphism (2.1) therefore to differential forms in $b^*(\Omega^{4n+3} \oplus \Omega^{4n+3})$. Thus $z' = (0, (\pi_{\hat{CC}} \circ h \circ j_*)(z)) \in Fil^{4n+3}_{\text{Hodge}} X_1(\hat{Q}A)^-.$

Altogether we have shown that the homology class $\Psi(\alpha)$ is of weight at least $4n + 3$ with respect to the Hodge filtration. As by the theorem of Cuntz and Quillen (2.2) there exists a chain map which is homotopy inverse to $\Psi$ and preserves the Hodge filtrations, we deduce that $\alpha$ itself is of weight at least $4n + 3$ which contradicts our assumptions.

The proof for classes of odd degree is similar. The case of classes of infinite dimension is a direct consequence of (3.4).
The reader might wonder why we used the tedious calculations with explicit elements instead of the more conceptual use of diagrams of chain maps. In fact these diagrams necessarily involve iterated mapping cones. A proof using diagrams appears to be therefore not only much longer but also more difficult to follow than the one we decided to give here.

\[ \square \]

### 3.3.2 Rigidity of the universal invertible extensions

The periodic cyclic (co)homology of the universal algebras \( E_A \) and \( \varepsilon A \) \((1.3)\) is well known [Ze]. There is a natural \( \mathbb{Z}/2\mathbb{Z} \)-equivariant chain homotopy equivalence

\[
\widehat{CC}(E_A) \xrightarrow{\sim} \widehat{CC}(A)
\]

where \( \mathbb{Z}/2\mathbb{Z} \) acts trivially on the right hand side. Thus one obtains from the excision theorem [CQ3] the following natural diagram of chain homotopy equivalences

\[
\begin{array}{ccc}
\widehat{CC}(A \oplus A)^{-} & \xrightarrow{\delta} & \widehat{CC}(\varepsilon A)[1]^{-} \\
& & \longrightarrow \widehat{CC}(\varepsilon A)[1]
\end{array}
\]

**Theorem 3.7.** Let \( A \) be a unital algebra and let

\[
\alpha_A^0 : \ 0 \longrightarrow \varepsilon A \longrightarrow E' A \longrightarrow A \longrightarrow 0
\]

be the associated universal invertible extension.

Then every class \( \beta \in HP(A) \) satisfies

\[
dim(\delta_*(\beta)) = \psi(dim(\beta))
\]

where \( \psi : \mathbb{N} \rightarrow \mathbb{N} \) is given by

\[
\psi(4n) = \psi(4n - 2) = 2n - 1 \quad \text{and} \quad \psi(4n + 1) = \psi(4n - 1) = 2n
\]

for \( n \geq 1 \), and \( \psi(0) = 1 \), \( \psi(1) = 0 \), \( \psi(\infty) = \infty \).

**Proof:**

Let \( \alpha \in HP_0(A) \) be a class of dimension \( 4n \) or \( 4n - 2 \) for some \( n \geq 1 \) and let \( u \in \text{Fil}_{Hodge}^{l_{-2}} \widehat{CC}(A) \) be a cycle which represents it.

By proposition (3.5) \( \dim(\delta_*(\alpha)) \geq 2n - 1 \). We want to show that this estimate is sharp. Otherwise there exists a cycle \( z \in \text{Fil}_{Hodge}^{l_{-2}} \widehat{CC}_1(\varepsilon A)^{-} \subseteq \text{Fil}_{Hodge}^{l_{-2}} \widehat{CC}_1(\varepsilon A)^{-} \) which represents \( \delta_*(\alpha) \in HP_1(\varepsilon A) \). Identify \( z \) with its image in \( \widehat{CC}(E, A \oplus A)^{-} \).

The strategy will be the same as in the proof of the previous theorem. We use \( z \) to construct a cycle \( z' \in X(E, F) \) of high weight with respect to the Hodge filtration,
which represents the image of $\alpha$ under the Cuntz-Quillen map $\chi$. The results of Cuntz and Quillen lead again to a contradiction with our assumption.

