ON PHASE SEMANTICS AND DENOTATIONAL SEMANTICS:  
THE SECOND-ORDER

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Abstract. In this paper, we extend the non-uniform denotational semantics defined by Bucciarelli and Ehrhard in [BuEhr.00] to second-order linear logic. For this purpose, we first define a relational model of second-order linear logic. Then we define a notion of variable M-spaces for second-order formulae and show that the interpretation of proofs in the relational model are precisely morphisms in the category of variable M-spaces, which leads to a non-uniform denotational semantics of LL2.

Introduction

In [BuEhr.99] and [BuEhr.00], Bucciarelli and Ehrhard investigate a new approach to denotational semantics: phase semantics. They define a phase space parameterized coherent semantics where formulae are interpreted by a set (the web) together with a fact valued coherent predicate over the families of points of the web. The web of the interpretation of a formula is the purely relational interpretation1 of this formula and thus is independent of coherence (unlike in coherence spaces). The model therefore is non-uniform in the sense that the interpretation of a proof of \( !A \rightarrow B \) contains informations about the behavior of this proof on arguments which are not accepted by the model (see section 3.3 for a simple example where \( A = B = 1 \oplus 1 \)). This non-uniform approach leads to models which radically differ from the usual denotational settings. For instance, in [BuEhr.00], a non-uniform coherence semantics of linear logic is introduced as a particular case of the general construction and in that model, a point of a web can be strictly coherent or strictly incoherent with itself. Among other consequences of non-uniformity, the resulting models are closer to the syntax (see section 3.3 for an example), leading to various completeness results (see [BuEhr.99] and [Ehr.00a]). We extend this non-uniform approach to the second-order.

For that purpose, we extend first the purely relational model, following the guidelines proposed by Jean-Yves Girard in [Gi.86]: our objects will be stable functors, that is, functors preserving directed limits and pullbacks over the category of sets and injections. This will allow to recover the “normal form theorem”, which is the basic ingredient for interpreting the second-order quantifiers in this simple relational setting: this theorem provides a simple relational representation of “objects of variable types”.

Proofs of a given formula \( F \) will be interpreted as objects of variable type, that is, set indexed families \( t = (t_X)_{X \in Z} \) with \( t_X \subseteq F(X) \) subject to a regularity condition called mutilation: given an injection \( f \) from \( X \) into \( Y \), \( t_X \) and \( t_Y \) must be equal up to the renaming and restriction operation induced by \( f \), \( t_X = F(f)^{-1}t_Y \). Stability

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1In this model of linear logic, types are interpreted as sets and proofs as relations. Additives are translated as disjoint unions, multiplicatives as cartesian products and the “of course” of a set \( E \) is the set of all the finite multisets of \( E \). The orthogonal of a set \( E \) is \( E \) itself, in that sense the model is “trivial”.

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and mutilation provide an almost syntactical grasp of these \textit{a priori} quite abstract objects which interpret formulae and proofs, see section 2.4.3.

Non-uniformity leads to a serious difficulty for defining the composition of morphisms. Given \( F \) and \( G \) two stable functors, a morphism \( s \) from \( F \) to \( G \) is an object of variable type \( F \times G = F \to G \) (written \( s : F \to G \)). Given \( s : F \to G \) and \( t : G \to H \) the most obvious way of defining \( t \circ s \) is to set \( (t \circ s)_X = t_X \circ s_X \) (relational composition “on the nose”), as in second-order coherence semantics. Unfortunately, this does not work: \( t \circ s \) so defined does not satisfy the mutilation property in general (see section 1.6 for a counter-example). In coherence spaces, mutilation for the composite is proved using two artefacts:

- the equivalence between mutilation and the following separability property
  (this lemma is due to Eugenio Moggi). Given a pullback:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{f'} \\
Y' & \xrightarrow{g'} & Z
\end{array}
\]

- the coherence relation in the space interpreting a linear implication.

In the present setting, coherence can no longer be used and so, we are led to defining composition of morphisms in another way. We then prove that the category \( \text{VRel} \) so defined is a model of second-order linear logic.

In section 3, we recall part of the material developed in [BuEh89] and [BuEh00]. We first recall the notions of product phase spaces and of symmetric product phase spaces and for any connective of linear logic, we recall the corresponding operation on facts. Given a symmetric product phase space \( M = (P^I_0, \perp) \) (where \( I \) is an infinite countable set), we recall how to build a \textit{category of M-spaces} which is a model of linear logic generalizing in some sense coherence spaces. A \( M \)-space is a pair \( X = (\{X\}, \hat{X}) \) where \( \{X\} \) is a denumerable set called the web of \( X \) and \( \hat{X} = (\hat{X}_J)_{J \subseteq I} \) is a family of mappings such that \( \hat{X}_J \) associates to any \( J \)-indexed family a factual coherence value.

