On the homology of algebras of Whitney functions

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Introduction

Methods originating from noncommutative differential geometry have proved to be very successful not only for the study of noncommutative algebras, but also gave new insight to the geometric analysis of smooth manifolds, which are the typical objects of commutative differential geometry. As three particular examples for this let us mention the following results:

1. the isomorphy between the de Rham homology of a smooth manifold and the periodic cyclic cohomology of its algebra of smooth functions (Connes [5, 6]),
2. the local index formula in noncommutative geometry by Connes–Moscovici [7],
3. the algebraic index theorem of Nest–Tsyzg [30].

It is a common feature of these examples that the underlying space has to be smooth, so the natural question arises, whether noncommutative methods can also be successfully applied to the study of singular spaces. This is exactly the question we want to address in this work. More precisely, we will show among other things that the periodic cyclic homology of the algebra $\mathcal{E}^\infty(X)$ of Whitney functions on a subanalytic set $X \subset \mathbb{R}^n$, the de Rham cohomology of $\mathcal{E}^\infty(X)$ (which we call the Whitney–de Rham cohomology of $X$) and the singular cohomology of $X$ naturally coincide. Let us briefly explain, why we have chosen to study algebras of Whitney functions in a noncommutative setting. First, the theory of jets and Whitney functions has
become an indispensable tool in real analytic geometry and the differential analysis of spaces with singularities [1, 2, 28, 38, 40]. Second, we have been inspired by the algebraic de Rham theory of Grothendieck [17] (see also [18, 19]) and by the work of Feigin–TSygan [11] on the (periodic) cyclic homology of the formal completion of the coordinate ring of an affine algebraic variety. Recall that the formal completion of the coordinate ring of an affine complex algebraic variety $X \subset \mathbb{C}^n$ is the $\mathbb{Z}$-adic completion of the coordinate ring of $\mathbb{C}^n$ with respect to the vanishing ideal of $X$ in $\mathbb{C}^n$. Thus the formally completed coordinate ring of $X$ can be interpreted as the algebraic analogue of the sheaf of Whitney functions on $X$. Now, Grothendieck [17] has proved that the de Rham cohomology of the formal completion coincides with the complex cohomology of the variety, and Feigin–TSygan [11] have shown that the periodic cyclic cohomology of the formal completion coincides with the algebraic de Rham cohomology, if the affine variety is locally a complete intersection. By the analogy between algebras of formal completions and algebras of Whitney functions it was natural to conjecture that these two results should also hold for Whitney functions over appropriate singular spaces. Theorems 6.4 and 7.1 confirm this conjecture for the case, where the underlying space is subanalytic. Besides the de Rham cohomology and the periodic cyclic homology of an algebra of Whitney functions we also study its Hochschild homology and cohomology. In fact, we compute these homology theories at first by application of a variant of the localization method of Teleman [36] and then derive the (periodic) cyclic homology from the Hochschild homology.

Our article is set up as follows. In the first section we have collected some basic material from the theory of jets and Whitney functions. Later on we also explain necessary results from Hochschild resp. cyclic homology theory. We have been fairly explicit in the presentation of the preliminaries, so that a noncommutative geometer will find himself easily through the singularity theory used in this article and vice versa. Since the localization method we use in this article can also be applied to the computation of the Hochschild (co)homology of other (sheaves of) commutative algebras over singular spaces, we introduce it in a separate section, namely Section 2. After that we treat Peetre-like theorems for local operators on spaces of Whitney functions and on spaces of $G$-invariant functions in Section 3. These results will later be used for the computation of the Hochschild cohomology of Whitney functions, but may be of interest of their own. Section 4 is dedicated to the computation of the Hochschild homology of $\mathcal{E}^\infty(X)$. Using localization methods we first prove that it is given by the homology of the so-called diagonal complex. This complex is naturally isomorphic to the tensor product of $\mathcal{E}^\infty(X)$ with the Hochschild chain complex of the algebra of formal power series. The homology of the latter complex can be computed via a Koszul-resolution, so we obtain the Hochschild homology of $\mathcal{E}^\infty(X)$. In the next section we consider the cohomological case. Interestingly, the Hochschild cohomology of $\mathcal{E}^\infty(X)$ is more difficult to compute, as several other tools besides localization methods are involved, like for example a generalized Peetre’s theorem and operations on the Hochschild cochain complex. In Section 5 we derive the cyclic and periodic cyclic homology from the Hochschild homology by standard arguments of noncommutative geometry.

In Section 7 we prove that the Whitney–de Rham cohomology over a subanalytic set coincides with the singular cohomology of the underlying topological space. The claim follows essentially from a Poincaré lemma for forms of Whitney functions over subanalytic sets. This Poincaré lemma will be proved with the help of a so-called bimeromorphic subanalytic triangulation of the underlying subanalytic set. The existence of such a triangulation will also be shown in the last section.
With respect to the above list of (some of) the achievements of noncommutative geometry in geometric analysis we have thus shown that the first result can be carried over to a wide class of singular spaces with the structure sheaf given by Whitney functions. It would be interesting and tempting to examine whether the other two results have also singular analogues involving Whitney functions.

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1 Preliminaries on Whitney functions

1.1 Jets By \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( \beta \) we will always denote multiindices lying in \( \mathbb{N}^m \). The variables \( x, x_0, x_1, \ldots, y \) and so on will always stand for elements of some \( \mathbb{R}^n \); the coordinates are respectively denoted by \( x, x_0, x_1, \ldots, y \) and so on, where \( i = 1, \ldots, n \). Moreover, we write \( |\alpha| = \alpha_1 + \cdots + \alpha_m \) and \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). By \( |x| \) we denote the euclidean norm of \( x \), and by \( d(x, y) \) the euclidean distance of two points.

In this article \( X \) will always mean a locally closed subset of some \( \mathbb{R}^n \) and, if not stated differently, \( U \subset \mathbb{R}^n \) an open subset such that \( X \subset U \) is relatively closed. By a jet of order \( m \) on \( X \) (with \( m \in \mathbb{N} \cup \{\infty\} \)) we understand a family \( F = (F^\alpha)_{|\alpha| \leq m} \) of continuous functions on \( X \). The space of jets of order \( m \) on \( X \) will be denoted by \( J^m(X) \). We write \( F(x) = F^\alpha(x) \) for the evaluation of a jet at some point \( x \in X \), and \( F_k \) for the restricted family \((F^\alpha(x))_{|\alpha| \leq m} \).

More generally, if \( Y \subset X \) is locally closed, the restriction of continuous functions gives rise to a natural map \( J^m(X) \to J^m(Y) \), \((F^\alpha)_{|\alpha| \leq m} \mapsto (F^\alpha|_Y)_{|\alpha| \leq m} \). Given \( |\alpha| \leq m \), we denote by \( D^\alpha : J^m(X) \to J^{m-|\alpha|}(X) \) the linear map, which associates to every \((F^\beta)_{|\beta| \leq m}\) the jet \((F^\beta \cdot \beta)|_{|\beta| \leq m-|\alpha|}\). If \( \alpha = (0, \ldots, 1, \ldots, 0) \) with 1 at the \( i \)-th spot, we denote \( D^\alpha \) by \( D_i \).

For every natural number \( r \leq m \) and every \( K \subset X \) compact, \( |F|^r = \sup_{x \in K} |F^\alpha(x)| \) is a seminorm on \( J^m(X) \). Sometimes, in particular if \( K \) consists only of one point, we write only \( |\cdot|_K \) instead of \( |\cdot|_K \). The topology defined by the seminorms \( |\cdot|_K \) gives \( J^m(X) \) the structure of a Fréchet space. Moreover, \( D^n \) and the restriction maps are continuous with respect to these topologies.

The space \( J^m(X) \) carries a natural algebra structure where the product \( FG \) of two jets has components \((FG)^\alpha = \sum_{\beta \subseteq \alpha} \binom{\alpha}{\beta} F^\beta G^{\alpha - \beta} \). One checks easily that \( J^m(X) \) with this product becomes an initial Fréchet algebra.

For \( U \subset \mathbb{R}^n \) open we denote by \( C^m(U) \) the space of \( C^m \)-functions on \( U \). Then \( C^m(U) \) is a Fréchet space with topology defined by the seminorms

\( |f|^r = \sup_{x \in K} |D^\alpha f(x)| \),

where \( K \) runs through the compact subsets of \( U \) and \( r \) through all natural numbers \( \leq m \). Note that for \( X \subset U \) closed there is a continuous linear map \( J^m_X : C^m(U) \to J^m(X) \) which associates to every \( C^m \)-function \( f \) the jet \( J^m_X(f) = (D^\alpha f|_X)|_{|\alpha| \leq m} \).

1.2 Whitney functions Given \( y \in X \) and \( F \in J^m(X) \), the Taylor polynomial (of order \( m \)) of \( F \) is defined as the polynomial

\( T_y^m F(x) = \sum_{|\alpha| \leq m} \frac{F^\alpha(y)}{\alpha!} (x - y)^\alpha, \quad x \in U. \)

Moreover, one sets \( R_y^m F = F - J^m(T_y^m F) \). Then, if \( m \in \mathbb{N} \), a Whitney function of class \( C^m \) on \( X \) is an element \( F \in J^m(X) \) such that for all \( |\alpha| \leq m \)

\( (R_y^m F)(x) = o(|y - x|^{-m+1}) \) for \( |x - y| \to 0, \ x, y \in X. \)
The space of all Whitney functions of class $C^m$ on $X$ will be denoted by $\mathcal{E}^m(X)$. It is a Fréchet space with topology defined by the seminorms

$$
\|F\|_m^K = |F|_m^K + \sup_{\substack{x,y \in K \\
x \neq y}} \frac{|(R^m F)^s(x) - (R^m F)^s(y)|}{|y - x|^{m+1}}
$$

where $K$ runs through the compact subsets of $X$. The projective limit $\lim_{\rightarrow} \mathcal{E}^r(X)$ will be denoted by $\mathcal{E}^\infty(X)$; its elements are called Whitney functions of class $C^\infty$ on $X$. By construction, $\mathcal{E}^\infty(X)$ can be identified with the subspace of all $F \in J^m(X)$ such that $\forall F \in \mathcal{E}^r(X)$ for every natural number $r$. Moreover, the Fréchet topology of $\mathcal{E}^\infty(X)$ then is given by the seminorms $\|\cdot\|_m^K$ with $K \subset X$ compact and $m \in \mathbb{N}$. It is not very difficult to check that for $U \subset \mathbb{R}^n$ open, $\mathcal{E}^m(U)$ coincides with $C^m(U)$ (even for $m = \infty$). Each one of the spaces $\mathcal{E}^m(X)$ inherits from $J^m(X)$ the associative product; thus $\mathcal{E}^m(X)$ becomes a subalgebra of $J^m(X)$ and a Fréchet algebra.

It is straightforward that the spaces $\mathcal{E}^m(V)$ with $V$ running through the open subsets of $X$ form the sectional spaces of a sheaf $\mathcal{E}_X^\infty$ of Fréchet algebras on $X$ and that this sheaf is fine. We will denote by $\mathcal{E}_X^{m,\alpha}(x)$ the stalk of this sheaf at some point $x \in X$ and by $[F]_\alpha \in \mathcal{E}_X^{m,\alpha}$ the germ (at $x$) of a Whitney function $F \in \mathcal{E}^m(V)$ defined on a neighborhood $V$ of $x$.

For more details on the theory of jets and Whitney functions the reader is referred to the monographs of MALGRANGE [28] and TOUGERON [38], where he will also find explicit proofs.

1.3 Regular sets For an arbitrary compact subset $K \subset \mathbb{R}^n$ the seminorms $|\cdot|_m^K$ and $\|\cdot\|_m^K$ are in general not equivalent. But if $K$ is $p$-regular for some positive integer $p$, there exists a constant $C_p > 0$ such that $\|F\|_m^K \leq C_p |F|_m^K$ for all $F \in \mathcal{E}^m(K)$. Recall [38, Def. 3.10], where a compact set $K$ is defined to be $p$-regular, if it is connected by rectifiable arcs and if the geodesic distance $\delta(s, y) \leq C|s - y|^{1/p}$ for all $s, y \in K$ and some $C > 0$ depending only on $K$. Clearly, if $K$ is 1-regular, the seminorms $|\cdot|_m^1$ and $\|\cdot\|_m^1$ have to be equivalent and $\mathcal{E}^m(K)$ is a closed subspace of $J^m(K)$.

Generalizing the notion of regularity to not necessarily compact locally closed subsets one calls a closed subset $X \subset U$ regular, if for every point $x \in X$ there exists a positive integer $p$ and a $p$-regular compact neighborhood $K \subset X$. For $X$ regular $\mathcal{E}^m(X)$ is a closed subspace of $J^m(X)$ which means in other words that the topology given by the seminorms $|\cdot|_m^K$ is equivalent to the original topology defined by the seminorms $\|\cdot\|_m^K$.

1.4 Whitney’s extension theorem Let $Y \subset X$ be closed and denote by $\mathcal{E}^m(Y; X)$ the ideal of all Whitney functions $F \in \mathcal{E}^m(X)$ which are flat of order $m$ on $Y$ that means which satisfy $F_{\mid Y} = 0$. The Whitney extension theorem (WHITNEY [40], see also [28, Thm. 3.2, Thm. 4.1] and [38, Thm. 2.2, Thm. 3.1]) then says that for every $m \in \mathbb{N} \cup \{\infty\}$ the sequence

$$
0 \rightarrow \mathcal{E}^m(Y; X) \rightarrow \mathcal{E}^m(X) \rightarrow \mathcal{E}^m(Y) \rightarrow 0
$$

is exact, where the third arrow is given by restriction. For finite $m$ and compact $X$ such that $Y$ lies in the interior of $X$ there exists a linear splitting or in other words an extension map $W : \mathcal{E}^m(Y) \rightarrow \mathcal{E}^m(X)$ which is continuous in the sense that $|W(F)|_m^X \leq C|F|_m^Y$ for all $F \in \mathcal{E}^m(Y)$. If in addition $X$ is 1-regular this means that the sequence (1.1) is split exact. These complements on the continuity of $W$ are due to GLAESER [14]. Note that for $m = \infty$ a continuous linear extension map does in general not exist.

Under the assumption that $X$ is 1-regular, $m$ finite and $Y$ in the interior of $X$, the subspace of all Whitney functions of class $C^\infty$ on $X$ which vanish in a neighborhood of $Y$ is dense in $\mathcal{E}^m(Y; X)$ (with respect to the topology of $\mathcal{E}^m(X)$).

Assume to be given two relatively closed subsets $X \subset U$ and $Y \subset V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ are open. Further let $g : U \rightarrow V$ be a smooth map such that $g(X) \subset Y$. Then, by Whitney’s extension theorem, there exists for every $F \in \mathcal{E}^\infty(Y)$ a uniquely determined
Whitney function \( g^*(F) \in \mathcal{E}^\infty(X) \) such that for every \( f \in \mathcal{C}^\infty(V) \) with \( I_X^\infty(f) = F \) the function \( f \circ g \in \mathcal{C}^\infty(U) \) satisfies \( I_X^\infty(f \circ g) = g^*(F) \). The Whitney function \( g^*(F) \) will be called the \textit{pull-back} of \( F \) by \( g \).

