CENTRAL SYMMETRIES OF PERIODIC BILLIARD ORBITS IN RIGHT TRIANGLES

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Abstract. We show that the midpoint of each periodic perpendicular beam hits the right-angle vertex of the triangle. The beam returns to itself after half its period with the opposite orientation, i.e. a Möbius band.

1. Introduction

A billiard ball, i.e. a point mass, moves inside a polygon $Q \subset \mathbb{R}^2$ with unit speed along a straight line until it reaches the boundary $\partial Q$, then instantaneously changes direction according to the mirror law: “the angle of incidence is equal to the angle of reflection,” and continues along the new line. If the trajectory hits a corner of the polygon, in general it does not have a unique continuation and thus by definition it stops there.

Cipra, Hanson and Kolan have shown that almost every orbit which is perpendicular to the base of a right triangle is periodic [CiHaKo, Ta]. Here the almost every is with respect to the length measure on the side of the triangle in question. Periodic billiard orbits always come in strips, i.e. for any $x = (q, v)$ whose billiard orbit is periodic where $q \in \partial Q$ and $v$ is any inward pointing direction there is an open interval $I \subset \partial Q$ such that $q \in I$ and for any $q' \in I$ the orbit of $x' = (q', v)$ visits the same sequence of sides as $(q, v)$ and thus in particular is periodic (see Figure 1). A maximal width strip will be called a beam. All orbits in a periodic beam have the same period except perhaps the middle orbit which has half the period in the case its period is odd. The results of [CiHaKo] imply that a set of full measure of the base of the triangle

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{beam.png}
\caption{A periodic strip.}
\end{figure}
is covered by an at most countable union of intervals such that each of these intervals forms a periodic beam.

They mention “our computational evidence also suggest that the trajectory at the middle of each (perpendicular periodic) beam hits the right-angle vertex of the triangle.” In this short note we we present an elementary proof of this fact.

Any billiard trajectory which hits a right-angled vertex (or more generally a vertex with angle $\pi/n$ for some positive integer $n$) has a unique continuation. Reflect the triangle in the sides of right angle to obtain a rhombus. The study of the billiard in the triangle reduces to that in the rhombus (see next section for details). Throughout the article all beams considered will be with respect to the coding in the rhombus. With this interpretation we can now state the main result of this note

**Theorem 1.** The midpoint of each periodic perpendicular beam hits the right-angle vertex of the triangle (i.e. the mid point of the rhombus). The beam returns to itself after half its period with the opposite orientation.

The theorem implies that when viewed as an object in the phase space of the billiard flow in the triangle the beam is a Möbius band. This is not in contradiction with the well known construction, of invariant surfaces (see for example [Gu1, Ta] in the rational case and [GuTr] in the irrational case) since the directional billiard flow is isomorphic to the geodesic flow on the invariant surface for any direction except for directions which are parallel to a side of the polygon. The direction we are considering is such a direction.

We remark that the symmetry of the beam is reminiscent of the symmetry of the beam of periodic orbits around a periodic orbit of odd period mentioned above.

Before turning to the proof of Theorem 1 I want to make several historical comments. The result of Cipra, Hanson and Kolan was strengthened by Guttin and myself in [GuTr]. We showed that for any right triangle and any fixed direction $\theta$, almost every orbit which begins in the direction $\theta$ returns parallel to itself. This result also holds for a more general class of polygons called generalized parallelograms. Cipra et. al.’s result follows since the billiard orbit of any point which begins perpendicular to a side of a polygon and at a later instance hits some side perpendicularly retraces its path infinitely often in both senses between the two perpendicular collisions and thus is periodic. A similar theorem was proven for a larger class of polygons by myself in [Ta]. The results of [GuTr] were further strengthened, estimates of the dimension of the set of non-periodic perpendicular orbits where given by Schmeling and myself in [ScTr] exploiting a link to number theory.
2. Definitions and Proofs

We begin by discussing some of the structure developed in [ClHaKo, GuTr, Ta, ScTr]. There is a nice introductory book on billiards by Tabachnikov [Ta] and several survey articles [Gu1, Gu2, MaTa] which can be consulted for details on polygonal billiards in general.

The proof is based on the procedure of unfolding of a billiard trajectory. Instead of reflecting the trajectory with respect in a side of a polygon reflect the polygon in this side. Thus the trajectory is straightened to a line with a number of isometric copies of the polygon skewered on it (Figure 2a).