In section 4, we extend this semantics to second-order linear logic (LL\(^2\)). For this purpose, we build a category of variable \( M \)-spaces and show that it is a model of second-order linear logic. A \textit{variable M-space} is a pair \( F = ([F], \hat{F}) \) where \( [F] \) is a \( n \)-ary stable functor that we call the \textit{variable web} of \( F \) and \( \hat{F} \) associates to any family of \( M \)-spaces \( X_1, \ldots, X_n \) an \( M \)-space structure \( \hat{F}(X_1, \ldots, X_n) \) over the web \( [F](\{X_1\}, \ldots, \{X_n\}) \). Then it is possible to define, for each logical connective, a corresponding construction on variable \( M \)-spaces. The operations on variable webs are nothing but the corresponding operations in the relational semantics. When \( F \) is a formula of \( \text{LL} \) the operation \( \hat{F} \) is defined “on the nose” as described in section 3. As for the interpretation of second-order quantifiers: universal quantifications are interpreted as intersections and existential quantifications are interpreted dually. In this setting, given a variable \( M \)-space \( F \), we call \textit{variable clique} any object \( f \) of variable type \( F \) such that for any family of \( M \)-spaces \( X_1, \ldots, X_n \), the set \( f_{X_1, \ldots, X_n} \) is a clique of \( \hat{F}(X_1, \ldots, X_n) \). We then define the \textit{category of variable M-spaces}: its objects are the variable \( M \)-spaces and its morphisms between \( F \) and \( G \) are the variable cliques of \( F \to G \). We prove that this category can be endowed with the structure required for being a model of second-order linear logic.

Last, we illustrate these second-order constructions in the particular case of \textit{non-uniform coherence space semantics} introduced in [BuEh00]. As an example, we
will see that the uniform interpretation of \textbf{Bool} can be retrieved from its non-uniform interpretation, as the set of all self-coherent points. Indeed, the usual coherence semantics is uniform as we have seen in the interpretation of the “of course” connective and also in the interpretation of the quantifiers. In constructing the web of the coherence space interpreting a “for all” formula, we reject all points which are not “coherent with themselves”. In variable $M$-spaces, no such discrimination is applied and this is another form of non-uniformity.

The present paper requires from the reader some familiarity with the non-uniform semantics of linear logic defined in [BuEh,00], as well as with the semantics of system $F$ defined by Girard in [Gi,86].

1. Definition of the relational model

In the sequel, we will denote by $I$ the category of sets and injections and given a set $X$, we will denote by $\mathcal{P}(X)$ the set of all the subsets of $X$.

**Definition 1.1.** Let $f : X \hookrightarrow Y$ be a morphism of $I$. One defines:

\[
\begin{align*}
  f^- : \mathcal{P}(Y) & \to \mathcal{P}(X) \\
  B & \mapsto f^{-1}(B) \\
  f^+ : \mathcal{P}(X) & \to \mathcal{P}(Y) \\
  A & \mapsto f(A)
\end{align*}
\]

Then $f^- \circ f^+ = Id_{\mathcal{P}(X)}$ and $f^+ \circ f^- = \pi_{f(X)}$.

We aim at defining a purely relational model of second-order linear logic, but we would like to preserve the notion of normal form developed by Girard in [Gi,86].

For this purpose, given a formula $F$ having only one free variable $X$ and given a set $U$ and an element $a$ of $F(U)$, we should be able to compute $a$ from a finite amount of data. As a consequence, we are led to consider continuous functors (that is functors preserving directed limits). Moreover, we would like to be able to compute $a$ from a minimal amount of data. In other words, there should exist a minimal finite subset $U_0$ of $U$ such that $a \in F(U_0)$. This leads us to consider stable functors (that is, continuous functors which preserve pullbacks). For convenience, we will also demand that stable functors preserve inclusions.

**Definition 1.2.** A stable functor is a functor $F$ from $I^n$ to $I$ which preserves inclusions, directed limits and pullbacks (fibered products).

Observe that if $F$ is a $n + 1$-ary stable functor, given $X_1, \ldots, X_n$ a family of sets, the functor $Y \mapsto F(\overline{X}, Y)$ is still a stable functor. In fact, the composite of stable functors is a stable functor and the constant functors are obviously stable.

**Theorem 1.3.** Normal Form Theorem

Let $F : I \to I$ be a stable functor, $X$ be a set and $a$ be a point of $F(X)$. There exist $X_0$ a finite set, $a_0 \in F(X_0)$ and $f : X_0 \hookrightarrow X$ such that

(i) $a = F(f)(a_0)$

(ii) for any $Y \in I$ and $f' : Y \hookrightarrow X$, for any $b \in F(Y)$ such that $a = F(f')(b)$, there exists an injection $g : X_0 \hookrightarrow Y$ such that $b = F(g)(a_0)$.

We will say that $(X_0, a_0)$ is a normal form of $(X, a)$ with respect to $F$ (notice that it is unique only up to isomorphisms).

**Proof.**

Let $\mathcal{X}$ be the set of all finite subsets of $X$. $\mathcal{X}$ is directed for inclusion and $X = \bigcup_{x_0 \in \mathcal{X}} X_0$. By hypothesis, $a \in F(X)$ and as $F$ is a stable functor,

\[
F\left(\bigcup_{x_0 \in \mathcal{X}} X_0\right) = \bigcup_{x_0 \in \mathcal{X}} F(X_0)
\]
Therefore there exists \( X_0 \in \mathcal{X} \) such that \( a \in F(X_0) \). Let us assume that \( X_0 \) has been chosen minimal for inclusion. Let \( Y \in \mathcal{I} \) be a set, \( b \in F(Y) \) be a point and \( f' : Y \to X \) be an injection such that \( a = F(f')(b) \). Then \( f' = \iota \circ \tilde{f} \) where \( \tilde{f} \) is the bijection induced by \( f' \) onto its range and \( \iota \) is the inclusion from \( Z = f'(Y) \) into \( X \). As \( F \) preserves inclusions, we have \( a = F(\tilde{f})(b) \). Therefore, \( a \) belongs to both \( F(X_0) \) and \( F(Z) \) with \( X_0, Z \subseteq X \) and as \( F \) preserves pullbacks, we have \( a \in F(X_0 \cap Z) \). Thus, by minimality of \( X_0 \), we conclude that \( X_0 \subseteq Z \), and thus \( g = (\tilde{f})^{-1}|_{X_0} \) is an injection from \( X_0 \) into \( Y \) such that \( b = F(g)(a) \).