We use (2.9) to exhibit a chain map which is chain homotopic to $\chi$ and behaves in a controlled way with respect to the Hodge filtration. The quotient map $(\pi_{EA})_{\ast} : \widehat{CC}(EA)^{-} \to \widehat{CC}(A \oplus A)^{-}$ is a chain homotopy equivalence by Goodwillie’s theorem. Fix a contracting chain homotopy $h$ of $\widehat{CC}(EA, A \oplus A)^{-}$ which satisfies the assumptions of proposition (3.1) (this is possible by (3.3)). Let $s$ be a $\mathbb{Z}/2\mathbb{Z}$-equivariant linear section of $\pi_{EA} : EA \to A \oplus A$ and denote by $j : EA \to \widehat{EA}$ the canonical inclusion. A chain homotopy inverse of $\pi_{EAs}$ is given by

$$\nu = j_{s} \circ s_{s} - h \circ j_{s} \circ (\partial \circ s_{s} - s_{s} \circ \partial)$$

According to (2.9) the chain map

$$\chi' = \pi_{rel} \circ \pi_{\widehat{CC}} \circ \nu \circ (i_{0s} - i_{1s})$$

is then chain homotopic to the Cuntz-Quillen map $\chi$.

We claim that $z' = - (\pi_{rel} \circ \pi_{\widehat{CC}} \circ h \circ j_{s})(z) \in X_{0}(\widehat{EA}, F)$ is a cycle which is at the same time homologous to $\chi'(u)$ and to a cycle in $\text{Fil}_{Hodge}^{m+2}X_{0}(\widehat{EA}, F)^{-}$.

By assumption there exists $v \in \widehat{CC}_{0}(EA)^{-}$ such that

$$(\partial \circ s_{s} - s_{s} \circ \partial)(i_{0s} - i_{1s})(u) - z = \partial(v)$$

It follows that

$$\chi'(u) - z' = \pi_{rel} \circ \pi_{\widehat{CC}} \circ j_{s} \circ s_{s} \circ (i_{0s} - i_{1s})(u) - \pi_{rel} \circ \pi_{\widehat{CC}} \circ h \circ j_{s} \circ \partial(v) =$$

$$= \pi_{rel} \circ \pi_{\widehat{CC}} \circ j_{s}(y) + \partial(\pi_{rel} \circ \pi_{\widehat{CC}} \circ h \circ j_{s}(v))$$

with $y = s_{s} \circ (i_{0s} - i_{1s})(u) - v \in \widehat{CC}_{0}(EA)^{-}$.

In order to show that $\pi_{rel} \circ \pi_{\widehat{CC}} \circ j_{s}(y)$ is a boundary we consider the commutative diagram

$$\begin{array}{ccc}
\widehat{CC}(EA)^{-} & \xrightarrow{\pi_{\widehat{CC}}} & X(EA)^{-} \\
\downarrow j_{s} & & \downarrow j_{s} \\
\widehat{CC}(EA)^{-} & \xrightarrow{\pi_{\widehat{CC}}} & X(EA)^{-}
\end{array}$$

The element $\pi_{\widehat{CC}}(y)$ in $X_{0}(EA)^{-}$ is a cycle because

$$\partial(y) = z \in \text{Fil}_{Hodge}^{2}\widehat{CC}_{1}(EA)^{-}$$

By corollary (2.7) of Cuntz and Quillen the chain complex $X(EA)^{-}$ is contractible. Thus $\pi_{\widehat{CC}} \circ j_{s}(y) = j_{s} \circ \pi_{\widehat{CC}}(y)$ is a boundary and $z'$ is homologous to $\chi'(u)$ as claimed.