1.5 \textbf{Regularly situated sets} Two closed subsets \( X, Y \) of an open subset \( U \subset \mathbb{R}^n \) are called \textit{regularly situated} [38, Def. 4.4], if either \( X \cap Y = \emptyset \) or if for every pair of compact sets \( K \subset X \) and \( L \subset Y \) there exists a pair of constants \( C > 0 \) and \( \lambda > 0 \) such that

\[
d(x, L) \geq C d(x, X \cap Y)^\lambda \quad \text{for all } x \in K.
\]

It is a well-known result by \textsc{Lojasiewicz} [26] that \( X, Y \) are regularly situated if and only if the sequence

\[
0 \longrightarrow \mathcal{E}^\infty(X \cup Y) \overset{\delta}{\longrightarrow} \mathcal{E}^\infty(X) \oplus \mathcal{E}^\infty(Y) \overset{\pi}{\longrightarrow} \mathcal{E}^\infty(X \cap Y) \longrightarrow 0
\]

is exact, where the maps \( \delta \) and \( \pi \) are given by \( \delta(F) = (F|_X, F|_Y) \) and \( \pi(F, G) = F|_{X \cap Y} - G|_{X \cap Y} \).

1.6 \textbf{Multipliers} By \( \mathcal{M}^\infty(X; U) \) we denote the set of all \( f \in \mathcal{C}^\infty(U \setminus X) \) which satisfy the following condition: For every compact \( K \subset U \) and every \( \alpha \in \mathbb{N}^p \) there exist constants \( C > 0 \) and \( \lambda > 0 \) such that

\[
|\partial^\alpha f(x)| \leq \frac{C}{(d(x, X))^\lambda} \quad \text{for all } x \in X \setminus K.
\]

The space \( \mathcal{M}^\infty(X; U) \) is an algebra of \textit{multipliers} for \( \mathcal{J}^\infty(X; U) \) that means for every \( f \in \mathcal{J}^\infty(X; U) \) and \( g \in \mathcal{M}^\infty(X; U) \) the product \( gf \) on \( U \setminus X \) has a unique extension to an element of \( \mathcal{J}^\infty(X; U) \). More generally, if \( X \) and \( Y \) are closed subsets of \( U \) which are regularly situated, then \( \mathcal{M}^\infty(X; U) \) is also an algebra of multipliers for \( \mathcal{J}^\infty(X \cap Y; X) \) (see [28, IV.1]).

1.7 \textbf{Subanalytic Sets} A set \( X \subset \mathbb{R}^n \) is called \textit{subanalytic} [20, Def. 3.1], if for every point \( x \in X \) there exists an open neighborhood \( U \) of \( x \) in \( \mathbb{R}^n \), a finite system of real analytic maps \( f_{ij} : U_{ij} \rightarrow U \) \((i = 1, \ldots, p; j = 1, 2)\) defined on open subsets \( U_{ij} \subset \mathbb{R}^{n+2} \) and a family of closed analytic subsets \( A_{ij} \subset U_{ij} \) such that every restriction \( f_{ij}|_{A_{ij}} : A_{ij} \rightarrow U \) is proper and

\[
X \cap U = \bigcup_{i=1}^p f_{i1}(A_{i1}) \setminus f_{i2}(A_{i2}).
\]

The set of all subanalytic sets is closed under the operations of finite intersection, finite union and complement. Moreover, the image of a subanalytic set under a proper analytic map is subanalytic. For details and proofs see \textsc{Ihara} [20] or \textsc{Bierstone-Milman} [3].

Note that every subanalytic set \( X \subset \mathbb{R}^n \) is regular [21, Cor. 2], and that any two relatively closed subanalytic sets \( X, Y \subset U \) are regularly situated [3, Cor. 6.7].

1.8 \textbf{Topological tensor products and nuclearity} Recall that on the tensor product \( V \otimes W \) of two locally convex real vector spaces \( V \) and \( W \) one can consider many different locally convex topologies arising from the topologies on \( V \) and \( W \) (see \textsc{Gromov} [16] or \textsc{Trèves} [39, Part. III]). The for our purposes most natural topology is the \( \pi\text{-topology} \), i.e. the finest locally convex topology on \( V \otimes W \) for which the natural mapping \( \otimes : V \times W \rightarrow V \otimes W \) is continuous.

\( V \otimes W \) with this topology is denoted by \( V \otimes_\pi W \), its completion by \( V\oplus W \). In fact, the \( \pi\text{-topology} \) is the strongest topology compatible with \( \otimes \) in the sense of \textsc{Gromov} [16, I, §3, n° 3]. The weakest topology compatible with \( \oplus \) is usually called the \( \varepsilon\text{-topology} \); in general it is different from the \( \pi\text{-topology} \). A locally convex space \( V \) is called \textit{nuclear}, if all the compatible topologies on \( V \otimes W \) agree for every locally convex spaces \( W \).
1.9 Proposition The algebra $\mathcal{E}^\infty(X)$ of Whitney functions over a locally closed subset $X \subset \mathbb{R}^n$ is nuclear. Moreover, if $X' \subset \mathbb{R}^{n'}$ is a further locally closed subset, then $\mathcal{E}^\infty(X) \otimes \mathcal{E}^\infty(X') \cong \mathcal{E}^\infty(X \times X')$.

Proof: For open $U \subset \mathbb{R}^n$ the Fréchet space $\mathcal{C}^\infty(U)$ is nuclear [16, II. §2, n° 3], [39, Chap. 51]. Choose $U$ such that $X$ is closed in $U$. Recall that every Hausdorff quotient of a nuclear space is again nuclear. Moreover, by Whitney’s extension theorem, $\mathcal{E}^\infty(X)$ is the quotient of $\mathcal{C}^\infty(U)$ by the closed ideal $\mathcal{I}^\infty(X; U)$, hence one concludes that $\mathcal{E}^\infty(X)$ is nuclear. Now choose an open set $U' \subset \mathbb{R}^{n'}$ such that $X'$ is closed in $U'$. Then we have the following commutative diagram of continuous linear maps:

$$
\begin{array}{ccc}
\mathcal{C}^\infty(U) \otimes \mathcal{C}^\infty(U') & \longrightarrow & \mathcal{E}^\infty(X) \otimes \mathcal{E}^\infty(X') \\
\downarrow & & \downarrow \\
\mathcal{C}^\infty(U \times U') & \longrightarrow & \mathcal{E}^\infty(X \times X').
\end{array}
$$

Clearly, the horizontal arrows are surjective and the vertical arrows injective. Since the completion of $\mathcal{C}^\infty(U) \otimes \mathcal{C}^\infty(U')$ coincides with $\mathcal{C}^\infty(U \times U')$, the completion of $\mathcal{E}^\infty(X) \otimes \mathcal{E}^\infty(X')$ coincides with $\mathcal{E}^\infty(X \times X')$. This proves the claim. □

1.1.0 Remark Note that for finite $m$ and nonfinite but compact $X$ the space $\mathcal{E}^m(X)$ is not nuclear, since a normed space is nuclear if and only if it is finite dimensional [39, Cor. 2 to Prop. 90.2].

2 Localization techniques

2.1 In this section we introduce a localization method for the computation of the Hochschild homology of a fine commutative algebra. This method generalizes the approach of TELEMAN [36] and works also for the computation of (co)homology groups with values in a module.

Let $X \subset \mathbb{R}^n$ be a locally closed subset and $d$ the euclidean metric. Let $A$ be a sheaf of commutative unital $\mathbb{R}$-algebras on $X$ and denote by $A = A(X)$ its space of global sections. We assume that $A$ is an $\mathcal{E}^\infty(X)$-module sheaf, which implies in particular that $A$ is a fine sheaf. Additionally, we assume that the sectional spaces of $A$ carry the structure of a Fréchet algebra, that all the restriction maps are continuous and that for every open $U \subset X$ the action of $\mathcal{E}^\infty(U)$ on $A(U)$ is continuous. This means in particular that $A$ is a commutative Fréchet algebra. The premises on $A$ are satisfied for example in the case, when $A$ is the sheaf of Whitney functions or the sheaf of smooth functions on $X$.

From $A$ one constructs for every $k \in \mathbb{N}^*$ the exterior tensor product sheaf $A^\otimes k$ over $X^k$. Its space of sections over a product of the form $U_1 \times \ldots \times U_k$ with $U_j \subset X$ open is given by the completed $\pi$-tensor product $A(U_1) \hat{\otimes} \ldots \hat{\otimes} A(U_k)$. Using the fact that $A$ is a topological $\mathcal{E}^\infty(X)$-module sheaf and that $\mathcal{E}^\infty(X)$ is fine one checks immediately that presheaf uniquely defined by these conditions is in fact a sheaf, hence $A^\otimes k$ is well-defined. Throughout this article we will often make silent use of the sheaf $A^\otimes k$ by writing an element of the topological tensor product $A^\otimes k$ as a section $c(x_0, \ldots, x_{k-1})$, where $c \in A^\otimes k(X^k)$ and $x_0, \ldots, x_{k-1} \in X$.

Next we introduce a few objects often used in the sequel. First choose a smooth function $\varphi : \mathbb{R} \to [0, 1]$ with supp $\varphi = (-\infty, \frac{1}{2}]$ and $\varphi(s) = 1$ for $s \leq \frac{1}{2}$. For every $t > 0$ denote by $\varphi_t$ the rescaled function $\varphi_t(s) = \varphi(t^2 s)$, $s \in \mathbb{R}$. By $\Delta_k : \mathbb{R}^n \to \mathbb{R}^{kn}$ or briefly $\Delta$ we denote the diagonal map $x \mapsto (x, \ldots, x)$, and by $d_k : \mathbb{R}^{k+1} \to \mathbb{R}$ the distance to the diagonal:

$$d_k(x_0, x_1, \ldots, x_k) = \sqrt{d^2(x_0, x_1) + d^2(x_1, x_2) + \cdots + d^2(x_k, x_0)}.$$
Finally, let $U_{k,t} = \{(x_0, \ldots, x_k) \in X^{k+1} \mid d_k^t(x_0, \ldots, x_k) < t\}$ be the so-called $t$-neighborhood of the diagonal $\Delta_{k+1}(X)$.

In the following we want to show how the computation of the Hochschild homology of $A$ can be essentially reduced to the computation of the local Hochschild homology groups of $\mathcal{A}$. Since we consider the topological version of Hochschild homology theory, we will use in the definition of the Hochschild (co)chain complex the completed $\pi$-tensor product $\hat{\otimes}$ and the functor $\text{Hom}_A$ of continuous $A$-linear maps between $A$-Fréchet modules.

2.2 Now assume to be given an $A$-module sheaf $M$ of symmetric Fréchet modules and denote by $M = M(X)$ the Fréchet space of global sections. Denote by $C^*_a(A, M)$ the Hochschild chain complex with components $M \hat{\otimes} A^\otimes k$ and by $C^*(A, M)$ the Hochschild cochain complex, where $C^k(A, M)$ is given by $\text{Hom}_A(C_k(A, M), M)$. Denote by $b_k : C_k(A, M) \to C_{k-1}(A, M)$ the Hochschild boundary and by $b^k : C^k(A, M) \to C^{k+1}(A, M)$ the Hochschild coboundary. This means that $b_k = \sum_{i=0}^{k} (-1)^i \langle b_{k,i} \rangle$, and $b^k = \sum_{i=0}^{k-1} (-1)^i b^k_{i+1} : C_k \to C_{k-1}$, where $b_{k,i} : C_k \to C_{k-1}$ with $C_k := C_k(A, M)$ are the face maps which act on an element $c \in C_k$ as follows:

$$b_k, c(x_0, \ldots, x_{k-1}) = \begin{cases} c(x_0, x_1, \ldots, x_{k-1}), & \text{if } i = 0, \\ c(x_0, \ldots, x_i, \ldots, x_{k-1}), & \text{if } 1 \leq i < k, \\ c(x_0, \ldots, x_{k-1}, x_0), & \text{if } i = k. \end{cases}$$

Hereby, $x_0, \ldots, x_{k-1}$ are elements of $X$, and the fact has been used that $C_k$ can be identified with the space of global sections of the sheaf $A^{\otimes k+1}$. The Hochschild homology of $A$ with values in $M$ now is the homology $H_*^*(A, M)$ of the complex $(C^*_a(A, M), b)$. Likewise, the Hochschild cohomology $H^*(A, M)$ is given by the cohomology of the cochain complex $(C^*(A, M), b^*)$. As usual we will denote the homology space $H_*^*(A, A)$ briefly by $HH_*(A)$.

2.3 Remark In general, the particular choice of the topological tensor product used in the definition of the Hochschild homology of a topological algebra is crucial for the theory to work well (see Taylor [35] for general information on this topic and Brassemlet–Legrand–Telemann [4] for a particular example of a topological algebra, where the $\pi$-tensor product has to be used in the definition of the topological Hochschild complex). But since the Fréchet space $E^\infty(X)$ is nuclear, this question does not arise in the main application we are interested in, namely the definition and computation of the Hochschild homology of $E^\infty(X)$.

2.4 As $C_k(A, M)$ is the space of global sections of a sheaf, the notion of support of a chain $c \in C_k(A, M)$ makes sense: $\text{supp} c = \{ x \in X^{k+1} \mid c|_x \neq 0 \}$. To define the support of a cochain note first that $C^k$ inherits from $A$ the structure of a commutative algebra and secondly that $C^k$ acts on $C^\bullet(A, M)$ by $c(f \otimes h) = f(c \otimes h)$, where $c \otimes h \in C^k$ and $f \in C^\bullet(A, M)$. The support of $f \in C^\bullet(A, M)$ then is given by the complement of all $x \in X^{k+1}$ for which there exists an open neighborhood $U$ such that $c|_x = 0$ for all $c \in C_k$ with $\text{supp} c \subset U$.

The following two observations are fundamental for localization à la Teleman.

1. Localization on the first factor: For $a \in A$ the chain $a_k = a \otimes 1 \otimes \cdots \otimes 1 \in A^{\otimes [k+1]}$ acts in a natural way on $C_k(A, M)$ and $C^\bullet(A, M)$. Since $A$ is commutative and $M$ a symmetric $A$-module, the resulting endomorphisms give rise to chain maps $a_k : C_k(A, M) \to C_k(A, M)$ and $a^* : C^\bullet(A, M) \to C^\bullet(A, M)$ such that $\text{supp} a \subset (\text{supp} \times X^k) \cap \text{supp} c$ and $\text{supp} a^* f \subset (\text{supp} \times X^k) \cap \text{supp} f$.

2. Localization around the diagonal: For any $t > 0$ and $k \in \mathbb{N}$ let $\Psi_{k,t} : A^{\otimes [k+1]} \to A^{\otimes [k+1]}$ be defined by

$$\Psi_{k,t}(x_0, \cdots, x_k) = \prod_{j=0}^{k} a \langle d^t(x_j, x_{j+1}) \rangle, \quad \text{where } x_{k+1} := x_0. \quad (2.1)$$
Then the action by $\Psi_{k,t}$ give rise to chain maps $\Psi_{\bullet,t} : C_\bullet(A, M) \to C_\bullet(A, M)$ and $\Psi^\bullet_t : C^\bullet(A, M) \to C^\bullet(A, M)$ such that $\text{supp}(\Psi_{\bullet,t}c) \subset U_{k,t}$ and $\text{supp}(\Psi^\bullet_t f) \subset U_{k,t}$.