**Notation.** In the sequel, given an integer \( n \), we will write \( \pi \) for \( \{0, \ldots, n - 1\} \).

By theorem 1.3, one can associate to any unary stable functor a trace defined as follows.

**Definition 1.4.** Let \( F : \mathcal{I} \to \mathcal{I} \) be a stable functor. The pre-trace \( \| \text{Tr}(F) \| \) of \( F \) is defined as the set of all pairs \((n, a)\) such that \( n \) is an integer, \( a \) belongs to \( F(\pi) \), and \((\pi, a)\) is a normal form.

Let \( \sim_F \) be the following equivalence relation: if \( (n, a), (m, b) \in \| \text{Tr}(F) \| \), then \( (n, a) \sim_F (m, b) \) if and only if \( n = m \) and there exists \( \sigma \in \mathcal{G}_n \) such that \( b = F(\sigma)(a) \).

Then the trace of \( F \) is defined by:

\[
\text{Tr}(F) = \| \text{Tr}(F) \| / \sim_F
\]

In the sequel, we will denote by \( \langle n, a \rangle \) the class of \((n, a)\) for the \( \sim_F \) equivalence relation.

The formulae of \( \text{LL}^2 \) will be interpreted by stable functors. Given \( F \) and \( G \) two \( n \)-ary stable functors, let us define the functor \( \rightarrow G \) by:

- \((F \rightarrow G)(X_1, \ldots, X_n) = F(X_1, \ldots, X_n) \times G(X_1, \ldots, X_n)\) for any family of \(1\)-sets \( X_1, \ldots, X_n\).
- \((F \rightarrow G)(f_1, \ldots, f_n)(a, b) = (F(\tilde{f})(a), G(\tilde{f})(b))\) for any family of injections \((f_i : X_i \to Y_i)_{i \in I}\) and any \((a, b) \in F(\tilde{X}) \times G(\tilde{X})\).

One easily checks that \( F \rightarrow G \) so defined is stable.

Let us now define the notion of object of variable type.

**Definition 1.5.** Let \( F \) be a \( n \)-ary stable functor. An object of variable type \( F \) is a family \( t = (t_{x_1, \ldots, x_n})_{x_1, \ldots, x_n \in \mathcal{X}} \) such that:

(i) for any family of sets \( X_1, \ldots, X_n \), we have \( t_{x_1, \ldots, x_n} \subseteq F(X_1, \ldots, X_n) \),
(ii) \( t \) enjoys the following mutilation property:

given \((f_i : X_i \to Y_i)_{i \in \{1, \ldots, n\}}\) a family of injections

\[
t_{x_1, \ldots, x_n} = F(f_1, \ldots, f_n)^{-1}(t_{y_1, \ldots, y_n})
\]

The mutilation property is crucial to interpret second-order in a uniform way (and therefore, to get soundness). Now, let us consider for a while the category of sets and relations. Let us associate to any stable functor \( F \) a contravariant functor \( F \) of \( \text{Set} \) with the same object part as \( F \) and whose morphism part is defined by \( F(f) = F(f)^{-1} \) for any injection \( f : X \to Y \). Then observe that the mutilation property does not imply the naturality of \( t : F \to G \) (whereas it was the case in system \( F \), see [PAT,89]). Indeed, let us consider the morphism \( ev = ev_{X, 1} : (X \to \pi) \otimes X \to 1 \) defined by \( (ev)_X = \{(((a, \ast), a), \ast) : a \in X\} \) for any set \( X \) (see proposition 2.4). Let us prove that the morphism \( ev \) so defined is not a natural transformation. Let \( f \) be the only application from \( \emptyset \) into \( \{0\} \). Then
the following diagram is not commutative

\[
\begin{array}{ccc}
((X \rightarrow 1) \otimes X) (f) & - & 1 (f) \\
\downarrow & & \downarrow \\
((X \rightarrow 1) \otimes X) (\emptyset) & - & \emptyset (\emptyset)
\end{array}
\]

since we have

\[
(\emptyset) \circ ((X \rightarrow 1) \otimes X) (f) (\emptyset) = \emptyset
\]

whereas

\[
(1 (f)) \circ (\emptyset (\emptyset)) = \emptyset.
\]