Fix an orbit segment. There is a strip around this trajectory segment such that the same sequence of reflections are made by all trajectories in the strip (Figure 2b). The number of reflections is called the length of the strip. We will call a maximal width strip a beam. The boundary of a periodic beam consists of one or more trajectory segments which hit a vertex of the polygon. If this vertex is the right-angle one, then the sequence of reflections on both sides of it are essentially the same since the singularity due to such a vertex is removable. This enables us to consider the billiard inside a rhombus which consists of four copies of the right triangle and unfold the rhombus (Figure 3).

Throughout the article unfoldings will be of the billiard in the rhombus and thus billiard orbits through its center will be considered as defined. As we follow the reflections along a straight line trajectory we see that each flip rotates the rhombus by $2 \alpha$ where $\alpha$ is one of the interior angles of the rhombus. Thus we can code the rhombi with integers according to the total number of (clockwise) rotations by $2 \alpha$ (see Figure 3). All reference to the length of a beam will pertain to the number of rhombi it crosses which differs from its length in the right triangle.

The other essential ingredient of the proof are the central and reflectional symmetries of the rhombus.

We begin the proof of the theorem by proving

**Lemma 2.** The two statements in Theorem 1 are equivalent.
Proof. Perpendicular periodic beams must have even period since they must retrace their orbits between the two perpendicular hits. Fix a perpendicular periodic beam of period $2p$ Consider the beam of length $p$ between the two perpendicular collisions (see Figure 3).

Suppose first that the midpoint of this beam hits the right-angle vertex. The central symmetry of the rhombus at this collision point implies that the beam itself is centrally symmetric around this point. Label by $I$ the interval of depart and by $J$ the interval of arrival. Both intervals are contained in the rhombus labelled 0, the central symmetry of the beam implies that $J$ and $I$ are centrally symmetric as well. This completes the proof of the first implication.

Now suppose that the beam returns to itself after half its period with the opposite orientation. The label of the beam starts and ends
with 0 and at each step increases or decreases by 1, thus $p$ must be odd. Viewed from the unfolding one sees that the beam is centrally symmetric around its center $c$. Since $p$ is odd the point $c$ must lie in the interior of a rhombus. However viewed in the rhombus one sees that the beam can only be centrally symmetric around the center of the rhombus and no other point, thus $c$ must be the center of the rhombus. \hfill \Box

Remark: Galperin, Stepin and Vorobets showed that any perpendicular orbit whose period is not a multiple of 4 is unstable under perturbation [GaStVo]. Thus the fact that $p$ is odd implies that the periodic orbit is unstable under perturbation (as viewed as an orbit in a rhombus). A simple argument shows that the period is also not a multiple of 4 when the orbit is viewed as an orbit in a right triangle [GaZv].

We will also need the following fact.

**Lemma 3.** Suppose the $A^+$ and $B^+$ are beams (not necessarily perpendicular or periodic) with the same codes, the $A^+ = B^+$.

**Proof.** The proof is by induction. The base case is clear, there is a single beam with code 0 (a single beam with code 01, a single beam with code 0−1, etc). Assume that every beam with length $n$ is the unique beam with its code. Since in exiting the $j$th rhombus, there are only two rhombi to enter, $j \pm 1$, each of these beam can be split into at most two sub-beams with length $n+1$. If a beam gets split, then resulting sub-beams code differs in the $n+1$st place. \hfill \Box

**Proof of Theorem 1.** We will prove the second statement of the theorem, the first statement then follows from Lemma 2. Fix one of the bases of the triangle and consider a beam of periodic trajectories perpendicular to this side. We call the interval of depart $I$. Consider the code of the beam, for example in Figure 3 the beam $A^+$’s code $\{a_i\}$ begins with 0121210…. The beam has period 14 since after 7 steps it returned perpendicularly. If one draws the continuation of the unfolding in Figure 3, one sees that the initial segment of the code of the beam is 0121210−1−2−1−2−1 and that this segment is then repeated periodically.

Consider the interval $J$ which is obtained from $I$ by applying the reflectional symmetry of the rhombus with respect to the vertical direction. Consider the perpendicular beam $B^+$ which starts the interval $J$. The symmetry in question implies that the code $\{b_i\}$ of $B^+$ satisfies $b_i = -a_i$ for all $i$ (see Figure 3).

Next we consider the central symmetry of the rhombus. This also takes the interval $I$ to the interval $J$ and thus the beam $B^+$ and the backwards beam $A^-$ are identified.
Suppose the period of $A^+$ is $2p$. This identification implies that the code of $A^+$ satisfies $a_i = -a_i = -a_{2p-i}$ for $i = 0, 1, \ldots, p$.

Furthermore the code of $f^pB^-$ is the same as the code of $A$. Thus Lemma 3 implies the result. \hfill \Box

REFERENCES


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