We now will construct a homotopy operator between the identity and $\Psi_{\bullet,t}$ resp. $\Psi^\bullet_t$. To this end define $A$-module maps $\eta_{k,i,t} : C_k \to C_{k+1}$ for every integer $k \geq -1$ and $i = 1, \ldots, k + 2$ by

$$\eta_{k,i,t}(c)(x_0, \ldots, x_{k+1}) = \begin{cases} 
\Psi_{k+1,i,t}(x_0, \ldots, x_{k+1}) c(x_0, \ldots, x_{i-1}, x_i, \ldots, x_{k+1}) & \text{for } i < k + 1, \\
\Psi_{k+1,i+1,t}(x_0, \ldots, x_{k+1}) c(x_0, \ldots, x_k) & \text{for } i = k + 1, \\
0 & \text{for } i = k + 2 
\end{cases}$$

(2.2)

where $c \in C_k$, $x_0, \ldots, x_{k+1} \in X$ and, using $x_{k+2} := x_0$, the functions $\Psi_{k+1,i,t}$, $i = 1, \ldots, k + 2$ are given by $\Psi_{k+1,i,t}(x_0, \ldots, x_{k+1}) = \prod_{j=0}^{i-1} \partial_d (d^2 (x_j, x_{j+1}))$. For $i = 2, \ldots, k - 1$ one then computes

$$\begin{align*}
(b_{k+1} + \eta_{k,i-1,t} + \eta_{k-1,i,t}) c(x_0, \ldots, x_k) &= 
\Psi_{k,i-1,t}(-1)^{i-1} c(x_0, \ldots, x_k) + \Psi_{k,i-1,t} \sum_{j=0}^{i-2} (-1)^j c(x_0, \ldots, x_{j-1}, x_j, \ldots, x_{i-2}, x_i, \ldots, x_k) + \\
&\quad (-1)^j \Psi_{k,i} c(x_0, \ldots, x_k) + \Psi_{k,i} \sum_{j=0}^{i-1} (-1)^j c(x_0, \ldots, x_j, x_{j+1} \ldots, x_{i-1}, x_i, \ldots, x_k) 
\end{align*}$$

(2.3)

and for the two remaining cases $i = 1$ and $i = k + 1$

$$\begin{align*}
((b_{k+1} + \eta_{k,1,t} + \eta_{k-1,1,t}) b_k c)(x_0, \ldots, x_k) &= 
\Psi_{k,1} c(x_0, x_k) - \Psi_{k,1} c(x_0, x_2, \ldots, x_k), \\
((b_{k+1} + \eta_{k,k+1,t} + \eta_{k-1,k+1,t}) b_k c)(x_0, \ldots, x_k) &= 
\Psi_{k,k+1}(-1)^k c(x_0, \ldots, x_k) + \Psi_{k,k+1} \sum_{j=0}^{k-1} (-1)^j c(x_0, \ldots, x_j, x_{j+1}, \ldots, x_{k-1}) + \\
&\quad (-1)^k \Psi_{k,k+1} c(x_0, \ldots, x_k). 
\end{align*}$$

(2.4)

(2.5)

Note that by definition every $\eta_{k,i,t}$ is a morphism of $A$-modules, which means that one can apply the functors $M \otimes -$ and $\text{Hom}_A(-, M)$ to these morphisms. By the computations above we thus obtain our first result.

**2.5 Proposition.** The map

$$H_{k,t} = \sum_{i=1}^{k+1} (-1)^{i+1} (\eta_{k,i,t}) : C_k(A, M) \to C_{k+1}(A, M)$$

resp.

$$H^k_t = \sum_{i=1}^{k} (-1)^{i+1} \eta^k_{i-i,t} : C^k(A, M) \to C^{k+1}(A, M)$$

gives rise to a homotopy between the identity and the localization morphism $\Psi_{\bullet,t}$ resp. $\Psi^\bullet_t$. More precisely,

$$\eta_{k,1,t} c = c - \Psi_{k,t} c \quad \text{for all } c \in C_k(A, M) \text{ and}$$

$$\eta^{k-1}_{k,t} f = f - \Psi_{k,1,t} f \quad \text{for all } f \in C^k(A, M).$$

(2.6)

(2.7)
2.6 Remark The localization morphisms given in Teleman, which form the analogue of the
morphisms \( \eta_{k,j} \) defined above, are not \( A \)-linear, hence can be used only for localization of
the complex \( C^*_n(A, M) \) but not for the localization of Hochschild cohomology or of Hochschild
homology with values in an arbitrary module \( M \).

Following Teleman [36] we denote by \( C^k(A, M) \subseteq C_k(A, M) \) resp. \( C^*_n(A, M) \subseteq C^*_n(A, M) \) the
space of Hochschild (co)chains with support disjoint from \( U_{k,j} \) and by \( C^k(A, M) \) resp. \( C^*_n(A, M) \)
the inductive limit \( \bigcup_{k \geq 0} C^k(A, M) \) resp. \( \bigcup_{k \geq 0} C^*_n(A, M) \). Finally denote by \( \mathcal{H}_n \) the sheaf associated
to the presheaf with sectional spaces \( \mathcal{H}_n(A(V), \mathcal{M}(V)) \), where \( V \) runs through the open
subsets of \( X \). The proposition then implies the following results.

2.7 Corollary The complexes \( C^*_n(A, M) \) and \( C^*_n(A, M) \) are acyclic.

2.8 Corollary The Hochschild homology of \( A \) coincides with the global sections of \( \mathcal{H}_n \) that
means \( H_n(A, M) = \mathcal{H}_n(X) \).

3 Peetre-like theorems

3.1 In this section we will show that a continuous local operator acting on Whitney functions
of class \( \mathcal{C}^\infty \) and with values in \( \mathcal{E}^m \), \( m \in \mathbb{N} \) is locally given by a differential operator. Thus we
obtain a generalization of Peetre’s theorem [32] which says that every local operator acting on
the algebra of smooth functions on \( \mathbb{R}^n \) has to be a differential operator, locally.

Recall that a \( k \)-linear operator \( D : \mathcal{E}^m(X) \times \ldots \times \mathcal{E}^m(X) \to \mathcal{E}^r(X) \) (with \( m, r \in \mathbb{N} \cup \{ \infty \} \))
is said to be \textit{local} if for all \( F_1, \ldots, F_k \in \mathcal{E}^m(X) \) and every \( x \in X \) the value \( D(F_1, \ldots, F_k)(x) \in \mathcal{E}^r([x]) \) depends only on the germs \( [F_1]_x, \ldots, [F_k]_x \). In other words this means that \( D \) can be
regarded as a morphism of sheaves \( \Delta^*_r(\mathcal{E}^m_X \otimes \ldots \otimes \mathcal{E}^m_X) \to \mathcal{E}^r_X \).

The following result forms the basic tool for our proof of a Peetre-like theorem for Whitney
functions.

3.2 Proposition Let \( E \) be a Banach space with norm \( \| \cdot \| \) and \( W \stackrel{q}{\to} V \to 0 \) an exact sequence
of Fréchet spaces and continuous linear maps such that the topology of \( W \) is given by a countable
family of norms \( \| \cdot \|_n, n \in \mathbb{N} \). Then for every continuous \( k \)-linear operator \( f : V \times \ldots \times V \to E \)
there exists a constant \( C > 0 \) and a natural number \( r \) such that

\[
\| f(v_1, \ldots, v_k) \| \leq C \| v_1 \|_r \cdots \| v_k \|_r \quad \text{for all } v_1, \ldots, v_k \in V,
\]

where \( \| \cdot \|_r \) is the seminorm \( \| v \|_r = \inf_{n \leq r} \| v \|_n \). sup_{l \leq r} \| u \|_l \).

Proof: Let us first consider the case, where \( W = V \) and \( q \) is the identity map. Assume that
in this situation the claim does not hold. Then one can find \( v_l \in E \) for \( i = 1, \ldots, k \) and \( l \in \mathbb{N} \)
such that

\[
\| f(v_{i1}, \ldots, v_{il}) \| > l \| v_{i1} \|_l \cdots \| v_{il} \|_l \quad \text{for all } l \in \mathbb{N}.
\]

Let \( w_l = \frac{1}{\sqrt{l!}} v_{il} \). Then \( \lim_{l \to \infty} (w_1, \ldots, w_k) = 0 \), but \( \| f(w_1, \ldots, w_k) \| \geq 1 \) for all \( l \in \mathbb{N} \),
which is a contradiction to the continuity of \( f \). Hence the claim must be true for \( W = V \) and
\( q = id \).

Let us now consider the general case of an exact sequence \( W \stackrel{q}{\to} V \to 0 \), where the topology
of \( W \) is given by a countable family of norms. Define \( F : W \times \ldots \times W \to E \) by

\[
F(w_1, \ldots, w_k) = \sum \text{ terms involving } w_{il} \text{ for } i = 1, \ldots, k, \quad l = 1, \ldots, n.
\]
\[ f(q(w_1), \ldots, q(w_k)), w_i \in W. \text{ By the result proven so far one concludes that there exists a } C > 0 \text{ and a natural } r \text{ such that } \]
\[ \|F(w_1, \ldots, w_k)\| \leq C \sup_{t \leq r} \|w_1\| \ldots \sup_{t \leq r} \|w_k\| \text{ for all } w_1, \ldots, w_k \in V. \]

But this entails
\[ \|f(v_1, \ldots, v_k)\| = \inf_{w_1 \in q^{-1}(v_1)} \ldots \inf_{w_k \in q^{-1}(v_k)} \|F(w_1, \ldots, w_k)\| \leq C \|v_1\| \ldots \|v_k\|, \]

hence the claim follows. \(\square\)

3.3 **Peetre’s theorem for Whitney functions** Let \(X\) be a regular locally closed subset of \(\mathbb{R}^n, m \in \mathbb{N}\) and \(D : \mathcal{E}^\infty(X) \times \cdots \times \mathcal{E}^\infty(X) \to \mathcal{E}^m(X)\) a \(k\)-linear continuous and local operator. Then for every compact \(K \subset X\) there exists a natural number \(r\) such that for all Whitney functions \(F_1, G_1, \ldots, F_k, G_k \in \mathcal{E}^\infty(X)\) and every point \(x \in K\) the relation \(J'F_i(x) = J'G_i(x)\) for \(i = 1, \ldots, k\) implies \(D(F_1, \ldots, F_k)_{|\mathbb{P}} = D(G_1, \ldots, G_k)_{|\mathbb{P}}\).

**Proof:** By a straightforward partition of unity argument one can reduce the claim to the case of compact \(X\). So let us assume that \(X\) is compact and \(p\)-regular for some positive integer \(p\).

Then \(\mathcal{E}^m(X)\) is a Banach space with norm \(\| \cdot \|_m\), and \(\mathcal{E}^\infty(X)\) is Fréchet with topology defined by the seminorms \(\| \cdot \|_l, l \in \mathbb{N}\). Choose a compact cube \(Q\) such that \(X\) lies in the interior of \(Q\). Then the sequence \(\mathcal{E}^\infty(Q) \to \mathcal{E}^\infty(X) \to 0\) is exact by Whitney’s extension theorem and the topology of \(\mathcal{E}^\infty(Q)\) is generated by the norms \(\| \cdot \|_l, l \in \mathbb{N}\). Since the sequence \(\mathcal{E}^l(Q) \to \mathcal{E}^l(X) \to 0\) is exact and the topology of \(\mathcal{E}^l(X)\) is defined by the norm \(\| \cdot \|_l\), Proposition 3.2 entails that the operator \(D\) extends to a continuous \(k\)-linear map \(D : \mathcal{E}^\ell(X) \times \cdots \times \mathcal{E}^\ell(X) \to \mathcal{E}^m(X)\), if \(r\) is chosen sufficiently large. Now assume that \(F_1, G_1 \in \mathcal{E}^\infty(X)\) are Whitney functions with \(J\, P F_i(x) = J\, P G_i(x)\) for \(i = 1, \ldots, k\). According to 1.4 we can then choose \(d_i \in \mathcal{E}^\infty(X)\) for \(i = 1, \ldots, k\) and \(l \in \mathbb{N}\) such that \(d_i\) vanishes in a neighborhood of \(x\) and such that \(\|F_i - G_i - d_i\|_{\ell\, p} < 2^{-l}\). But then \(G_i + d_i\) converges to \(F_i\) in \(\mathcal{E}^\ell(X)\), hence by continuity \(\lim_{t \to -\infty} D(G_{1, d_1}, G_{2, d_2}, \ldots, G_k + d_k)_{|\mathbb{P}} = D(F_1, F_2, \ldots, F_k)_{|\mathbb{P}}\). On the other hand we have \(D(G_{1, d_1}, G_{2, d_2}, \ldots, G_k + d_k)_{|\mathbb{P}} = D(G_{1, d_1}, G_{2, d_2}, \ldots, G_k)_{|\mathbb{P}}\) for all \(l\) by the locality of \(D\). Hence the claim follows. \(\square\)

3.4 **Remark** In case \(m = \infty\), a continuous and local operator \(D : \mathcal{E}^\infty(X) \to \mathcal{E}^m(X)\) need not be a differential operator, as the following example shows. Let \(X\) be the \(x_2\)-axis of \(\mathbb{R}^2\) and let \(D\) be the operator \(D = \sum_{k \in \mathbb{N}} d_k D_{x_2}^k\) where \(d_k = J_{x_2}^{\infty} x_2^k\). Then \(D\) is continuous and local, but \(DF\) depends over every compact set on infinitely many jets of the argument \(F\).

The following theorem will not be needed in the rest of this work but appears to be of independent interest. Since the proof goes along the same lines like the one for Peetre’s theorem for Whitney functions, we leave it to the reader.

3.5 **Peetre’s theorem for \(G\)-invariant functions** Let \(G\) be a compact Lie group acting by diffeomorphisms on a smooth manifold \(M\) and let \(E, E_1, \ldots, E_k\) be smooth vector bundles over \(M\) with an equivariant \(G\)-action. Let \(D : \Gamma^\infty(E_1)^G \times \cdots \times \Gamma^\infty(E_k)^G \to \Gamma^\infty(E)^G\) be a \(k\)-linear continuous and local operator. Then for every compact set \(K \subset M\) there exists a natural \(r\) such that for all sections \(s_1, t_1, \ldots, s_k, t_k \in \Gamma^\infty(E_i)^G\) and every point \(x \in K\) the relation \(J' s_i(x) = J' t_i(x)\) for \(i = 1, \ldots, k\) implies \(D(s_1, \ldots, s_k)(x) = D(t_1, \ldots, t_k)(x)\).
4 Hochschild homology of Whitney functions

4.1 Our next goal is to apply the localization techniques established in Section 2 to the computation of the Hochschild homology of the algebra $\mathcal{E}^\infty(X)$ of Whitney functions on $X$. Note that this algebra is the space of global sections of the sheaf $\mathcal{E}^\infty_X$, hence the premises of Section 2 are satisfied. Throughout this section we will assume that $X$ is a regular subset of $\mathbb{R}^n$ and that $X$ has regularly situated diagonals. By the latter we mean that $X^k$ and $\Delta_k(\mathbb{R}^n)$ are regularly situated subsets of $\mathbb{R}^{kn}$ for every $k \in \mathbb{N}$. Denote by $C_\bullet$ the complex $C_\bullet(\mathcal{E}^\infty(X), \mathcal{E}^\infty(X))$. By Proposition 1.9 we then have $C_k = \mathcal{E}^\infty(X^k+1)$. Now let $J_\bullet \subset C_\bullet$ be the subspace of chains infinitely flat on the diagonal that means that $J_k = \mathcal{J}^\infty(\Delta_0(X); X^{k+1})$. Obviously, every face map $b_{k,j}$ maps $J_k$ to $J_{k-1}$, hence $J_\bullet$ is a subcomplex of $C_\bullet$.