Therefore, naturality implies mutilation but the converse is not true. In fact, mutilation does even not imply naturality in the coherence space semantics of LL² (the object of variable type \((ev) X 1\) is still a counter example). However, in system \(F\), naturality is equivalent to mutilation and as we shall see, the intuitionistic hypothesis is indispensable. Let \(F\) and \(G\) be \(n\)-ary stable functors, let \(t : F \Rightarrow G\) be an object of variable type and let \(f : X \rightarrow Y\) be an injection. Let us prove that \(G(f) \circ t \in (F(f))^{-1}\). Let \((\mu, a) \in G(f) \circ t \in (F(f))^{-1}\). Now, let \(\nu = F(f)^{-1}(\mu)\): observe that in both the set and the multisemantics, \(\nu\) is well defined, even if possibly empty, this is precisely why an object of variable type \(F \rightarrow G\) is not a natural transformation, a point (contrarily to a set) does not necessarily have an antecedent by \(F(f)^{-1}\). Now as \((\mu, G(f)(a)) \in ty\), we have \((1 F \rightarrow G)(f)^{-1}(\mu, G(f)(a)) \in (1 F \rightarrow G)(f)^{-1}(ty) = t_X\) by mutilation. Moreover \((1 F \rightarrow G)(f)^{-1}(\mu, G(f)(a)) = (\nu, a)\), thus \((\mu, \nu) \in (F(f)^{-1}\) and \((\nu, a) \in t_X\) and as a consequence, \((\mu, a) \in t_X\) \(\circ (F(f))^{-1}\).

Last, observe that when \(f\) is an inclusion \(X \subseteq Y\), the mutilation property can be rephrased as \(t_X = t_Y \cap F(X)\).

Let us now define the category \(VRel\) where we will interpret LL². The point is that without coherence, the usual relational composition operation is not well defined. Indeed, the relational composite of two objects of variable type does not necessarily satisfy the mutilation property. For instance, consider the following objects of variable type: let \(t = \pi^*\) and \(s = \rho^*\) where \(\pi\) is the proof

\[
\begin{array}{c}
\vdash X^*, X \\
\vdash X \rightarrow X \\
\vdash 1 \rightarrow 0 (X \rightarrow X)
\end{array}
\]

and \(\rho\) is the proof

\[
\begin{array}{c}
\vdash X^*, X \\
\vdash X \rightarrow X \\
\vdash (X \rightarrow X), 0 \\
\vdash (X \rightarrow X) \rightarrow 0 \rightarrow 0
\end{array}
\]

with \(X\) a second-order variable. Given a set \(X\), we have \(t_X = \{((*, a, a)); a \in X\}\) and \(s_X = \{((a, a), *); a \in X\}\). As \(t\) and \(s\) are interpretations of proofs, they are respectively objects of variable type \(1 \rightarrow (X \rightarrow X)\) and \((X \rightarrow X)^+ \rightarrow \bot\) (which is equal to \((X \rightarrow X) \rightarrow \bot\), since in the relational model, any variable type \(T\) is equal to \(T^+\)). Let us now compose \(s\) and \(t\) with the relational composition operation. First, we have \(s_0 = t_0 = \emptyset\) therefore \((t \circ s)_0 = \emptyset\). Moreover, \(s_{(0,0)} = \{((*, (0,0))\}\)

and
\[ t_{(0)} = \{ (0,0), \ast \} \] and therefore \((t \circ s)_{(0)} = \{ (\ast, \ast) \} \). Let now \( f \) be the only injection from the empty set into \( \{0\} \). We have \((1 \rightarrow \bot)(f) = \{ (\ast, \ast), (\ast, \ast) \} \) (indeed, observe that for any set \( X \) we have \((1 \rightarrow \bot)(X) = \{ (\ast, \ast) \} \)). As a consequence, \((1 \rightarrow \bot)(f)^{-1}(t \circ s)_{(0)} = \{ (\ast, \ast) \} \neq (t \circ s)_{(0)} \) and defining \((t \circ s)\) in that way, we could not obtain an object of variable type \( 1 \rightarrow \bot \).

**Definition 1.6.** Let us define the following \( \text{VRrel} \) category:

- **Objects:** stable functors from \( T^n \) to \( I \) (for \( n \in \mathbb{N} \)).
- **Morphisms:** if \( F, G \) are \( n \)-ary stable functors, a morphism from \( F \) to \( G \) is an object of variable type \( F \rightarrow G \) (written \( t \colon F \rightarrow G \)).
  - **Identity:** \((\text{Id}_F)_{X_1, \ldots, X_n} = \{ (a,a) \colon a \in F(X_1, \ldots, X_n) \} \).
  - **Composition:** let \( F, G, H \) be \( n \)-ary stable functors and \( t \colon F \rightarrow G \), \( s \colon G \rightarrow H \) be objects of variable type, then: \( (a,c) \in (s \circ t)_{X_1, \ldots, X_n} \) if there exist sets \( Y_1, \ldots, Y_n \), injections \( f_1 : X_1 \hookrightarrow Y_1, \ldots, f_n : X_n \hookrightarrow Y_n \) and \( b \in G(Y_1, \ldots, Y_n) \) such that:
    
    \[
    (F(f_1, \ldots, f_n)(a), b) \in t_{Y_1, \ldots, Y_n} \quad \text{and} \quad (b, H(f_1, \ldots, f_n)(c)) \in s_{Y_1, \ldots, Y_n}.
    \]

We will denote by \( \text{VRrel}[n] \) the subcategory of \( \text{VRrel} \) whose objects are the \( n \)-ary functors of \( \text{VRrel} \).

Now let us come back to the previous example. With our definition of composition, one uses \( f : \emptyset \hookrightarrow \{0\} \) and \( b = (0,0) \in (X \rightarrow X)((\emptyset)) \) to compose at level \( \emptyset \), and therefore, \((t \circ s)_{\emptyset} = \{ (\ast, \ast) \} \).