We calculate now the weight of this cycle. As $z \in \text{Fil}_{Hodge}^{2m+1}\widehat{CC}(\epsilon A)^{-}$ we find

$$j_{s}(z) \in \text{Fil}_{Hodge}^{2m+1} \text{Fil}_{\text{can}}^{2m+1}\widehat{CC}_{1}(EA)^{-} \subset \text{Fil}_{\text{can}}^{2m+1}\widehat{CC}_{1}(EA)^{-}$$

29
and therefore

\((h \circ j_\ast)(z) \in \Fil_{can}^{n+1} \overline{CC}_0(\widehat{E}A)^- \subset \Fil_{cA}^{n+1} \overline{CC}_0(\widehat{E}A)^- + \Fil_{Hodge}^1 \overline{CC}_0(\widehat{E}A)^-\)

by (3.1).

It follows that the homology class of \(z' = - (\pi_{rel} \circ \pi_{CC} \circ h \circ j_\ast)(z)\) can be represented by a cycle in \(\Fil_{cA}^{n+1} \chi_0(\widehat{E}A, F)^-\). The latter space is spanned by elements of the form \(a^0[F, a^1] \ldots [F, a^k]\) and \(Fa^0[F, a^1] \ldots [F, a^{k+1}]\) for \(k \geq 4n + 1\) odd (1.3). Now note that in the relative \(X\)-complex \(X(\widehat{E}A, F)^-\)

\[
2a^0[F, a^1] \ldots [F, a^k] = F[F, a^0] \ldots [F, a^k]
\]

for \(k\) odd. A look at (2.5) shows therefore that \(\chi(\alpha) \in H_0(\widehat{X(EA)}^-)\) can be represented by a cycle of weight at least \(4n + 2\) with respect to the Hodge filtration. By theorem (2.6) of Cuntz and Quillen one may find a chain homotopy inverse of \(\chi\) which preserves the Hodge filtration. Therefore the class \(\alpha\) itself can be represented by a cycle in \(\Fil_{Hodge}^{n+2} \overline{CC}(A)\) which contradicts our assumption.

A similar argument works in dimension 0 and for classes of odd degree. The case of classes of infinite dimension follows directly from (3.5).

\[\square\]

There are no similar theorems in periodic cyclic cohomology. As the shift functions \(\varphi\) and \(\psi\) of the previous theorems are not injective one gets only estimates but no exact formulas for the dimension shift under the excision map in periodic cyclic cohomology.

### 3.4 Universal bounds for dimension shifts

The previous rigidity theorems allow to understand the dimension shifts under excision isomorphisms and boundary maps for general splitting respectively invertible extensions (compare also with [Pu], (4.7)).

**Theorem 3.8.** Let \(\iota^{-1}_\ast\) be the excision isomorphism in periodic cyclic homology. There exists a splitting extension of algebras

\[
0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0
\]

and a class \(\alpha \in HP_n(A, B)\) such that

\[
\dim(\alpha) = n, \quad \dim(\iota^{-1}_\ast(\alpha)) = m,
\]

if and only if

\[
m \equiv n(\text{mod } 2) \quad \text{and} \quad \varphi(n) \leq m \leq n
\]

where \(\varphi\) is the numerical function introduced in (3.6).
Proof:

It is clear that $\iota_s$ preserves the parity and may increase but cannot decrease the dimension of homology classes so that necessarily $m \leq n$. The necessity of the condition $\varphi(n) \leq m$ follows from (3.4). We show now that the conditions are also sufficient. Recall that there exists an algebraic suspension $\Sigma$ functor from unital algebras to unital algebras such that one has a natural isomorphism

$$\gamma : HP_n(\Sigma A) \xrightarrow{\cong} HP_{n-1}(A)$$

which shifts dimensions by exactly one, i.e.,

$$\dim(\gamma(\alpha)) = \dim(\alpha) - 1$$

for all $\alpha \in HP_n(\Sigma A)$ (see for example [Pu] (4.6)). Let $(n, m)$ be a pair of nonnegative integers such that $m \equiv n (\text{mod} \ 2)$ and $\varphi(n) \leq m \leq n$. If $n = m$ then the trivial extension

$$0 \longrightarrow \Sigma^m \mathfrak{g} \longrightarrow \Sigma^m \mathfrak{g} \oplus \Sigma^m \mathfrak{g} \longrightarrow \Sigma^m \mathfrak{g} \longrightarrow 0$$