4.2 Proposition Assume that $M$ is a finitely generated projective $\mathcal{E}^\infty_X$-module sheaf of symmetric Fréchet modules, $M$ the $\mathcal{E}^\infty(X)$-module $M(X)$ and $m \in \mathbb{N} \cup \{\infty\}$. Then the complexes $J_\bullet \otimes_{\mathcal{E}^\infty(X)} M$ and $\text{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, M \otimes_{\mathcal{E}^\infty(X)} \mathcal{E}^m(X))$ are acyclic.

Before we can prove the proposition we have to set up a few preliminaries. First let us denote for $i = 1, \ldots, k+1$ by $e_{k,j} : C_k \to C_{k+1}$ the extension morphism such that $(e_{k,j}(x_0, \ldots, x_{k+1}) = e(x_0, \ldots, x_{i-1}, x_i, \ldots, x_{k+1})$. Clearly, $e_{k,j}$ is continuous and satisfies $e_{k,j}(J_k) \subset J_{k+1}$. Secondly recall the definition of the functions $\Psi_{k,j}$ and $\Phi_{k,j}$ in 2.4. The following two lemmas now hold true.

4.3 Lemma Let $\varphi_{k,t} \in C^\infty(\mathbb{R}^{k+1})$ be one of the functions $\varphi_{k,t}$ or $\varphi_{k,t}(\partial_t \Phi_{k-1,t})$, where $\varepsilon > 0$, $t > 0$ and $i = 1, \ldots, k$. Then for every compact set $K \subset \mathbb{R}^{k+1}$, $T > 0$ and $\alpha \in \mathbb{N}^{k+1}$ there exists a constant $C > 0$ and an $m \in \mathbb{N}$ such that

$$|D^\alpha \varphi_{k,t}(x)| \leq C \left( \frac{t}{d(x, \Delta_0(\mathbb{R}^n))^m} \right) \quad \text{for all } x \in K \setminus \Delta_0(\mathbb{R}^n) \text{ and } t \in (0, T).$$

(4.1)

Proof: If $\varphi_{k,t} = \Phi_{k,t}$ and $\alpha = 0$ the estimate (4.1) is obvious since $\Phi_{k,t}(x)$ is bounded as a function of $x$ and $t$. Now assume $|\alpha| \geq 1$ and compute

$$(D^\alpha \Phi_{k,t})(x) = \sum_{|\beta| = |\alpha|} \frac{k!}{\beta!} \left( \frac{d^\beta(x_j, x_{j+1})}{t} \right) d_{\beta,\alpha}(x_j, x_{j+1}),$$

(4.2)

where $x = (x_0, \ldots, x_k)$, $x_{k+1} := x_0$ and the functions $d_{\beta,\alpha}$ are polynomials in the derivatives of the euclidean distance, so in particular bounded on compact sets. Now, by definition of the function $\varphi_{k,t}$ we have $\varphi_{t}(s) = 0$ for $0 < s \leq \frac{t}{2}$ and $\varphi_{t}(s) = 0$ for $s > t$, hence

$$(D^\alpha \varphi_{k,t})(x) = 0 \quad \text{for all } x \in U_{k,t} \text{ and all } x \in \mathbb{R}^{k+1} \setminus U_{k,t}.$$ (4.3)

On the other hand by Eq. (4.2) there exists a constants $C' > 0$ such that for all $t \in (0, T]$ and $x \in (K \cap U_{k,t} \setminus U_{k,t})$,

$$|D^\alpha \varphi_{k,t}(x)| \leq C' \left( \frac{1}{d(x_0, \ldots, x_k)^{k+1}} \right) \left( \frac{t}{d(x_0, \ldots, x_k)^{k+1/2}} \right).$$ (4.4)

But the estimates (4.3) and (4.4) imply that (4.1) holds true for appropriate $C$ and $m$, hence the claim follows for $\Phi_{k,t}$. By a similar argument one shows the claim for the functions $\Phi_{k,t} e_{k-1,t}(\partial_t \Phi_{k-1,t})$. 

\hfill \Box

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4.4 Lemma Each one of the mappings
\[
\mu_k : J_k \times [0,1] \rightarrow J_k, \quad (c, t) \mapsto \begin{cases} 
\Psi_{k,t} c & \text{if } t > 0, \\
0 & \text{if } t = 0,
\end{cases}
\]  
\[
(4.5)
\]
\[
\mu_{k,j} : J_k \times [0,1] \rightarrow J_k, \quad (c, t) \mapsto \begin{cases} 
\Psi_{k,j,t} c + \partial_t \Psi_{k-1,j,t} c & \text{if } t > 0, \\
0 & \text{if } t = 0,
\end{cases}
\]  
\[
(4.6)
\]
is continuous.

Proof: As \(X^{k+1} \) and \(\Delta(\mathbb{R}^n)\) are regularly situated there exists a smooth function \(\tilde{c} \in \mathcal{C}^\infty(\Delta(\mathbb{R}^n); \mathbb{R}^{k+1})\) such that the image of \(\tilde{c}\) in \(E^\infty(X^{k+1})\) equals \(c\). By Taylor’s expansion one then concludes that for every compact set \(K \subset \mathbb{R}^{k+1}, \alpha \in \mathbb{N}^{k+1} n\) and \(N \in \mathbb{N}\) there exists a second compact set \(L \subset \mathbb{R}^{k+1} n\) and a constant \(C_{k,N}\) such that
\[
|D^\alpha \tilde{c}(x)| \leq C_{k,N} (d(x, \Delta(\mathbb{R}^n)))^N |A_{k+1}^x| \quad \text{for all } x \in K.
\]  
\[
(4.7)
\]
By Leibniz rule and Lemma 4.3 the continuity of \(\mu_{k,j}\) follows immediately. Analogously, one shows the continuity of \(\mu_k\).

Proof of Proposition 4.2: By the assumptions on \(\mathcal{M}\) it suffices to show that the complexes \(J_*\) and \(\text{Hom}_{E^\infty(X)}(J_*, E^m(X))\) are acyclic. To prove the claim in the homology case we will construct a (continuous) homotopy \(K_k : J_k \rightarrow J_{k+1}\) such that
\[
(b_{k+1}^{-1} + K_{k+1}^{-1})(b_k)c = \Psi_{k+1,c} c \quad \text{for all } c \in J_k.
\]  
\[
(4.8)
\]
By Prop. 2.5 the complex \(J_*\) then has to be acyclic. Using the homotopy \(H_{k,*}\) of Prop. 2.5 we first define \(K_{k,*} : C_k \rightarrow C_{k+1}\) by
\[
K_{k,*} c = \int_t^1 H_{k+1}(\partial_t \Psi_{k,t} c) \, ds, \quad c \in C_k.
\]
Since \(\Psi_{k,*}\) is a chain map, we obtain by Eq. (2.6)
\[
(b_{k+1} + K_{k+1}^{-1} + b_k) c = \int_t^1 b_{k+1} H_k(\partial_t \Psi_{k,t} c) + H_{k+1}(\partial_t \Psi_{k,t} c) + b_k(\partial_t \Psi_{k,t} c) \, ds
\]  
\[= \int_t^1 \partial_t \Psi_{k,t} c - \Psi_{k+1,t} c + \partial_t \Psi_{k,t} c \, ds = \int_t^1 \partial_t \Psi_{k,t} c \, ds = \Psi_{k+1} c - \Psi_{k} c.
\]  
\[
(4.9)
\]
Hereby we have used the relation \(\Psi_{k,t} c = \Psi_{k+1,t} c = 0\) which follows from the fact that \(\partial_t \Psi_{k,t} c = 0\) vanishes on \(U_{k+1}\) and that \(\supp \Psi_{k,t} c \subset U_k\). Let us now assume that \(c \in J_k\). Since
\[
K_{k,*} c = \sum_{i=1}^{k+1} (-1)^{i+1} \int_t^1 \Psi_{k+1,i}(\partial_t \Psi_{k,i} c) e_k j(c) \, ds = \sum_{i=1}^{k+1} (-1)^{i+1} \int_t^1 \mu_{k+1,i}(c, s) \, ds
\]
and \(e_k j(c) \in J_{k+1}\) one concludes by Lemma 4.4 that the map \(K_k : J_k \rightarrow J_{k+1}, c \mapsto \lim_{t \rightarrow 0} K_{k,t} c\) is well-defined and continuous. So we can pass to the limit \(t \rightarrow 0\) in Eq. (4.9) and obtain (4.8), because \(\lim_{t \rightarrow 0} \Psi_{k,t} c = 0\) by Lemma 4.4.

Since every \(K_k\) is continuous and \(E^\infty(X)\)-linear, the map
\[
K^k : \text{Hom}_{E^\infty(X)}(J_k, E^m(X)) \rightarrow \text{Hom}_{E^\infty(X)}(J_{k+1}, E^m(X)), \quad f \mapsto f \circ K_{k+1}
\]
gives rise to a homotopy such that
\[(
\delta^{k-1}K_k + K^{k+1}\delta^k \right) f = \Psi_{k,1} f \quad \text{for all } f \in \text{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X)).\] (4.10)

Hence the complex $\text{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X))$ is acyclic as well. □

Consider now the following short exact sequence of complexes:
\[0 \rightarrow J_k \rightarrow C_k \rightarrow E_k \rightarrow 0,\] (4.11)

where $E_k = C_k / J_k$. As a consequence of the proposition the homology of the complexes $C_k$ and $E_k$ have to coincide. Following Teleman [36] we call $E_k$ the diagonal complex. By Whitney’s extension theorem its $k$th component is given by $E_k = \mathcal{E}^\infty(\Delta_{k+1}(X))$. Since $M$ is a finitely generated projective $\mathcal{E}^\infty(X)$-module, the tensor product of $M$ with the above sequence remains exact. We thus obtain the following result.

**4.5 Corollary** The Hochschild homology $H_k(\mathcal{E}^\infty(X), M)$ is naturally isomorphic to the homology of the tensor product of the diagonal complex and $M$, i.e., to the homology of the complex $E_k \otimes M$.

The following proposition can be interpreted as a kind of Borel lemma with parameters.

**4.6 Proposition** There is a canonical topologically linear isomorphism of $\mathcal{E}^\infty(X)$-modules
\[j^\infty_\Delta : \mathcal{E}^\infty(\Delta_{k+1}(X)) \rightarrow \mathcal{E}^\infty(X) \otimes_\pi \mathcal{F}^\infty(\mathbb{R}^n), \quad F \mapsto \sum_{\alpha_1, \ldots, \alpha_k \in \mathbb{N}^n} F_{\alpha_1, \ldots, \alpha_k} y_1^{\alpha_1} \cdots y_k^{\alpha_k}, \quad F_{\alpha_1, \ldots, \alpha_k} = \Delta_{k+1}(D^{\alpha_1} \cdots D^{\alpha_k} F),\]

where $\mathcal{F}^\infty(\mathbb{R}^n)$ denotes the formal power series algebra in $n$ (real) indeterminates and, for every $i = 1, \ldots, k$, the symbols $y_i = (y_{i,1}, \ldots, y_{i,n})$ denote indeterminates.

**Proof:** Clearly, the map $j^\infty_\Delta$ is continuous and injective. By an immediate computation one checks that $j^\infty_\Delta$ is a morphism of $\mathcal{E}^\infty(X)$-modules. So it remains to prove surjectivity; since $\mathcal{E}^\infty(X) \otimes_\pi \mathcal{F}^\infty(\mathbb{R}^n)$ is a Fréchet space the claim then follows by the open mapping theorem.

To prove surjectivity we use an argument similar to the one used in the proof of Borel’s lemma. For simplicity we assume that $X$ is compact; the general case can be deduced from that by a partition of unity argument. Given a series $\sum F_{\alpha_1, \ldots, \alpha_k} y_1^{\alpha_1} \cdots y_k^{\alpha_k}$ let us define a Whitney function $F \in \mathcal{E}^\infty(\Delta_{k+1}(X))$ by
\[F(y_0, x_1, \ldots, x_k) = \sum_{\alpha_1, \ldots, \alpha_k \in \mathbb{N}^n} \frac{F_{\alpha_1, \ldots, \alpha_k}}{\alpha_1! \cdots \alpha_k!} \mu \left( A_{\alpha_1, \ldots, \alpha_k} d_k(x_0, \ldots, x_k) \right) (x_1 - x_0)^{\alpha_1} \cdots (x_k - x_0)^{\alpha_k},\]

where $\mu$ is a $C^\infty$-function whose value is 1 in a neighborhood of 0 and whose support is contained in $[-1, 1]$, $d_k(x_0, \ldots, x_k)$ is the distance to the diagonal previously defined and
\[A_{\alpha_1, \ldots, \alpha_k} = \sup \left( 1, \frac{\sup_{|\beta_1| \leq \alpha_1, \ldots, |\beta_k| \leq \alpha_k, |\alpha| \leq |\beta_1| + \cdots + |\beta_k|} |F_{\beta_1, \ldots, \beta_k}| \right),\]

The function $\mu(A_i d_k(x_0, \ldots, x_k))$ is $C^\infty$ because $\mu(t) = 1$ near $t = 0$. It is straightforward to check that the above series converges to an element $F \in \mathcal{E}^\infty(\Delta_{k+1}(X))$ indeed and that this $F$ satisfies $j^\infty_\Delta(F) = \sum F_{\alpha_1, \ldots, \alpha_k} y_1^{\alpha_1} \cdots y_k^{\alpha_k}$. □
4.7 Before we formulate a Hochschild-Kostant-Rosenberg type theorem for Whitney functions, let us briefly explain what we mean by the space of Whitney differential forms. Recall that the space of Kähler differentials of \( \mathcal{E}^\infty(X) \) is the (up to isomorphism uniquely defined) \( \mathcal{E}^\infty(X) \)-module \( \Omega^k_{\mathcal{E}^\infty}(X) \) having a derivation \( d : \mathcal{E}^\infty(X) \to \Omega^1_{\mathcal{E}^\infty}(X) \) which is universal with respect to derivations \( \delta : \mathcal{E}^\infty(X) \to M \), where \( M \) is an \( \mathcal{E}^\infty(X) \)-module (see MATSUMURA [29, Ch. 10]). Given an open \( U \subset \mathbb{R}^n \) with \( X \subset U \) closed, the space of smooth differential 1-forms over \( U \) and \( \Omega^k_{\mathcal{E}^\infty}(X) \) are related by the following exact sequence for Kähler differentials [29, Thm. 58]:

\[
\mathcal{J}^\infty(X; U)/\left( \mathcal{J}^\infty(X; U) \right)^2 \cong \mathcal{E}^\infty(X) \otimes_{\mathcal{E}^\infty(U)} \Omega^1(U) \to \Omega^k_{\mathcal{E}^\infty}(X) \to 0.
\]

Since \( \mathcal{J}^\infty(X; U) = \left( \mathcal{J}^\infty(X; U) \right)^2 \) this entails that there is a canonical isomorphism

\[
\Omega^k_{\mathcal{E}^\infty}(X) \cong \mathcal{E}^\infty(X) \otimes_{\mathcal{E}^\infty(U)} \Omega^k(U).
\] (4.12)

Hereby, \( \Omega^k_{\mathcal{E}^\infty}(X) \) is the exterior power \( \Lambda^k \Omega^1_{\mathcal{E}^\infty}(X) \) and will be called the space of Whitney differential forms over \( X \). The differential \( d : \mathcal{E}^\infty(X) \to \Omega^1_{\mathcal{E}^\infty}(X) \) extends naturally to \( \Omega^k_{\mathcal{E}^\infty}(X) \) and gives rise to the Whitney-de Rham complex:

\[
0 \to \mathcal{E}^\infty(X) \xrightarrow{d} \Omega^1_{\mathcal{E}^\infty}(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k_{\mathcal{E}^\infty}(X) \xrightarrow{d} \cdots .
\]

Its cohomology \( H^\bullet_{\mathcal{E}^\infty}(X) \) will be called the Whitney-de Rham cohomology of \( X \) and will be computed for subanalytic \( X \) later in this work. Clearly, the spaces \( \Omega^k_{\mathcal{E}^\infty}(V) \), where \( V \) runs through the open subsets of \( X \), are the sectional spaces of a fine sheaf over \( X \) which we denote by \( \mathcal{E}^\infty_{\mathcal{E}^\infty}(X) \). We thus obtain a sheaf complex and, taking global sections, again the Whitney-de Rham complex.