In fact, in coherence spaces, \( t \) is an object of variable type \( (X \rightarrow X)^\perp \rightarrow \perp \), which is not isomorphic to \((X \rightarrow X) \rightarrow \perp \). Therefore, \( s \) and \( t \) can not be composed which prevents such problems.

Now let us mention another important point. As the relational model is not uniform (see [BrEh00]), given two morphisms \( s : F \rightarrow G \) and \( t : G \rightarrow H \), if \((a,c) \in t_X \circ s_X \) the point \( b \in G(X) \) such that \((a,b) \in s_X \) and \((b,c) \in t_X \) is not unique in general. In fact, there could even be an infinite number of such points. For instance, in the previous example, composing \( s : 1 \rightarrow (X \rightarrow X) \) and \( t : (X \rightarrow X) \rightarrow 1 \) at level \( n \) we have \((\ast, (n,n)) \in s_n \) and \(((n,n), \ast) \in t_n \) for any integer \( n \). However, remember that in coherence spaces, as \( s \) and \( t \) are cliques respectively of \( F \rightarrow G \) and \( G \rightarrow H \), this point is unique. In [Eh00,b], Thomas Ehrhard shows that however, a finiteness condition on the number of these points can be semantically imposed.

**Proposition 1.7.**

(i) **For any \( n \)-ary stable functor \( F \), the family \( \text{Id}_F \) is an object of variable type \( F \rightarrow F \) (ie. enjoys the mutilation property) and is neutral for the composition operation.**

(ii) **Given \( n \)-ary stable functors \( F,G,H \) and two objects of variable type \( t : F \rightarrow G \) and \( s : G \rightarrow H \) we have**

\[ s \circ t : F \rightarrow H. \]

(iii) **The composition operation is associative.**

**Proof.**

The first and last points are immediate. For notational convenience, let us assume that \( F, G \) and \( H \) are unary stable functors. Given \( f : X \rightarrow Y \), let us prove that \( (s \circ t)_X \subseteq (F \rightarrow H)(f)^{-1}(s \circ t)_Y \).
Let \((a, c) \in (s \circ t)_X\). By definition, there exist a set \(Z\), an injection \(g : X \hookrightarrow Z\) and a point \(b \in G(Z)\) such that \((F(g)(a), b) \in t_Z\) and \((b, H(g)(c)) \in s_Z\). One has the following commutative diagram (which is in fact a pullback):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{f'} \\
Z & \xrightarrow{g'} & Y \oplus_{\overline{X}} Z \\
\end{array}
\]

where \(Y \oplus_{\overline{X}} Z\) is the coalesced sum of \(Y\) and \(Z\) over \(X\) defined by:

\[
Y \oplus_{\overline{X}} Z = Y \amalg Z / \equiv
\]

with \((1, y) \equiv (2, z)\) if there exists a point \(x \in X\) such that \(y = f(x)\) and \(z = g(x)\) (here \(f'\) and \(g'\) denote the canonical injections). Let \(b' = G(g')(b)\), then by mutuality we have \((F(g')F(g)(a), G(g')(b)) \in (F \circ \ell G)(g')(Z) \subseteq t_W\). Moreover, by functoriality of \(F\), we have \(F(g')F(g)(a) = F(f')F(f)(a)\), which in turn implies that \((F(f')F(f)(a), b') \in t_W\). One proves in the same way that \((b', H(f')H(f)(c)) \in s_W\). Consequently, \((F \circ \ell H)(f)(a, c) \in (s \circ t)_Y\).

Conversely, let us check that \((F \circ \ell H)(f)(s \circ t)_Y \subseteq (s \circ t)_X\). Let \((a, c) \in (F \circ \ell H)(f)(s \circ t)_Y\). By definition of composition, there exist a set \(Z\), an injection \(g : Y \hookrightarrow Z\) and a point \(b \in G(Z)\) such that \((F(g)F(f)(a), b) \in t_Z\) and \((b, H(g)H(f)(c)) \in s_Z\). Thus, \((F(g \circ f)(a), b) \in t_Z\) and \((b, H(g \circ f)(c)) \in s_Z\), with \(g \circ f : X \hookrightarrow Z\), which implies that \((a, c) \in (s \circ t)_X\).

Now, let \(F\) be a stable functor and let \(t = (t_X)_{X \in \mathcal{I}}\) be a family such that for any set \(X\), \(t_X \subseteq F(X)\). The closure under mutilation of \(t\) (let us denote it by \(\overline{t}\)) is the smallest object of variable type containing \(t\). At level \(X\), \(\overline{t}_X\) can be described as the union of all the sets \(F(f)^{-1}(t_Y)\), where \(Y\) is a set and \(f : X \hookrightarrow Y\) is an injection. Observe that \(s \circ t\) can be equivalently defined as the closure under mutilation of \((s_X \circ t_X)_{X \in \mathcal{I}}\) (where \(s_X \circ t_X\) denotes the relational composition), that is

\[
(s \circ t)_X = \{(F \circ \ell G)(f)^{-1}(a, c) ; Y \in \mathcal{I}, (a, c) \in s_Y \circ t_Y, f : X \hookrightarrow Y\}
\]

**Proposition 1.8.** Let \(F, G, H\) be \(n\)-ary stable functors, \(t\) be an object of variable type \(F \circ \ell G\) and \(s\) be an object of variable type \(G \circ \ell H\). Then given a set \(X\), a couple \((a, c)\) belongs to \((s \circ t)_X\) if and only if there exists a set \(Z\) with \(X \subseteq Z\), and a point \(b \in G(Z)\) such that \((a, b) \in t_Z\) and \((b, c) \in s_Z\).

