Self-adjoint negation

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Abstract

If $X$ is a cartesian closed category with an initial object, the following
three conditions are equivalent: (1) involutiveness of the negation functor
$\neg$ up to isomorphism, (2) self-adjointness of $\neg$, (3) existence of a natural
transformation from $\neg\neg$ to $\text{id}_X$. This strengthens a result of Joyal saying
that condition (1) makes $X$ a preorder.

If $X$ is a cartesian closed category with an initial object $\bot$, $\neg$ denotes the functor,
called negation, from $X$ to $X^{\text{op}}$ (or from $X^{\text{op}}$ to $X$) with object function given
by $\neg x = \bot^x$.

It is a well-known result of Joyal that in such a category $X$, there is at most
one arrow from any given object $x$ to $\bot$: see, e.g., Proposition 8.3 in [1]. As an
immediate consequence, if the canonical natural transformation from $\text{id}_X$ to $\neg\neg$
is a natural isomorphism, then $X$ is a preorder. This is why the straightforward
extension of intuitionistic denotational semantics to classical logic identifies all
proofs of a given formula. However, the requirement to be a cartesian closed
category with an initial object is just a matter of adjoints, so instead of asking
negation to be involutive (up to isomorphism), it is more natural to ask it to be
self-adjoint. The functor $\neg : X \to X^{\text{op}}$ is always left-adjoint to $\neg : X^{\text{op}} \to X$:
indeed, $X^{\text{op}}(\neg x, y) = X(y, \neg x) \cong X(y \times x, \bot) \cong X(x \times y, \bot) \cong X(x, \neg y)$ and
all bijections are natural in $x$ and $y$. The opposite adjunction, namely that
$\neg : X^{\text{op}} \to X$ is left-adjoint to $\neg : X \to X^{\text{op}}$

\[
X \cong X^{\text{op}} \quad \text{adjunction}
\]

need not hold in general, and is a priori a weaker condition; the following lemma
shows that this adjunction is actually equivalent to involutiveness, and is even
equivalent to the bare existence of a natural transformation from $\neg\neg$ to $\text{id}_X$.

Lemma 1. Let $X$ be a cartesian closed category with an initial object $\bot$. The
following are equivalent:

1. The canonical natural transformation from $\text{id}_X$ to $\neg\neg$ is a natural isomor-
phism.

2. The functor $\neg : X^{\text{op}} \to X$ is left-adjoint to $\neg : X \to X^{\text{op}}$.  

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3. There is a natural transformation from \( \neg \neg \) to \( \text{id}_X \).

As a consequence, either of these conditions implies that \( X \) is a preorder.

In the posetal case, this implies that a Heyting algebra \( H \) is Boolean if, and only if, \( \neg : H^{\text{op}} \to H \) is left-adjoint to \( \neg : H \to H^{\text{op}} \). Observe that this statement differs from the remark after Proposition 1.8.4 in [2].

**Proof** — Obviously, condition 1 implies condition 2. To show that condition 2 implies condition 1, observe that the terminal object \( T \) is isomorphic to \( \neg \bot \): indeed, the image of the second projection \( T \times \bot \to \bot \) under the bijection \( X(x \times y, z) \cong X(x, z^y) \) is an arrow from \( T \) to \( \neg \bot \); the composite \( T \to \neg \bot \to T \) has to be the unique arrow \( \text{id}_T \) from \( T \) to \( T \), and the composite \( \neg \bot \to T \to \neg \bot \) is \( \text{id}_{\neg \bot} \) as an arrow to an object \( \neg x \). As a consequence,

\[
X(x, y) \cong X(T \times x, y) \cong X(T, y^x) \cong X(\neg \bot, y^x).
\]

Now, the adjunction from \( \neg : X^{\text{op}} \to X \) to \( \neg : X \to X^{\text{op}} \) gives \( X(\neg \bot, y^x) \cong X(\neg(y^x), \bot) \), so \( X(x, y) \cong X(\neg(y^x), \bot) \) and \( X(x, y) \) has cardinality at most one. In particular, \( X(x, x) = \{ \text{id}_x \} \), so the composite

\[
x \xrightarrow{1} \neg \neg x \xrightarrow{\epsilon_x} x
\]

with \( \epsilon : \neg \neg \to \text{id}_X \) the counit of the given adjunction, is the identity. Since the other composite \( \neg \neg x \to x \to \neg x \) is the unique arrow \( \text{id}_{\neg x} : \neg x \to \neg x \), we conclude that the two natural transformations are inverses of each other.

To show that condition 2 is equivalent to condition 3, recall that an adjunction from \( \neg : X^{\text{op}} \to X \) to \( \neg : X \to X^{\text{op}} \) amounts to natural transformations \( \epsilon \) from \( \neg \neg \) to \( \text{id}_X \) and \( \eta \) from \( \text{id}_{X^{\text{op}}} \) to \( \neg \neg \) (i.e., \( \eta \) from \( \neg \neg \) to \( \text{id}_X \)) such that \( \neg \epsilon \circ \eta \) is the identity natural transformation on \( \neg : X \to X^{\text{op}} \) and \( \epsilon \circ \neg \eta \) is the identity on \( \neg : X^{\text{op}} \to X \). The last two equations are always satisfied because hom-sets to an object \( \neg x \) have cardinality at most one. Hence, such an adjunction simply amounts to a pair of natural transformations \( \epsilon \) and \( \eta \) from \( \neg \neg \) to \( \text{id}_X \). Moreover, \( \epsilon \) and \( \eta \) have to be equal because \( X(\neg \neg x, x) \cong X(\neg x, x) \cong X(\neg x \times x, \bot) \) has cardinality at most one. As a consequence, an adjunction from \( \neg : X^{\text{op}} \to X \) to \( \neg : X \to X^{\text{op}} \) just amounts to one natural transformation from \( \neg \neg \) to \( \text{id}_X \). \( \Box \)

**References**