produces the desired shift whereas in the case $n > m$ we consider the extension

$$0 \longrightarrow \Sigma^j q(\Sigma^k \mathfrak{g}) \longrightarrow \Sigma^j Q(\Sigma^k \mathfrak{g}) \longrightarrow \Sigma^{j+k} \mathfrak{g} \longrightarrow 0$$

with $k = 2(n - m - 1)$ and $j = 2(m + 1) - n$. The generator $\alpha$ of $HP(\Sigma^j Q(\Sigma^k \mathfrak{g}))$ satisfies then $\dim(\alpha) = k + j = n$ and $\dim(\iota_{m}^{-1}(\alpha)) = \varphi(k) + j = m$. 

\begin{theorem}
Let $\delta_s$ be the boundary map in periodic cyclic homology. There exists an invertible extension of algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

and a class $\alpha \in HP_n(B)$ such that

$$\dim(\alpha) = n, \ \dim(\delta_s(\alpha)) = m,$$

if and only if

$$m \equiv n (\text{mod} \ 2) \ \text{and} \ m \geq \psi(n)$$

where $\psi$ is the numerical function introduced in (3.7). In particular there are no upper bounds for the dimension shift under excision in periodic cyclic homology.

\begin{proof}
The condition is necessary by (3.5). It is also sufficient: Let $(n, m)$ be a pair of nonnegative integers such that $m \geq \psi(n)$. Choose an integer $k$ such that $\psi(n + 2k) = m$ and let $R \to \Sigma^{2k} \mathfrak{g}$ be a quasifree presentation (2.1.3). Consider the pullback diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \epsilon(\Sigma^{n+2k} \mathfrak{g}) & \longrightarrow & E' & \longrightarrow & \Sigma^n R & \longrightarrow & 0 \\
0 & \longrightarrow & \epsilon(\Sigma^{n+2k} \mathfrak{g}) & \longrightarrow & E'_1(\Sigma^{n+2k} \mathfrak{g}) & \longrightarrow & \Sigma^{n+2k} \mathfrak{g} & \longrightarrow & 0
\end{array}
$$

The generator $\alpha$ of $HP_n(\Sigma^n R) \simeq \mathfrak{g}$ satisfies then $\dim(\alpha) = n$ and $\dim(\delta(\alpha)) = \psi(n+2k) = m$ by theorem (3.7) which establishes the other implication.

\end{proof}

By dualizing one obtains the following results in cohomology.
Theorem 3.10. There exists a splitting extension of algebras

\[ 0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0 \]

and a class \( \beta \in HP^n(I) \) such that

\[ dim(\beta) = n, \quad dim((\gamma^*)^{-1}(\beta)) = m, \]

if and only if

\[ m \equiv n \mod 2 \quad \text{and} \quad n \leq m \leq 2n + 2 \]

Theorem 3.11. There exists an invertible extension of algebras

\[ 0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0 \]

and a class \( \beta \in HP^n(I) \) such that

\[ dim(\beta) = n, \quad dim(\delta^*(\beta)) = m, \]

if and only if

\[ m \equiv n + 1 \mod 2 \quad \text{and} \quad m \leq 2n + 2 \]

Remark 3.12. All the results of this section carry directly over to continuous periodic cyclic (co)homology of locally convex algebras with jointly continuous multiplication. The only difference is that in the definition of invertible extensions the existence of a bounded linear splitting has to be assumed.

Finally we come back to the example mentioned in the introduction.