**Proof.**

Given a set \(X\), let us consider \((a, c) \in (s \circ t)_X\). By definition of composition, there exist a set \(Y\), an injection \(f : X \hookrightarrow Y\) and a point \(b \in G(Y)\) such that \((F(f)(a), b) \in t_Y\) and \((b, H(f)(c)) \in s_Y\). The function \(f\) is an injection, therefore, \(f = 1 \circ \overline{f}\) where \(\overline{f}\) is the bijection induced by \(f\) onto its range, and \(1\) is the inclusion from \(f(X)\) into \(Y\). Now let \(Z\) be a superset of \(X\) and \(g : Z \hookrightarrow Y\) be a bijection
such that the following diagram is commutative,
\[
\begin{array}{c}
Z \xrightarrow{g} Y \\
\downarrow \quad \quad \quad \downarrow \\
X \xrightarrow{f} f(X) \\
\subseteq \quad \quad \quad \subseteq
\end{array}
\]

We obtain that \( F(f)(a) = F(g)(a) \); thus \((F(g)(a), b) \in t_Y \) and as \( g \) is a bijection, we conclude by mutation that \((a, F(g^{-1})(b)) \in t_X \). Let us denote \( F(g^{-1})(b) \) by \( b' \), we have \( X \subseteq Z \), with \((a, b') \in t_Z \) and \((b', c) \in s_Z \).

Consequently, one easily proves the following lemma.

**Lemma 1.9.** Let \( F, G, H \) be \( n \)-ary stable functors, let \( s : F \to G \), \( s' : G \to H \) and \( t : F \to H \) be objects of variable type. The following propositions are equivalent

(i) \( s' \circ s = t \)

(ii) for any set \( X \), we have \( s'_X \circ s_X \subseteq t_X \) and moreover, for any \((a, c) \in t_X \), there exists a set \( Y \) with \( X \subseteq Y \) such that \((a, c) \in s'_Y \circ s_Y \).

**Proof.**

For notational convenience, let us assume that \( n = 1 \). Let us prove \((ii) \Rightarrow (i)\).

Let \( X \) be a set, and let \((a, c)\) be a point of \( (s' \circ s)_X \). By proposition 1.8, there exist a set \( Y \supseteq X \) and a point \( b \in G(Y) \) such that \((a, b) \in s'_Y \) and \((b, c) \in s_Y \). Thus \((a, c) \in s'_Y \circ s_Y \) as \( s'_Y \circ s_Y \subseteq t_Y \), we get \((a, c) \in t_Y \). Moreover, as \( t \) is an object of variable type and \((a, c) \in (F \to H)(X) \), we conclude by mutation that \((a, c) \in t_X \). The converse inclusion (that is \( t_X \subseteq (s' \circ s)_X \)) is trivial.

Observe that as a corollary, if \((s' \circ s)_X \in \mathcal{I}_X \) is an object of variable type, then \( s' \circ s = (s' \circ s)_X \in \mathcal{I}_X \).

Now, given a stable functor \( F \), let us represent the objects of variable type \( F \) as subsets of \( \text{Tr}(F) \).

**Proposition 1.10.** Let \( F : \mathcal{I} \to \mathcal{I} \) be a stable functor. There is a canonical bijection between the objects of variable type \( F \) and the subsets of \( \text{Tr}(F) \) and this correspondence is defined as follows:

- To an object \( t \) of variable type \( F \) we associate \( \text{Tr}(t) = \{(n, a) \in \text{Tr}(F) : a \in t_\pi\} \)

- To a subset \( x \) of \( \text{Tr}(F) \) we associate the object of variable type \( x \in \mathcal{I}_X \)

\[\text{defined by} \quad x_X = \{F(f)(b) ; (n, b) \in x \text{ and } F(f) \in x\} \]

**Proof.**

Let \( x \subseteq \text{Tr}(F) \). Let us prove that \( t = (x \in \mathcal{I}_X \) is an object of variable type \( F \). Let \( X, Y \) be sets and let \( f : X \leftrightarrow Y \) be an injection. We show first that \( x_X \subseteq F(f)^{-1}(x_Y) \).

Let \( c = F(g)(b) \in x_X \), with \( (n, b) \in x \) and \( g : \pi \leftrightarrow Y \). Then \( f \circ g : \pi \leftrightarrow Y \) and therefore \( F(f)(c) \in x_Y \).

Conversely let \( c \in F(f)^{-1}(x_Y) \). We have \( F(f)(c) \in x_Y \), therefore there exist \((n, b) \in x \) and \( g : \pi \leftrightarrow Y \) such that \( F(f)(c) = F(g)(b) \). But \((n, b) \in \text{Tr}(F) \), therefore \( (\pi, b) \) is a normal form of \((F(Y), F(f)(c))\) and by theorem 1.3, there exists a function \( h : \pi \leftrightarrow X \) such that \( c = F(h)(b) \). As a consequence \( c \in x_X \).

Now, it is obvious that \( \text{Tr}(t) = x \).
Conversely, let \( t : F \). For any set \( X \), \( \text{Tr}(t)\{X\} = tx \) (both inclusions are immediate using mutilation and theorem 1.3).