4.8 Theorem Let \( X \subset \mathbb{R}^n \) be a regular subset with regularly situated diagonals, and \( m \in \mathbb{N} \cup \{ \infty \} \). Assume that \( \mathcal{M} \) is a finitely generated projective \( \mathcal{E}^\infty_{\mathcal{E}^\infty} \)-module sheaf of symmetric Fréchet modules and that \( M \) is the \( \mathcal{E}^\infty(X) \)-module \( \mathcal{M}(X) \). The Hochschild homology of \( \mathcal{E}^\infty(X) \) with values in \( M \) then is given by

\[
H_\mathcal{M}(\mathcal{E}^\infty(X), M) = \Omega^k_{\mathcal{E}^\infty}(X) \otimes_{\mathcal{E}^\infty(U)} \mathcal{M} \cong M \otimes \Lambda^k(T^*_m \mathbb{R}^n).
\] (4.13)

4.9 Remark Since a subanalytic set \( X \subset \mathbb{R}^n \) is always regular and possesses regularly situated diagonals (the diagonal is obviously subanalytic and two subanalytic sets are always regularly situated), the statement of the theorem holds in particular for subanalytic sets.

Proof: Since \( \mathcal{M} \) is finitely generated projective we can reduce the claim to the case \( \mathcal{M} = \mathcal{E}^\infty_{\mathcal{E}^\infty} \). We will present two ways to prove the result for this case; both of them show that

\[
HH_\mathcal{M}(\mathcal{E}^\infty(X)) \cong \mathcal{E}^\infty(X) \otimes_{\mathcal{E}^\infty(U)} \Omega^k(U).
\]

The first proof follows TELEMAN’s procedure [37]. The homology of the diagonal complex \( E^n \) coincides with the homology of the non degenerated complex \( E^n \), i.e. the complex generated by non degenerated monomials (non lacunary in the terminology of [37]). The non degenerated complex \( E^n \) is itself identified with the direct product of its components \( E^k \) where \( E^k \) is the subcomplex of \( E^n \) generated by all monomials of (total) degree \( r \). Proposition 4.6 shows that the elements of \( E^n \) can be interpreted as infinite jets vanishing at the origin, regarding the variables \( y_1, \ldots, y_k \), and with coefficients in \( \mathcal{E}^\infty(X) \). An argument similar to TELEMAN’s spectral sequence computation [37], but here with coefficients in \( \mathcal{E}^\infty(X) \), proves that the homology of \( E^n \) is \( \mathcal{E}^\infty(X) \otimes \Lambda^k(T^*_m \mathbb{R}^n) \) and the result.

The second way to prove the result is to consider the isomorphism \( j^\infty_{\mathcal{E}^\infty} \) of Proposition 4.6 and carry the boundary map \( b_k \) from \( E_k \) to \( \mathcal{E}^\infty(X) \otimes_{\mathcal{E}^\infty} \mathcal{E}^\infty(\mathbb{R}^{nk}) \) such that \( b_k \circ j^\infty_{\mathcal{E}^\infty}(\mathcal{E}^\infty(\mathbb{R}^{nk})) = j^\infty_{\mathcal{E}^\infty}(b_k \mathcal{E}^\infty(\mathbb{R}^{nk})) \) for all
\[ F \in \mathcal{E}^\infty(\Delta_{k+1}(X)). \] Writing an element \( \sigma \in \mathcal{E}^\infty(X) \otimes \mathcal{F}^\infty(\mathbb{R}^n) \) as a section \( \sigma(x_0, y_1, \ldots, y_k) \) of the module sheaf \( \mathcal{E}^\infty_X \otimes \mathcal{F}^\infty(\mathbb{R}^n) \) one now computes

\[
b_k \sigma(x_0, y_1, \ldots, y_k) = \sigma(x_0, 0, y_1, \ldots, y_k) + \sum_{i=1}^{k-1} \sigma(x_0, y_1, \ldots, y_i, y_i, \ldots, y_k) + (-1)^k \sigma(x_0, y_1, \ldots, y_k, 0).
\]

This shows that the homology of the complex \( (\mathcal{E}^\infty_X \otimes \mathcal{F}^\infty(\mathbb{R}^n), b) \) is nothing else but the Hochschild homology \( H_*(\mathcal{E}^\infty(\mathbb{R}^n), \mathcal{E}^\infty(X)) \), if \( \mathcal{E}^\infty(X) \) is given the \( \mathcal{F}^\infty(\mathbb{R}^n) \)-module structure such that \( y_j F = 0 \) for each of the indeterminates \( y_1, \ldots, y_n \) and for every \( F \in \mathcal{E}^\infty(X) \). Now, since Hochschild homology can be interpreted as a derived functor homology (see [25, Prop. 1.1.13] in the algebraic and [34, Sec. 6.3] in the topological case), we can use the Koszul resolution for the computation of \( H_*(\mathcal{E}^\infty(\mathbb{R}^n), \mathcal{E}^\infty(X)) \), that means the following projective resolution

\[
K_*: 0 \leftarrow \mathcal{F}^\infty(\mathbb{R}^n) \leftarrow \mathcal{F}^\infty(\mathbb{R}^n) \otimes \Lambda^1(T^*_{0}(\mathbb{R}^n)) \leftarrow \cdots \leftarrow \mathcal{F}^\infty(\mathbb{R}^n) \otimes \Lambda^n(T^*_{0}(\mathbb{R}^n)) \leftarrow 0,
\]

where \( i_X \) denotes the insertion of the radial (formal) vector field \( X = y_1 \frac{\partial}{\partial y_1} + \cdots + y_n \frac{\partial}{\partial y_n} \) in an alternating form. Hence

\[
H_*(\mathcal{F}^\infty(\mathbb{R}^n), \mathcal{E}^\infty(X)) = H_*(K_* \otimes \mathcal{F}^\infty(\mathcal{E}^\infty(X)) = \mathcal{E}^\infty(X) \otimes \Lambda^n(T^*_{0}(\mathbb{R}^n)).
\]

The result is then a Hochschild-Kostant-Rosenberg type theorem for Whitney functions. \( \square \)

5 Hochschild cohomology of Whitney functions

5.1 After having determined the Hochschild homology of \( \mathcal{E}^\infty(X) \) we now consider its Hochschild cohomology. In particular we want to compute the cohomology of the Hochschild cochain complex \( C^*(\mathcal{E}^\infty(X), \mathcal{E}^\infty(X)) \) which we briefly denote by \( C^\bullet \). Like in the previous section we assume that \( X \subset \mathbb{R}^n \) is a regular locally closed subset and that \( X \) has regularly situated diagonals.

We then apply the functor \( \text{Hom}_{\mathcal{E}^\infty(X)}(\mathcal{E}^\infty(X) \hat{\otimes} \mathcal{E}^\infty(X), M) \) to (4.11) and obtain the following sequence

\[
0 \rightarrow \text{Hom}_{\mathcal{E}^\infty(X)}(E_\bullet, \mathcal{E}^m(X) \hat{\otimes} \mathcal{E}^\infty(X), M) \rightarrow \text{Hom}_{\mathcal{E}^\infty(X)}(C_\bullet, \mathcal{E}^m(X) \hat{\otimes} \mathcal{E}^\infty(X), M) \rightarrow \text{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, \mathcal{E}^m(X) \hat{\otimes} \mathcal{E}^\infty(X), M) \rightarrow 0.
\]

(5.1)

Since the Hom-functor is left exact, this sequence is exact at the first two (nontrivial) spots. For \( m = \infty \) it need not be exact at the third spot, but we have the following.

5.2 Proposition For every \( m \in \mathbb{N} \cup \{ \infty \} \) let \( Q^\bullet_m \) be the quotient complex making the following sequence exact:

\[
0 \rightarrow \text{Hom}_{\mathcal{E}^\infty(X)}(E_\bullet, \mathcal{E}^m(X) \hat{\otimes} \mathcal{E}^\infty(X), M) \rightarrow \text{Hom}_{\mathcal{E}^\infty(X)}(C_\bullet, \mathcal{E}^m(X) \hat{\otimes} \mathcal{E}^\infty(X), M) \rightarrow Q^\bullet_m \rightarrow 0.
\]

(5.2)

Then \( Q^\bullet_m \) is acyclic and coincides with \( \text{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, \mathcal{E}^m(X) \hat{\otimes} \mathcal{E}^\infty(X), M) \) for finite \( m \).

Proof: Note first that one can reduce the claim to compact \( X \) by a localization argument involving an appropriate partition of unity. Moreover, we can reduce the claim to the case, where \( M = \mathcal{E}^\infty(X) \), since \( M \) is finitely generated projective. So we assume without loss of generality that \( X \) is compact and that \( M = \mathcal{E}^\infty(X) \).
Let us now consider the case $m \in \mathbb{N}$. Under this assumption it suffices by Proposition 4.2 to show that the sequence (5.1) is exact at the third spot. So we have to check that for every $k$ the natural map

$$\text{Hom}_{\mathcal{E}^\infty(X)}(C_k, \mathcal{E}^m(X)) \to \text{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X))$$

(5.3)

is surjective. Choose a compact cube $Q$ such that $X$ lies in the interior of $Q$. Then $\mathcal{J}^\infty(\Delta_{k+1}(X); \mathcal{E}^{k+1})$ is a Fréchet space the topology of which is defined by the norms $\| \cdot \|^r$, $r \in \mathbb{N}$. Moreover, the Fréchet topology of $J_k$ is the quotient topology with respect to the canonical projection $\mathcal{J}^\infty(\Delta_{k+1}(X); \mathcal{E}^{k+1}) \to J_k$. Hence, given $f \in \text{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^m(X))$ there exists by Proposition 3.2 a natural number $r \geq m$ such that $f$ extends to a continuous $\mathcal{E}^r(X)$-linear map $f_r : \mathcal{J}^r(\Delta_{k+1}(X); \mathcal{E}^{k+1}) \to \mathcal{E}^r(X^{k+1})$. Now recall the map $j^\infty_\Delta$ and define, using the notion introduced in Proposition 4.6, a map $j^\infty_\Delta : \mathcal{E}^\infty(\Delta_{k+1}(X)) \to \mathcal{E}^\infty(X^{k+1})$ by

$$j^\infty_\Delta(F)[x_0, x_1, \ldots, x_r] = \sum_{|p_1, \ldots, p_r| \leq r} \frac{F_{a_1, \ldots, a_r}}{\alpha_1 \cdots \alpha_r} (x_1 - x_0)^{a_1} \cdots (x_r - x_0)^{a_r}.$$

Like $j^\infty_\Delta$ the map $j^\infty_\Delta$ is continuous, linear and a morphism of $\mathcal{E}^\infty(X)$-modules. Moreover, using Taylor's formula, one checks easily that

$$F - j^\infty_\Delta(F) \in \mathcal{J}^r(\Delta_{k+1}(X); X^{k+1}) \quad \text{for all } F \in \mathcal{E}^\infty(X^{k+1}).$$

(5.4)

Since $j^\infty_\Delta(G) = 0$ for $G \in \mathcal{J}^\infty(\Delta_{k+1}(X); X^{k+1})$, the map $\tilde{f} : \mathcal{E}^\infty(X^{k+1}) \to \mathcal{E}^m(X)$, $\tilde{f}(F) = f(F - j^\infty_\Delta(F))$ lies in $\text{Hom}_{\mathcal{E}^\infty(X)}(C_k, \mathcal{E}^m(X))$ and satisfies $\tilde{f}(G) = f(G)$ for all $G \in \mathcal{J}^\infty(\Delta_{k+1}(X); X^{k+1})$. This proves the claim for $m \in \mathbb{N}$.