Let \( M^{2n} \) be a smooth compact Riemannian manifold of dimension \( 2n \) and let \( \mathcal{P}(M^{2n}) \) be the locally convex algebra of bounded operators on \( \mathcal{H} = L^2(M) \) generated by the scalar pseudodifferential operators of nonpositive integral order and by the Schatten ideal \( \ell^{2n+1}(\mathcal{H}) \) of \((2n+1)\)-summable compact operators. Due to a result of Hermann Weyl the ideal of pseudodifferential operators of strictly negative integral order is contained in the Schatten ideals \( \ell^p(\mathcal{H}) \) for \( p > dim(M) \). Thus there exists an extension

\[ 0 \rightarrow \ell^{2n+1}(\mathcal{H}) \rightarrow \mathcal{P}(M^{2n}) \xrightarrow{\sigma_{pr}} C^\infty(S^*M^{2n}) \rightarrow 0 \]

where \( \sigma_{pr} \) denotes the principal symbol homomorphism. It is called the pseudodifferential operator extension [Do]. It is well known that the boundary map

\[ \delta : K_1(C^\infty(S^*M^{2n})) \rightarrow K_0(\ell^{2n+1}(\mathcal{H})) \simeq \mathbb{Z} \]

in the corresponding six-term exact sequence of topological K-groups associates to the class \( [\sigma_{pr}(D)] \in K_1(C^\infty(M)) \) of the principal symbol of an elliptic operator \( D \) on \( M \) its index \( ind(D) = dim(\text{Ker}(D)) - dim(\text{Ker}(D^*)) \in \mathbb{Z} \).

Proposition 3.13. The pseudodifferential operator extension is invertible and realizes among invertible extensions the maximal dimension shift of the boundary map in periodic cyclic homology.
\textbf{Proof:} It is well known that the pseudodifferential operator extension is invertible \cite{Do}. Strictly speaking it is proven there that the $C^*$-closure of the present extension is invertible, but the demonstration can easily be carried out in the present context by making use of a smooth partition of unity and the Fourier transform in each of the coordinate charts. The compatibility under the Chern character of the boundary maps in K-theory and periodic cyclic homology \cite{CQ3} and the Atiyah-Singer index theorem imply that the canonical generator \cite{CQ3}

$$\tau \in HR_0(\ell^{2n+1}(\mathcal{H})) \simeq \mathcal{C}$$

is mapped under the boundary map in periodic cyclic cohomology to the Poincaré dual of the total Todd class

$$\pi^*(Todd(TM \otimes \mathcal{C})) \cap [S^* M] \in H_*(S^* M, \mathcal{C}) \simeq HP^*(\mathcal{C}^\infty(S^* M))$$

The total Todd class is a genus in the sense of Hirzebruch and is therefore of the form

$$Todd(TM \otimes \mathcal{C}) = 1 + \text{terms of higher degree}$$

Its Poincaré dual is thus given by

$$\pi^*(Todd(TM \otimes \mathcal{C})) \cap [S^* M] = [S^* M] + \text{terms of lower degree}$$

As the Hodge filtration on $HP^*(\mathcal{C}^\infty(S^* M))$ corresponds to the degree filtration on $H_*(S^* M, \mathcal{C})$ we obtain

$$dim(\delta^*(\tau)) = dim(S^* M) = 4n - 1$$

where $\delta^*$ denotes the boundary map in periodic cyclic cohomology.

Let now $\alpha \in HP_1(\mathcal{C}^\infty(S^* M)) \simeq H^{odd}(S^* M, \mathcal{C})$ be the fundamental class. It is of dimension $4n - 1$. By proposition (3.5) we find

$$dim(\delta_*(\alpha)) \geq \psi(dim(\alpha)) = \psi(4n - 1) = 2n$$

On the other hand the generator $\tau \in HP_0(\ell^{2n+1}(\mathcal{H}))$ can be realized by the cyclic $2n$-cocycle

$$S^n(Trace)(a^0 da^1 \ldots da^{2n}) = Trace(a^0 \ldots a^{2n})$$

so that we deduce from

$$\langle \tau, \delta_*(\alpha) \rangle = \langle \delta^*(\tau), \alpha \rangle \neq 0$$

the inequality $dim(\delta_*(\alpha)) \leq 2n$. Thus finally

$$dim(\delta(\alpha)) = 2n = \psi(dim(\alpha))$$

which shows that the pseudodifferential extension realizes the maximal dimension shift. \hfill \Box

33
References


34