By the previous proposition, it is possible to rebuild all of the objects of variable type \( F \) from the trace of \( F \), using \( F \) itself. Notice that unlike the trace of stable functions, \( \text{Tr}(F) \) is not sufficient to determine the functor \( F' \): neither the object nor the morphism part of \( F \) can be described by means of \( \text{Tr}(F) \).

2. **VRel is a model of second-order linear logic**

In the sequel, \( n \) will denote an arbitrary integer, \( \vec{X} \) will stand for \( X_1, \ldots, X_n \), and \( F, G, H \ldots \) will denote \( n \)-ary stable functors.

### 2.1 Multiplicative fragment.

Given \( F \) and \( G \) two \( n \)-ary stable functors, let us define the functor \( F \otimes G \) by:

- \( (F \otimes G)(X_1, \ldots, X_n) = F(X_1, \ldots, X_n) \times G(X_1, \ldots, X_n) \) for any family of sets \( X_1, \ldots, X_n \).
- \( (F \otimes G)(f_1, \ldots, f_n)(a, b) = (F(f_1, \ldots, f_n)(a), G(f_1, \ldots, f_n)(b)) \) for any family of injections \( (f_i : X_i \rightarrow Y_i)_{i \in \{1 \ldots n\}} \).

One easily checks that \( F \otimes G \) is a stable functor.

Given \( t : F \rightarrow G \) and \( s : F' \rightarrow G' \), let us define for any family of sets \( X_1, \ldots, X_n \),

\[
(s \otimes t) x_{1}, \ldots, x_n = s x_{1}, \ldots, x_n \otimes t x_{1}, \ldots, x_n
\]

\[
= \{ ((a, d'), (b, b')) : (a, b) \in s x_{1}, \ldots, x_n, (a', b') \in t x_{1}, \ldots, x_n \}
\]

**Lemma 2.1.** Let \( s \) and \( t \) be objects of variable type \( F \rightarrow G \) and \( F' \rightarrow G' \), then

\[ s \otimes t : (F \otimes F') \rightarrow (G \otimes G') \]

The proof is straightforward.

The operation \( \otimes \) so defined is a functor from \( \text{VRel}[n] \times \text{VRel}[n] \) to \( \text{VRel}[n] \)

(proof is very similar to the proof of the functoriality of \( ! \) in proposition 2.7).

#### Proposition 2.2.

(i) Let \( ! \) be the object of \( \text{VRel}[n] \) defined by:

\[ 1(\vec{X}) = \{ x \} \] for any family of sets \( X_1, \ldots, X_n \)

\[ 1(f) = \{ (x, x) \} \] for any family of injections \( (f_i : X_i \rightarrow Y_i)_{i \in \{1 \ldots n\}} \)

\( ! \) is a stable functor.

(ii) Let \( \alpha_{\vec{X}} = \{ ((a, b, c), (a, b, c)) : a, b, c \in F(\vec{X}) \times G(\vec{X}) \times H(\vec{X}) \} \), then

\( \alpha : F \otimes (G \otimes H) \rightarrow (F \otimes G) \otimes H \) is an isomorphism.

(iii) Let \( \iota_{\vec{X}} = \{ ((a, a), a) : a \in F(\vec{X}) \} \), then

\( \iota : ! \otimes F \rightarrow F \) is an isomorphism.

(iv) Let \( \iota_{\vec{X}} = \{ ((a, a), a) : a \in F(\vec{X}) \} \), then

\( \iota_{\vec{X}} : F \otimes ! \rightarrow F \) is an isomorphism.

**VRel[n] equipped with \( \otimes \), \( ! \), \( \alpha \), \( \iota \) and \( \iota_r \) is monoidal.**

**Proof.** All the mutilation properties and diagram commutations are essentially trivial.

#### Proposition 2.3. Let \( \gamma_{F,G} : F \otimes G \rightarrow G \otimes F \) be an isomorphism,

\[ \gamma_{F,G} : F \otimes G \rightarrow G \otimes F \]

and \( \text{VRel}[n] \) is symmetric monoidal.

**Proof.** By the corollary of lemma 1.9, it suffices to show that for any set \( X \), we have

\[ (\gamma_{G,F})_X \circ (\gamma_{F,G})_X = (\text{Id}_{F \otimes G})_X \]

and this is proved “on the nose”. 

\[ \square \]
Proposition 2.4. Let \( s : (F \otimes G) \rightarrow H \) be an object of variable type. For any family of sets \( X_1, \ldots, X_n \), let us define \( \lambda^{F,G,H}(s) \in \{(a,(b,c)) ; (a,b,c) \in s_X \} \), then \( \lambda^{F,G,H}(s) : F \rightarrow (G \rightarrow H) \) and \( \lambda \) is a natural bijection.

Furthermore, let us define \( (\text{ev}_{F,G})_X = \{(a,b) \cdot b \in G(X)\} \), then \( \text{ev}_{F,G} : (F \rightarrow G) \otimes F \rightarrow G \) and the following diagram is commutative

\[
\begin{array}{ccc}
G \rightarrow H & \otimes & F \\
\downarrow \text{ev}_{G,H} & & \downarrow \text{id}_F \\
F & \otimes & G \\
\end{array}
\]

As a consequence, VRel[\([n]\)] is symmetric monoidal closed.

Proof. By the corollary of lemma 1.9, it suffices to show that for any family of sets \( X_1, \ldots, X_n \), we have \( (\text{ev}_{F,G})_X \circ (\lambda^{F,G,H}(s) \otimes \text{id}_G)_X = s_X \) and this is proved "on the nose".