Next we will show that $Q_\bullet^\bullet$ is an acyclic cochain complex. Observe that $Q_\bullet^\bullet$ is a subcomplex of $\text{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, \mathcal{E}^\infty)$, so the claim is proved, if we can show that $K^\bullet(Q_\bullet^\bullet) \subset Q_\bullet^\bullet$, where $K \bullet$ is the contracting homotopy of $\text{Hom}_{\mathcal{E}^\infty(X)}(J_\bullet, \mathcal{E}^\infty)$ constructed in Proposition 4.2. Let $\tilde{f} \in \text{Hom}_{\mathcal{E}^\infty(X)}(C_k, \mathcal{E}^\infty)$, $\tilde{f}$ its image in $\text{Hom}_{\mathcal{E}^\infty(X)}(J_k, \mathcal{E}^\infty)$ and $g = K^k f = f \circ K^k_{k-1}$. We have to show that there is an element $\tilde{g} \in \text{Hom}_{\mathcal{E}^\infty(X)}(C_k, \mathcal{E}^\infty)$ inducing $g$. By an argument similar to the one above one concludes that for every $m \in \mathbb{N}$ the composition $j^m \circ \tilde{f}$ has an extension to a continuous map $\tilde{f}_m \in \text{Hom}_{\mathcal{E}^\infty}(\mathcal{E}^m(X^{k+1}), \mathcal{E}^m(X))$ for an appropriate number $r_m \geq m$. Note that $\mathcal{E}^m(X)$ is a subspace of the jet space $\mathcal{J}^m(X)$. Since

$$j^m \circ (\tilde{f}_{m+1} - \tilde{f}_m)(F) = 0$$

(5.5)

for all $F \in \mathcal{E}^\infty(X^{k+1})$, one concludes that the series

$$\tilde{f} + \sum_{m=1}^\infty (\tilde{f}_{m+1} - \tilde{f}_m) \in \text{Hom}_{\mathcal{E}^\infty}(\mathcal{E}^\infty(X^{k+1}), \mathcal{J}^m(X))$$

(5.6)

converges uniformly over $\mathcal{E}^\infty(X^{k+1})$ to the original map $\tilde{f}$. At this point it is essential to observe that Eq. (5.5) holds also for all $F \in \mathcal{E}^m(X^{k+1})$, since $\mathcal{E}^\infty(X^{k+1})$ is dense in $\mathcal{E}^m(X^{k+1})$. Moreover, there exist natural numbers $s_m \geq r_m$ such that $K^r_{k-1}$ has a (unique) extension to a continuous map $K^r_{k-1} : \mathcal{J}^r(\Delta_{k}(X); X^{k}) \to \mathcal{J}^r(\Delta_{k+1}(X); X^{k+1})$. Now we can define $\tilde{g} \in \text{Hom}_{\mathcal{E}^\infty}(\mathcal{E}^\infty(X^{k}), \mathcal{J}^m(X))$ by

$$\tilde{g}(F) = \tilde{f}_1(K^1_{k-1}(F - j^1_\Delta F)) + \sum_{m=1}^\infty ((\tilde{f}_{m+1} - \tilde{f}_m)(K^{m+1}_{k-1}(F - j^{m+1}_\Delta F))$$

for all $F \in \mathcal{E}^\infty(X^{k})$. 

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Since \( \tilde{j}^m \circ (\tilde{f}_{m+1} - \tilde{f}_m) (K_{k+1}^{m+1} (F - j^{m+1}_u F)) = 0 \) for all such \( F \) the series defining \( \tilde{g} \) converges uniformly on \( \mathcal{E}^\infty (X^k) \), hence \( \tilde{g} \) lies in \( \text{Hom}_{\mathcal{E}^\infty} (C^\infty_{k-1}, \mathcal{E}^\infty (X)) \). On the other side, it is clear by construction that for \( G \in J^\infty (\Delta_k (X); X^k) \)

\[
\bar{g}(G) = \tilde{f}_1 (K_{k-1}^1 (G)) + \sum_{m=1}^{\infty} (\tilde{f}_{m+1} - \tilde{f}_m) (K_{k-1}^m (G)) = f (K_{k-1}^1 (G)) = g(G),
\]

since \( j^s_k (G) = 0 \) for all \( s \in \mathbb{N} \). This finishes the proof. \( \square \)

Propositions 5.2 and 4.2 now entail the following.

5.3 Corollary The Hochschild cohomology \( H^*(\mathcal{E}^\infty (X), \mathcal{E}^m (X) \otimes \mathcal{E}^\infty (X)) \) is naturally isomorphic to the cohomology of the cochain complex \( \text{Hom}_{\mathcal{E}^\infty (X)} (E_*, \mathcal{E}^m (X) \otimes \mathcal{E}^\infty (X)) \).

5.4 Before we come to the computation of the cohomology of \( \text{Hom}_{\mathcal{E}^\infty (X)} (E_*, \mathcal{E}^m (X) \otimes \mathcal{E}^\infty (X)) \) we will introduce two operations on the Hochschild cochain complex, namely the the cup product and the Gerstenhaber bracket. The latter has originally been defined in [12] and has its use in the deformation theory of algebras [9, 13, 23]. For two cochains \( f_1 \in C^{k_1} \) and \( f_2 \in C^{k_2} \) define \( f_1 \circ f_2 \in C^{k_1 + k_2} \) by \( f_1 \circ f_2 = 0 \), if \( k_1 = 0 \), and else by

\[
f_1 \circ f_2 (F_{0}, \ldots, F_{k_1+k_2-1}) = \sum_{j=1}^{\infty} (-1)^{j-1} |k_1-1| f_1 (F_{0}, \ldots, F_{j-1}, f_2 (1, F_j, \ldots, F_{j+k_2-1}), F_{k_1+j}, \ldots, F_{k_1+k_2-1}),
\]

where \( F_0, \ldots, F_{k_1+k_2-1} \in \mathcal{E}^\infty (X) \). The Gerstenhaber bracket of \( f_1 \) and \( f_2 \) then is defined by

\[
[f_1, f_2] = f_1 \circ f_2 - (-1)^{|k_1-1|} f_2 \circ f_1.
\]

Moreover, the cup product of \( f_1 \) and \( f_2 \) is given by

\[
f_1 \circ f_2 (F_0, \ldots, F_{k_1+k_2}) = f_1 (F_0, \ldots, F_{k_1}) f_2 (1, F_{k_1+1}, \ldots, F_{k_1+k_2}).
\]

It is well-known that the complex \( C^{\infty-1} \) together with the Gerstenhaber bracket becomes a graded Lie algebra and that \( C^* \) is a graded algebra with respect to the cup product. Note that the cup product \( f_1 \circ f_2 \) is also well-defined for \( f_1, f_2 \in C^* (\mathcal{E}^\infty (X), \mathcal{E}^m (X)) \) and that \( f \circ G \) even makes sense, if \( f \in C^* (\mathcal{E}^\infty (X), \mathcal{E}^m (X)) \) and \( G \in C^0 (\mathcal{E}^\infty (X), \mathcal{E}^m (X)) = \mathcal{E}^\infty (X) \).

Next recall the fact that the inclusion of the normalized cochain complex \( \overline{\text{C}}^* \to C^* \) is a quasi-isomorphism. Hereby, \( \overline{\text{C}}^* \) consists of all normalized cochains that means of all \( f \in C^k \)

such that \( f (F_0, \ldots, F_k) = 0 \), whenever one of the Whitney functions \( F_i \), \( i > 0 \) is constant. Likewise, the inclusion of the normalized cochain complex \( \overline{\text{Hom}}_{\mathcal{E}^\infty (X)} (E_*, \mathcal{E}^m (X) \otimes \mathcal{E}^\infty (X)) \) is a quasiisomorphism.

5.5 Let us proceed to the computation of the cohomology of the cochain complex \( E^{m*} := \text{Hom}_{\mathcal{E}^\infty (X)} (E_*, \mathcal{E}^m (X)) \) respectively of its normalization \( \overline{E}^{m*} \). We denote elements of \( E^{m*} \) by letters \( D, D_1, \ldots, \) since every \( D \in E^{m,k} \) can be regarded as a local and continuous \( k \)-linear operator \( \mathcal{E}^\infty (X) \times \cdots \times \mathcal{E}^\infty (X) \to \mathcal{E}^m (X) \) and, at least for finite \( m \), such a \( D \) is locally given by a differential cochain according to Peetre’s theorem 3.3 for Whitney functions. Recall that by a differential cochain of degree \( k \) and order \( \leq d \in \mathbb{N} \) (and class \( C^m \)) one understands an element \( D \in E^{m,k} \) such that

\[
D (F_0, \ldots, F_k) = \sum_{a_1, \ldots, a_k \in \mathbb{N}^m} \mu (a_1 + \cdots + a_k = d) F_0 (D^{a_1} F_1) \cdots (D^{a_k} F_k).
\]
where the coefficients $d_{a_1, \ldots, a_k}$ are elements of $\mathcal{E}^m(X)$. A differential cochain is called 
\textit{homogeneous} of order $d$, if it is a linear combination of monomial cochains of order $d$, i.e. of cochains of the form $d_{a_1, \ldots, a_k} \cdot D^{a_1} \cdot \ldots \cdot D^{a_k}$ with $d = |a_1| + \ldots + |a_k|$. In a first step we will now determine $H^\bullet_{(\mathcal{E}^m)\text{diff}}$ and then show in a second step that the cohomology of $\mathcal{E}^m_{\text{diff}}$ coincides with $H^\bullet(\mathcal{E}^m)$. For $m$ finite the second step follows trivially from the localization results of Section 2, but for $m = \infty$ we need some more arguments to prove that.

Denote by $\mathcal{X}^\infty$ the sheaf of smooth vector fields on $\mathbb{R}^n$ and let $V_1, \ldots, V_k$ be elements of $\mathcal{X}^\infty(X) := \mathcal{E}^m(X) \otimes_{\mathcal{C}^m(\Omega)} \mathcal{X}^\infty(\Omega)$. Such elements will be called \textit{Whitney vector fields of class $\mathcal{C}^m$} on $X$. Clearly, a Whitney vector field $V$ of class $\mathcal{C}^m$ on $X$ defines for every $F \in \mathcal{E}^m(X)$ a Whitney function $VF \in \mathcal{E}^m(X)$ by $VF = \sum_{j=1}^n v_j D_j F$, where the $v_j$ are the coefficient Whitney functions of $V$ with respect to the standard basis of $\mathbb{R}^n$. Hence the skew symmetric product $V_1 \wedge \ldots \wedge V_k$, which we regard as an element of $\Lambda^k \mathcal{X}^\infty(X) := \mathcal{E}^m(X) \otimes_{\mathcal{C}^m(\Omega)} \Lambda^k \mathcal{X}^\infty(\Omega)$, defines a Hochschild cocycle with values in $\mathcal{E}^m(X)$ by

\[ (F_0, \ldots, F_k) \mapsto \sum_{\sigma \in S_k} \text{sgn}(\sigma) F_0(V_{\sigma(1)} F_1) \cdot \ldots \cdot (V_{\sigma(k)} F_k). \]

In the following we will show that the inclusion $\Lambda^\bullet \mathcal{X}^\infty(X) \xrightarrow{\text{incl}} \mathcal{E}^m_{\text{diff}}$ is a quasiisomorphism by constructing an appropriate homotopy. To this end we will make use of the homotopy operator introduced by DEWILDE–LECOMTE in [10]. The principal idea hereby is to decrease the order of a differential Hochschild cocycle while staying in the same cohomology class till one arrives at a skew symmetric differential Hochschild cocycle of order 1 in each nontrivial argument or in other words at a linear combination of skew symmetric products of Whitney vector fields. Note that by a nontrivial argument of a cochain $(F_0, F_1, \ldots, F_k) \mapsto D(F_0, F_1, \ldots, F_k)$ we will understand one of the arguments $F_1, \ldots, F_k$, since $D$ is $\mathcal{E}^\infty(X)$-linear in $F_0$.

Following DEWILDE–LECOMTE [10] we will first define two maps on $\mathcal{E}^m_{\text{diff}}$, where $k \geq 1$. Put

\[ Q^k D(F_0, \ldots, F_{k-1}) = \sum_{l=1}^n \sum_{0 < j < k} (-1)^j D(F_0, \ldots, F_{l-1}, x_j, F_l, x_j \ldots, F_{k-1}) \]

and

\[ P^k D = \sum_{l=1}^n [x_l, D] \sim D_l = (-1)^k \sum_{l=1}^n (D \circ x_l) \sim D_l \]

The proof of Proposition 4.1 in [10] can now be literally transferred to the case of Whitney functions, so we obtain

5.6 Proposition If $D \in \mathcal{E}^m_{\text{diff}}$ with $k > 0$ is a differential cochain homogeneous of order $d$, then

\[ (Q^{k+1} + \delta^{-1} Q^k) D = -(d + P^k) D. \]

Next define for every $l \in \mathbb{N}$ a homogeneous map $P^l : \mathcal{E}^m_{\text{diff}} \to \mathcal{E}^m_{\text{diff}}$ of degree 0 as follows:

\[ P^l D = \begin{cases} D, & \text{if } l = 0, \\ \sum_{j_l \ldots j_1 = -1}^{n} \text{ad}_{x_{j_1}} \ldots \text{ad}_{x_{j_l}}(D) \sim D_{j_l} \sim \ldots \sim D_{j_1}, & \text{if } 1 \leq l \leq k, \\ 0, & \text{if } l > k. \end{cases} \]

Hereby, ad is the adjoint action $E^m_{\text{skew}} \ni D \mapsto [G, D] := (-1)^d D \circ G \in E^{m-k-1}$ of some element $G \in \mathcal{E}^\infty(X)$. Since ad $G_1$ ad $G_2 = -\text{ad} G_2 \text{ad} G_1$ for all $G_1, G_2 \in \mathcal{E}^\infty(X)$, the cochain $P^l D$ is skew symmetric in the nontrivial arguments, hence a linear combination of skew symmetric products of Whitney vector fields.
5.7 Proposition The operators $P^k_i$ satisfy the recursive relations $P^k_i_{i+1} = P^k_i P^k_i + lP^k_i$. Moreover, $\frac{(-1)^k}{m!} P^k : \mathcal{E}^m \to \mathcal{E}^m$ is a projection onto the space of normalized differential cochains which are homogeneous of order $k$ and skew symmetric in the nontrivial arguments. $P^k_i$ vanishes on every monomial cochain of order $> 1$ in some argument or which is symmetric with respect to at least two of its nontrivial arguments. Finally, $P^k_i$ is a chain map.

Proof: Repeating the proof of [10, Prop. 4.2] immediately gives the claim.

Using the maps $P^k$ and $Q^k$, DeWilde–Lecomte define iteratively operators $Q^k : \mathcal{E}^m \to \mathcal{E}^{m-k-1}$, $0 \leq l \leq k$ by

$$Q^k_0 D = D, \quad Q^k_1 D = \frac{1}{d} Q^k_0 D, \quad Q^k_{l+1} D = \frac{1}{d-l} ((P^k - d) Q^k_0 + Q^k_1) D,$$

where $D$ is homogeneous of order $d$. Note that $d \geq k$, since $D$ is normalized. The operator $Q^k_0$ will turn out to comprise a homotopy between the identity and the antisymmetrization. Let us show this by induction like in [10, Sec. 4]. By Proposition 5.6 the formula

$$D = \lambda_{k,l,d} P^k D = (\theta^{k-1} Q^k_{l} + Q^k_{l+1} \theta^k) D$$

holds true for $l = 1$ and $\lambda_{k,l,d} = \frac{1}{d}$. Assume that it is true for some $l$ with $1 \leq l < k$ and apply $P^k$ to both sides. By Proposition 5.7 and the definition of the $Q^k_0$ one concludes that it holds for $l + 1$ with $\lambda_{k, l+1, d} = \frac{1}{d-l} \lambda_{k, l}$. Hence the formula is true for $l = k$ and $\lambda_{k, k, k} = (-1)^k \frac{d-1}{d}$. Note that $P^k_0 D = 0$, if $d > k$, and that $\lambda_{k, k, k} = (-1)^k \frac{d-1}{d}$, so we finally obtain

5.8 Proposition Let $A^\bullet : \mathcal{E}^m \to A^\bullet \mathcal{E}_n(X)$ with $A^k = \frac{(-1)^k}{k!} P^k_i$ be the skew symmetrization operator. Then

$$(\theta^{k-1} Q^k_0 - Q^k_{l+1} \theta^k) D = D - A^k D \quad \text{for all} \; D \in \mathcal{E}^m.$$ (5.8)

Thus, the inclusion $A^\bullet \mathcal{E}_n(X) \to \mathcal{E}_n^m$ is a quasiisomorphism.

Since the subcomplex $A^\bullet \mathcal{E}_n(X)$ has coboundary 0, the proposition gives the cohomology of the complex $\mathcal{E}^m$. Let us show that it coincides with the cohomology of $\mathcal{E}^m_\text{diff}$. By Poitre's theorem 3.3 for Whitney functions one concludes that every element $D \in \mathcal{E}^m$ has a representation of the form

$$D = \sum_{a_1, \ldots, a_k \in \mathbb{N}} d_{a_1, \ldots, a_k} D^{a_1} \cdots D^{a_k},$$

where the $d_{a_1, \ldots, a_k}$ are uniquely determined elements of $\mathcal{E}^m(X)$ and where the differential operators $D_j$, $j \in \mathbb{N}$ with

$$D_j = \sum_{a_1, \ldots, a_k \in \mathbb{N}, b_j \in \mathbb{N}} d_{a_1, \ldots, a_k} D^{a_1} \cdots D^{a_k}$$

converge to the operator $D$ in such a way that for every natural $r \leq m$ and every compact $K \subset X$ there exists a number $j_{r, K}$ such that $\|D_i F - D_j F\|_{r, K} = 0$ for all $i, j \geq j_{r, K}$. Thus, the sequence of differential operators $D_j$ converges uniformly on its domain to $D$. If now $D \in \mathcal{E}^m$, the construction of the operators $P^k_i$ and $Q^k_0$ shows that the operator sequences $(P^k_i D)_{j \in \mathbb{N}}$ and $(Q^k_0 D)_{j \in \mathbb{N}}$ converge uniformly to $P^k_i D$ respectively to an operator $Q^k_0 D \in \mathcal{E}^{m-k-1}$. But this entails that Equation (5.8) holds for all $D \in \mathcal{E}^m$, so the inclusion $A^\bullet \mathcal{E}_n(X) \to \mathcal{E}^m$ is a quasiisomorphism as well. The main result of this section now follows immediately.
5.9 Theorem Let \( X \subset \mathbb{R}^n \) be a regular subset with regularly situated diagonals, and \( m \in \mathbb{N} \cup \{ \infty \} \). Assume that \( \mathcal{M} \) is a finitely generated projective \( \mathcal{E}^\infty \) algebra of symmetric Fréchet modules and that \( \mathcal{M} \) is the \( \mathcal{E}^\infty \) module \( \mathcal{M}(X) \). The Hochschild cohomology of \( \mathcal{E}^\infty(X) \) with values in \( \mathcal{E}^m(X) \otimes \mathcal{E}^\infty(X) \) then is given by

\[
H^*(\mathcal{E}^\infty(X), \mathcal{E}^m(X) \otimes \mathcal{E}^\infty(X) \otimes \Lambda^* \mathcal{E}^\infty(X)) = \Lambda^* \mathcal{E}^\infty(X) \otimes \mathcal{E}^m(X) \otimes \Lambda^* \mathcal{E}^\infty(X). \tag{5.9}
\]

5.10 Remark One can also prove this theorem using the Koszul-resolution introduced in the proof of Theorem 4.8. The advantage of the method presented above is, that in principle it gives for every Hochschild cocycle \( c \in C^k(\mathcal{E}^\infty(X), \mathcal{E}^\infty(X)) \) an explicit construction of an alternating Whitney vector field lying in the same cohomology class.

6 Cyclic homology of Whitney functions

6.1 Following the presentation by Loday [25, Chap. 2] let us recall the classical operators defining cyclic homology: the usual cyclic group action on the module \( (\mathcal{E}^\infty(X))^\oplus k+1 \) is denoted by \( t \), the classical norm operator by \( N = 1 + t + \cdots + t^k \) and the extra degeneracy operator by \( s \). More precisely:

\[
t(F_0 \oplus F_1 \oplus \cdots \oplus F_k) = (-1)^k F_k \oplus F_0 \oplus \cdots \oplus F_{k-1} \quad \text{and} \quad s(F_0 \oplus F_1 \oplus \cdots \oplus F_k) = 1 \oplus F_0 \oplus F_1 \oplus \cdots \oplus F_k \quad \text{for all } F_0, \ldots, F_k \in \mathcal{E}^\infty(X).
\]

Moreover, there is a canonical map

\[
\pi_k : C_k = (\mathcal{E}^\infty(X))^\oplus k+1 \rightarrow \Omega^k_{\mathcal{E}^\infty}(X), \quad F_0 \oplus F_1 \oplus \cdots \oplus F_k \mapsto F_0 dF_1 \wedge \cdots \wedge F_k,
\]

which, as a consequence of Theorem 4.8 and under the assumptions made there, induces an isomorphism \( HH_k(\mathcal{E}^\infty(X)) \rightarrow \Omega^k_{\mathcal{E}^\infty}(X) \), still denoted by \( \pi_k \).

On the one hand, the Connes boundary map \( B = (1-t)s : C_k \rightarrow C_{k+1} \) induces a boundary map \( B : E_k \rightarrow E_{k+1} \). This map gives rise to a map \( B_* : HH_k(\mathcal{E}^\infty(X)) \rightarrow HH_{k+1}(\mathcal{E}^\infty(X)) \) and there is a commutative diagram ([25, Prop. 2.3.4]):

\[
\begin{array}{ccc}
\Omega^k_{\mathcal{E}^\infty}(X) & \xrightarrow{\pi_k} & \Omega^k_{\mathcal{E}^\infty}(X) \\
\downarrow^{d} & & \downarrow^{d} \\
\Omega^{k+1}_{\mathcal{E}^\infty}(X) & \xrightarrow{B_*} & \Omega^{k+1}_{\mathcal{E}^\infty}(X)
\end{array}
\tag{6.1}
\]

where \( d : \Omega^k_{\mathcal{E}^\infty}(X) \rightarrow \Omega^{k+1}_{\mathcal{E}^\infty}(X) \) is the differential of the Whitney–de Rham complex. The factor \( (k+1) \) appears in the same way like in [25].

On the other hand, we have two mixed complexes (see [25, 25.13]), the first one \((C_*, b, B)\) defines the (topological) bicomplex \( B(\mathcal{E}^\infty(X)) \) ([25, 2.1.7])

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
(\mathcal{E}^\infty(X))^\oplus 3 & \xrightarrow{B} & (\mathcal{E}^\infty(X))^\oplus 2 \\
\downarrow & & \downarrow \\
(\mathcal{E}^\infty(X))^\oplus 2 & \xrightarrow{B} & \mathcal{E}^\infty(X) \\
\downarrow & & \downarrow \\
(\mathcal{E}^\infty(X))^\oplus 1 & \xrightarrow{B} & \mathcal{E}^\infty(X) \\
\downarrow & & \\
\mathcal{E}^\infty(X)
\end{array}
\]

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the second one \((\Omega^*_{\infty}(X), 0, d)\) determines the bicomplex

\[
\begin{array}{c}
\Omega^*_{\infty}(X) \\
\downarrow d \\
\Omega^*_{\infty}(X) \\
\downarrow d \\
\Omega^*_{\infty}(X)
\end{array}
\begin{array}{c}
0 \\
\downarrow 0 \\
0 \\
\downarrow 0 \\
0
\end{array}
\begin{array}{c}
\Omega^*_{\infty}(X) \\
\downarrow d \\
\Omega^*_{\infty}(X)
\end{array}
\begin{array}{c}
0 \\
\downarrow 0 \\
0
\end{array}
\begin{array}{c}
\Omega^*_{\infty}(X)
\end{array}
\]

Note that all spaces involved in these bicomplexes are Fréchet spaces and all maps are continuous. The (topological) cyclic homology of \(\mathcal{E}_\infty(X)\) is defined as the homology of the total complex of \(B(\mathcal{E}_\infty(X))\), in signs \(HC_\bullet(\mathcal{E}_\infty(X)) := \text{Tot}_*B(\mathcal{E}_\infty(X))\). Now, the map \((1/k)!\pi_k\) induces a map of mixed complexes \((C_\bullet, b, B) \to (\Omega^*_{\infty}(X), 0, d)\), hence by [25, Prop. 2.3.7] to a morphism

\[
HC_\bullet(\mathcal{E}_\infty(X)) \to \Omega^*_{\infty}(X)/d\Omega^*_{\infty}(X) \oplus H^{k-2}_{\text{wht}}(X) \oplus H^{k-4}_{\text{wht}}(X) \oplus \cdots.
\]

Since the maps \(\pi_k\) in the diagram (6.1) are isomorphisms, this morphism has to be an isomorphism. In the homological setting we thus obtain an equivalent of Connes’ result [6, III.2a] for Whitney functions.

6.2 Theorem For every regular subset \(X \subset \mathbb{R}^n\) having regularly situated diagonals the cyclic homology \(HC_\bullet(\mathcal{E}_\infty(X))\) coincides with

\[
\Omega^*_{\infty}(X)/d\Omega^*_{\infty}(X) \oplus H^{k-2}_{\text{wht}}(X) \oplus H^{k-4}_{\text{wht}}(X) \oplus \cdots.
\]

Arguing like in [25, 2.5.13] one obtains as a corollary a Connes’ periodicity exact sequence:

\[
\cdots \to HH_k(\mathcal{E}_\infty(X)) \xrightarrow{\iota} HC_k(\mathcal{E}_\infty(X)) \xrightarrow{\partial} HC_{k-2}(\mathcal{E}_\infty(X)) \xrightarrow{\text{B}} HH_{k-1}(\mathcal{E}_\infty(X)) \xrightarrow{\iota} \cdots.
\]

6.3 Let us finally determine the periodic cyclic homology of \(\mathcal{E}_\infty(X)\). It is given by the homology \(HC_\bullet(\text{Tot} \mathcal{E}_\infty(\mathcal{E}).)\) of the (product) total complex of the periodic bicomplex \(B(\mathcal{E}_\infty(X))^{\text{per}}\) below and will be denoted by \(H \mathcal{E}_\infty(X)\) (cf. [25, 5.1.7]).

\[
\begin{array}{c}
\cdots \\
\downarrow b \\
(\mathcal{E}_\infty(X))^{B/3} \\
\downarrow b \\
(\mathcal{E}_\infty(X))^{B/2} \\
\downarrow b \\
(\mathcal{E}_\infty(X)) \\
\downarrow b \\
\mathcal{E}_\infty(X)
\end{array}
\]

From the exact sequence [25, 5.1.9]

\[
0 \to \text{lim}_r HC_{k+2(r+1)} \to HH_k(\mathcal{E}_\infty(X)) \to \text{lim}_r HC_{k+2r} \to 0
\]

and the fact that the periodicity map \(S : HC_k(\mathcal{E}_\infty(X)) \to HC_{k-2}(\mathcal{E}_\infty(X))\) is surjective one can conclude by [25, 5.1.10] that \(HP_k(\mathcal{E}_\infty(X)) = \text{lim}_r HC_{k+2r}\). This proves the last result of this section.
6.4 Theorem For every regular $X \subset \mathbb{R}^n$ having regularly situated diagonals the periodic cyclic homology of $E^\infty(X)$ is given by $HP^0 = H^0_{\text{max}}(X)$ and $HP^1 = H^1_{\text{max}}(X)$, where $H^r_{\text{max}}(X)$ denotes the Whitney–de Rham cohomology in even resp. odd degree.

6.5 Remark The reader might ask, whether it would be possible to use the excision result in periodic cyclic homology of CUNTZ–QUILLEN [8] for the computation, since according to Whitney’s extension theorem there is an exact sequence

$$0 \longrightarrow \mathcal{J}^\infty(X;U) \longrightarrow C^\infty(U) \longrightarrow E^\infty(X) \longrightarrow 0,$$

(6.2)

and the periodic cyclic homology of $C^\infty(U)$ is well-known. But unfortunately it is not possible to apply excision to compute $HP^*(E^\infty(X))$, since the sequence (6.2) does in general not possess a continuous splitting.

7 Whitney–de Rham cohomology of subanalytic spaces

In this section we will compute the cohomology of the Whitney–de Rham complex over a subanalytic set by proving the following.

7.1 Theorem For every subanalytic $X \subset \mathbb{R}^n$ the sequence

$$0 \longrightarrow \mathbb{R}_X \longrightarrow E^\infty_X \xrightarrow{d} \Omega^1_{\mathbb{R}_X} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k_{\mathbb{R}_X} \xrightarrow{d} \cdots$$

(7.1)

comprises a fine resolution of the sheaf $\mathbb{R}_X$ of locally constant real valued functions on $X$.

Before we prepare the proof of the theorem observe that a subanalytic $X$ is a locally path connected and locally contractible locally compact topological Hausdorff space (this can be concluded for example from the fact that a subanalytic set is regular [24, Cor. 2] and possesses a Whitney stratification [20, Thm. 4.8]). Hence the complex of sheaves

$$0 \longrightarrow \mathbb{R}_X \longrightarrow S^0_X \longrightarrow S^1_X \longrightarrow \cdots \longrightarrow S^k_X \longrightarrow \cdots,$$

where $S^k_X$ is the sheaf associated to the presheaf of singular $k$-cochains on $X$, is a soft resolution of $\mathbb{R}_X$ (cf. GODMENT [15, Ex. 3.9.1]). By the theorem it thus follows that the Whitney–de Rham cohomology coincides with the cohomology of the complex of global sections of $S^*_X$. But the cohomology of the latter complex is nothing else but the singular cohomology of $X$ (with values in $\mathbb{R}$). So we obtain:

7.2 Corollary The Whitney–de Rham cohomology $H^r_{\text{max}}(X) = H^r(\Omega^*_\infty(X))$ coincides with the singular cohomology $H^r_{\text{sing}}(X; \mathbb{R})$.

7.3 The nontrivial part in the proof of the theorem is to show that the sequence (7.1) is exact or in other words that Poincare’s lemma holds true for forms of Whitney functions. The essential tool for proving Poincare’s lemma for Whitney functions will be a subanalytic triangulation of $X$ together with a particular system of tubular neighborhoods for the strata defined by the triangulation. Let us explain in some more detail, what we mean by that.

Recall that by a finite (resp. locally finite) subanalytic triangulation of a closed subanalytic set $X$ one understands a pair $T = (h, \mathcal{K})$, where $\mathcal{K}$ is a finite (resp. locally finite) simplicial complex in some $\mathbb{R}^n$ and $h : [\mathcal{K}] \rightarrow X$ is a subanalytic homeomorphism such that for every simplex $\Delta \in \mathcal{K}$ the following holds true:

(TRG1) The image $\Delta := h(\Delta)$ is a subanalytic leave that means a subanalytic and locally closed smooth real-analytic submanifold of $\mathbb{R}^n$. 


(TRG2) The homeomorphism $h$ induces a real-analytic isomorphism $h_{\Delta} : \Delta \to \tilde{\Delta}$.

If $X_1, \ldots, X_k \subset \mathbb{R}^n$ are subanalytic subsets, one calls $T$ compatible with the $X_j$, if every one of the sets $X_j$ is a union of simplices $h(\Delta)$, $\Delta \in \mathcal{K}$. The following result is well-known (see Lojasiewicz [27] and Hironaka [21]).

**7.4 Theorem** For every family $X_1, \ldots, X_k \subset \mathbb{R}^n$ of bounded subsets there exists a compact parallelotope $Q \subset \mathbb{R}^n$ containing the $X_j$ in its interior and a finite subanalytic triangulation $(h : Q \to Q, \mathcal{K})$ which is compatible with the $X_j$.

A subanalytic triangulation has the following further property which follows immediately from the lemma below.

(TRG3) The triangulation map $h$ is bi-Hölder, i.e. $h$ and its inverse are Hölder continuous.

**7.5 Lemma** Every subanalytic function $f : Y \to \mathbb{R}^N$ with compact graph is Hölder continuous.

**Proof:** Let $g_1(x, y) = |f(x) - f(y)|^p$ and $g_2(x, y) = |x - y|^p$ for $x, y \in Y$. Since these functions are subanalytic and $g_2^{-1}(0) \subset g_1^{-1}(0)$, the Lojasiewicz inequality [3, Thm. 6.4] immediately gives the claim. □

The triangulation we need for the proof of the theorem has to be a so-called bimeromorphic triangulation. Before we define this notion let us mention that by a tubular neighborhood of a subanalytic leave $M \subset \mathbb{R}^n$ we will understand a triple $(E_M, \varepsilon_M, \pi_M)$ such that the following holds:

**(TUB1)** $E_M \to M$ is a smooth real-analytic subbundle of $T_{|M} \mathbb{R}^n$ which is complementary to the tangent bundle $TM$.

**(TUB2)** $\varepsilon_M : M \to \mathbb{R}_{>0}$ is a subanalytic continuous map such that the map $\varphi_M : \{(x, v) \in E_M \mid \|v\| < 2 \varepsilon_M(x)\} \to \mathbb{R}^n$, $(x, v) \mapsto x + v$ is an analytic embedding.

**(TUB3)** $\pi_M : U_{\varepsilon_M} \to M$ is the projection $x + v \mapsto x$, where $U_{\varepsilon_M}$ denotes the open set $\varphi_M(\{(x, v) \in E_M \mid \|v\| < \min(\varepsilon(x), 2 \varepsilon_M(x))\})$ for every continuous $\varepsilon : M \to \mathbb{R}_{>0}$.

For the simplices $\Delta \in \mathcal{K}$ we have natural tubular neighborhoods $(E_{\Delta}, \varepsilon_{\Delta}, \pi_{\Delta})$, where $E_{\Delta}$ is the bundle normal to $T\Delta$ with respect to the euclidean metric and $\pi_{\Delta}$ is the orthonormal projection onto $\Delta$. Note that we can choose the $\varepsilon_{\Delta}$ in such a way that for two different simplices $\Delta, \Delta'$ of the same dimension we have $U_{\varepsilon_{\Delta}} \cap U_{\varepsilon_{\Delta'}} = \emptyset$. Moreover, we can achieve easily that $U_{\varepsilon_{\Delta}} \setminus \partial \Delta \subset U_{2\varepsilon_{\Delta}}$, where $\partial \Delta$ denotes the frontier of $\Delta$, i.e. the union of the faces of $\Delta$. By a bimeromorphic triangulation of $X$ we now understand a subanalytic triangulation $(h, \mathcal{K})$ of $X$ together with a system of tubular neighborhoods $(E_{\Delta}, \varepsilon_{\Delta}, \pi_{\Delta})$ for the leaves $\Delta$ such that the conditions (BMT1) to (BMT4) below are satisfied for every simplex $\Delta \in \mathcal{K}$.

**(BMT1)** The tubular neighborhoods satisfy $U_{\varepsilon_{\Delta}} \setminus \partial \Delta \subset U_{2\varepsilon_{\Delta}}$ and $U_{\varepsilon_{\Delta}} \cap U_{\varepsilon_{\Delta'}} = \emptyset$ for every $\Delta$ of the same dimension like $\Delta$ but disjoint from $\Delta$.

**(BMT2)** For a sufficiently small neighborhood $U$ of $U_{\varepsilon_{\Delta}}$ the map $h_{\Delta} \circ \pi_{\Delta} : U_{\varepsilon_{\Delta}} \setminus \partial \Delta \to \mathbb{R}^n$ can be extended to a Hölder continuous map $h^{\pi_{\Delta}}_{\varepsilon_{\Delta}} : U \to \mathbb{R}^n$ which lies in the multiplier algebra $\mathcal{M}^0(\partial \Delta; U)$.

**(BMT3)** For a sufficiently small neighborhood $\tilde{U}$ of $U_{\varepsilon_{\Delta}}$ the map $h^{-1}_{\Delta} \circ \pi_{\Delta} : U_{\varepsilon_{\Delta}} \setminus \partial \Delta \to \mathbb{R}^n$ can be extended to a Hölder continuous map $h^{-1}_{\pi_{\Delta}} : \tilde{U} \to \mathbb{R}^n$ which lies in the multiplier algebra $\mathcal{M}^0(\partial \Delta; \tilde{U})$.

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(BMT4) For a sufficiently small neighborhood $U$ of $\overline{\mathcal{C}}$, the map $(U_{\mathcal{C}} \setminus \partial \Delta) \times [0, 1] \to \mathbb{R}^n$, 
$(x, t) \mapsto \varphi_{\Delta}(t \varphi_{\Delta}^{-1}(x))$ can be extended to a H"older continuous homotopy $H_{\Delta} : U \times [0, 1] \to \mathbb{R}^n$ which lies in the multiplier algebra $\mathcal{M}^\infty(\partial \Delta \times [0, 1]; U \times [0, 1])$.

**Proof of Theorem 7.1:** As already mentioned we only have to show that for every $x_0 \in X$ there exists a basis of contractible neighborhoods $V \subset X$ of $x_0$ such that for all $\omega \in \Omega^k_{C^\infty}(V)$ with $d\omega = 0$ there exists a form $\eta \in \Omega^k_{C^\infty}(V)$ satisfying $d\eta = \omega$.

We can assume that $x_0$ lies in the interior of $X$, because otherwise the classical Poincaré lemma could be applied. Since the claim is essentially a local statement, we can even assume furthermore without loss of generality that $X$ is a compact and connected subanalytic set.

Now choose a holomorphic subanalytic triangulation $(h : Q \to Q, \mathcal{K})$ compatible with $X$ and the one point set $\{x_0\}$; in the following section we will show that this is possible indeed. Clearly, we can choose the triangulation in such a way that $0$ is a simplex of $\mathcal{K}$ and $h(0) = x_0$. As a further tool for our construction we need the integral operator $K_M : \Omega^k B(M) \times [0, 1] \to \Omega^k B(M)$, where $M$ is an arbitrary smooth manifold. This operator is defined by

$$K_M \omega = \int_0^1 \iota^*_s (\partial_y \omega) ds, \quad \text{for all } \omega \in \Omega^{k+1}(M),$$

where $s$ denotes the last coordinate of elements of $M \times [0, 1]$, $\partial_y$ means the insertion of the vector field $\partial_y$ in a differential form at the first position and $\iota_s : M \to M \times [0, 1]$ is the map $y \mapsto (y, s)$. By Cartan’s magic formula

$$dK_M \omega + K_M d \omega = \iota_1^* - \iota_0^*.$$  \hfill (7.2)

Now let $B \subset \mathbb{R}^n$ be an open ball around the origin such that $B$ does not contain any other $0$-simplex of $\mathcal{K}$ besides the origin and let $K_j$ for $j = 0, \ldots, n$ be the set of all $j$-simplexes of $\mathcal{K}$ which meet $B$ and $h^{-1}(X)$. Let $H : B \times [0, 1] \to B$ be the radial homotopy $(x, t) \mapsto tx$. Then, $H((\Delta \cap B) \times [0, 1]) \subset (\Delta \cap B) \cup \{0\}$ for all $\Delta$ meeting $B$. Next let $B = h(B)$, $V = B \cap X$ and let $\omega \in \Omega^k_{C^\infty}(V)$ be closed. Choose a smooth differential form $\omega \in \Omega^k B(M)$ which induces $\omega$ over $V$ according to Whitney’s extension theorem. We will construct inductively smooth differential forms $\eta_0, \ldots, \eta_n \in \Omega^k_{C^\infty}(V)$ such that

$$\tilde{\omega} - d(\tilde{\eta}_0 + \cdots + \tilde{\eta}_j) \in \mathcal{F}^\infty(K_0 \cup \cdots \cup K_j; \tilde{B}) \Omega^k B(M) \quad \text{for } j = 1, \ldots, n,$$  \hfill (7.3)

where $K_j = \bigcup_{\Delta \in K_j} h(\Delta)$. Clearly, this proves the claim, since the element $\eta \in \Omega^k_{C^\infty}(V)$ induced by $\eta_0 + \cdots + \eta_n$ satisfies $\omega = d\eta$.

To construct $\eta_0$ choose a smooth homotopy $G : \tilde{B} \times [0, 1] \to \tilde{B}$ such that $G(x_0, t) = x_0$, $G(x, 0) = x_0$ and $G(x, 1) = x$ for all $x \in B$ and $t \in [0, 1]$. Let $\widetilde{\eta}_0 = K_0 G^* \tilde{\omega}$. By Eq. (7.2) we then have $d\widetilde{\eta}_0 + K_0 G^* d\tilde{\omega} = \tilde{\omega}$. Using the assumption on $G$ and $d\tilde{\omega} = 0$ one concludes that $K_0 G^* d\tilde{\omega} \in \mathcal{F}^\infty((x_0), \tilde{B})$, hence $\widetilde{\eta}_0$ has the desired property. Now assume that we have constructed $\widetilde{\eta}_0, \ldots, \widetilde{\eta}_j$ for $0 \leq j < n$. If $K_{j+1} = \emptyset$ we are done, since we can then put $\widetilde{\eta}_{j+1} = \ldots = \eta_n = 0$. So assume $K_{j+1} \neq \emptyset$. Let $\tilde{\omega} = \tilde{\omega} - d(\tilde{\eta}_0 + \cdots + \tilde{\eta}_j)$. The following constructions can be performed separately for every $\Delta \in K_{j+1}$, so we assume for simplicity that there is only one simplex $\Delta \in K_{j+1}$. We proceed in three steps.

1. **Step:** Consider the homotopy $H_{\Delta} : \tilde{U} \times [0, 1] \to \mathbb{R}^n$ of (BMT4). After possibly changing $H_{\Delta}$ outside a sufficiently small neighborhood of $U_{\Delta}$ and extending the homotopy appropriately we can achieve that $H_{\Delta}$ is a homotopy which is defined on $\tilde{B} \times [0, 1]$, has values in $\tilde{B}$ and has the properties stated in (BMT4). Let $\tilde{\eta} = K_{\tilde{B}} H_{\Delta}^* \tilde{\omega}$. Since $\tilde{\omega} \in \mathcal{F}^\infty(\partial \Delta; \tilde{B}) \Omega^k B(M)$ and $H_{\Delta}(x, t) = x$ for all $x$ in the closure of $\Delta$ and all $t \in [0, 1]$ one concludes by (BMT4) that $\tilde{\eta}$ is
well-defined and lies in $\mathcal{I}^\infty(\partial \overline{\Delta}; \overline{B}) \Omega^{k-1}(\overline{B})$. Moreover, since $d\omega'$ we have by virtue of (7.2)

$$\widetilde{\omega}' - H_{z,D_0}^* \omega' - d\tilde{n}' \in \mathcal{I}^\infty(\overline{\Delta}; \overline{B}) \Omega^{k-1}(\overline{B}), \tag{7.4}$$

where $H_{z,D_0}^* = H_{z}(\cdot, 0)$. Note that the restriction of $H_{z,D_0}^*$ to $\overline{U_{\varepsilon_\Delta}}$ coincides with $\pi_{\Delta}^*$.

2. Step: Next consider the map $h_{\pi_{\Delta}}$ in (BMT2). Clearly, $h_{\pi_{\Delta}}(\overline{U_{\varepsilon_\Delta}} \cap B) \subset \overline{B}$, so after possibly redefining $h_{\pi_{\Delta}}$ outside a sufficiently small neighborhood of $\overline{U_{\varepsilon_\Delta}}$ and appropriate extension we can assume that $h_{\pi_{\Delta}}$ is defined on $B$ and has image in $\overline{B}$, while the properties of (BMT2) remain valid. By (BMT2), $h_{\pi_{\Delta}}(\partial \Delta) \subset \partial \overline{\Delta}$ and $\omega' \in \mathcal{I}^\infty(\partial \overline{\Delta}; \overline{B}) \Omega^{k}(\overline{B})$ the pull-back $h_{\pi_{\Delta}}^* \omega'$ has to be in $\mathcal{I}^\infty(\partial \overline{\Delta}; \overline{B}) \Omega^{k}(\overline{B})$. Moreover, $dh_{\pi_{\Delta}}^* \omega' \in \mathcal{I}^\infty(\overline{\Delta}; B) \Omega^{k+1}(B)$. Let $\tilde{\nu} = K_B h_{\pi_{\Delta}}^* \omega'$. Observe that $\tilde{\nu} \in \mathcal{I}^\infty(\partial \overline{\Delta}; B) \Omega^{k-1}(B)$ and $K_B h_{\pi_{\Delta}}^* d\omega' \in \mathcal{I}^\infty(\overline{\Delta}; B) \Omega^{k}(B)$. Hence by (7.2)

$$h_{\pi_{\Delta}}^* \omega' - d\tilde{\nu} \in \mathcal{I}^\infty(\overline{\Delta}; B) \Omega^{k}(B). \tag{7.5}$$

3. Step: Analogously to the second step we now consider the map $h_{\pi_{\Delta}}$ in (BMT3) and can assume that after appropriate alteration $h_{\pi_{\Delta}}$ is defined on $\overline{B}$, has image in $B$ and has the properties stated in (BMT3). Let $\tilde{\eta}' = (h_{\pi_{\Delta}}^*)^* \tilde{\nu}$. Similarly like in the second step one checks that $\tilde{\eta}' \in \mathcal{I}^\infty(\partial \Delta; B) \Omega^{k-1}(B)$. Since $(h_{\pi_{\Delta}}^*)^* h_{\pi_{\Delta}}^* \omega' = \pi_{\Delta}^* \omega'$ over $\overline{U_{\varepsilon_\Delta}}$, Eqs. (7.4) and (7.5) entail that

$$\omega' - d\tilde{\eta}' - d\tilde{n}' \in \mathcal{I}^\infty(\overline{\Delta}; B) \Omega^{k-1}(\overline{B}). \tag{7.6}$$

So, if we now put $\tilde{n}_{j+1} := \tilde{n}' + \tilde{\eta}'$, the induction step is finished and the theorem is proven. \qed

7.6 The existence of a bimeromorphic subanalytic triangulation is essentially a consequence of Pawlucki’s work [31] on Puiseux’s theorem applied to subanalytic mappings. In fact, Proposition 2 of [31] entails immediately that the subanalytic homeomorphism appearing in a subanalytic stratification has to satisfy the growth conditions (BMT2) to (BMT4). The existence of tubular neighborhoods needed for that and which have the desired properties has been proved in [34, Chap. 3].
References