In fact, most of the diagram commutations for LL will be proved as above: we will use the corollary of lemma 1.9, conclude "on the nose" and as a consequence, the heart of the proof will lie in showing that a given family is actually an object of variable type. In the previous proof, we use the hypothesis that \( s \) is an object of variable type.

Proposition 2.5. VRel[\([n]\)] is a \( \ast \)-autonomous category with dualizing object \( \bot = \mathbb{1} \).

Proof. One easily checks that \( (\text{ev}_{F,\bot} \circ \gamma_{F,F \rightarrow \bot})_X = (\text{ev}_{F,\bot})_X \circ (\gamma_{F,F \rightarrow \bot})_X \). As a consequence, one can prove "on the nose" that \( \lambda^{F,F + \bot,\bot} \circ (\text{ev}_{F,\bot} \circ \gamma_{F,F \rightarrow \bot})_X : F \rightarrow ((F \rightarrow \bot) \rightarrow \bot) \) is an isomorphism.

Therefore, by [Br,95], VRel[\([n]\)] is a model of MLL.

2.2. Additive fragment.

Given \( F,G : I^n \rightarrow I \) two stable functors, let us define the following functor:

1. \( (F \& G)(X_1, \ldots, X_n) = F(X_1, \ldots, X_n) + G(X_1, \ldots, X_n) \) for any family of sets \( X_1, \ldots, X_n \).
2. \( (F \& G)(f_1, \ldots, f_n) = F(f_1, \ldots, f_n) + G(f_1, \ldots, f_n) \) for any family of injections \( (f_i : X_i \hookrightarrow Y_i)_{i \in \{1, \ldots, n\}} \).

\( F \& G \) so defined is a stable functor.

Proposition 2.6. Let \( F,G,H \) be \( n \)-ary stable functors.

(i) Let \( p_1 : F \& G \rightarrow F \) and \( p_2 : F \& G \rightarrow G \) be defined by

\[ p_1_X = \{ (a,1,a) ; a \in F(\tilde{X}) \} \]

\[ p_2_X = \{ (b,2,b) ; b \in G(\tilde{X}) \} \]

(ii) Given \( f : H \rightarrow F \) and \( g : H \rightarrow G \), let us define \( \langle f,g \rangle : H \rightarrow F \& G \) with

\[ \langle f,g \rangle_X = \{ c, (1,a) \} ; (c,a) \in f_X \} \cup \{ c, (2,b) \} ; (c,b) \in g_X \} \]

(iii) Let \( \top \) be the empty stable functor (for any stable functor \( F \) there exists a unique object of variable type \( F \rightarrow \top \), namely \( \{0\}_X \)).

VRel[\([n]\)] so equipped is Cartesian (with terminal object \( \top \)).
Proof. All of the diagram commutations are straightforward. □

As a consequence, by [Bt,95] \( \text{VRel}[n] \) is a model of MALL.

2.3. Exponential fragment.
In the sequel, \( \mathcal{M}_{\text{fin}}(X) \) will denote the set of all finite multisets of \( X \).

Given a \( n \)-ary stable functor \( F \), let us define the following functor:

- \( !F(X_1, \ldots, X_n) = \mathcal{M}_{\text{fin}} \left( F(X) \right) \) for any family of sets \( (X_i)_{i \in \{1, \ldots, n\}} \).

- \( !F(f_1, \ldots, f_m)\{a_1, \ldots, a_m\} = \left[ F(f_1)(a_1), \ldots, F(f_m)(a_m) \right] \) for any family of injections \( (f_i : X_i \rightarrow Y_i)_{i \in \{1, \ldots, n\}} \).

\( !F \) so defined is a stable functor.

**Proposition 2.7.** Let \( F, G \) be stable \( n \)-ary functors and let \( t : F \rightarrow G \) be an object of variable type.

(i) Let \( !t : X_1, \ldots, X_n = \left\{ \left( \sum_{i=1}^{m} [a_i], \sum_{j=1}^{r} [b_j] \right) : (a_i, b_j) \in t_{X_1, \ldots, X_n} \right\} \), then

\[ !t : F \rightarrow !G. \]

(ii) Let \( \delta^F_{X_1, \ldots, X_n} = \{([a], a) : a \in F(X_1, \ldots, X_n) \} \), then

\[ \delta^F : !F \rightarrow F. \]

(iii) Let \( p^F_{X_1, \ldots, X_n} = \{(x_1 + \cdots + x_m, x_1, \ldots, x_m) : x_i \in F(X_1, \ldots, X_n) \} \), then

\[ p^F : !F \rightarrow !F. \]

(iv) \( (\delta, \beta, \gamma) \) is a comonad and the following objects of variable type are isomorphisms:

\[ s_X = \{ (\sum_{i=1}^{m} [a_i], \sum_{j=1}^{r} [b_j]) : a_i \in F(X), b_j \in G(X) \} \]

\[ s : !(F \otimes G) \rightarrow !F \otimes !G. \]

\[ t_X = \{([[\star]], \star) \} : !T \rightarrow !1 \]

Proof. Most of the diagram commutations are straightforward. We will only prove the functoriality of the \( ! \) operation. Let \( F, G \) and \( H \) be \( n \)-ary stable functors. Let \( s : F \rightarrow G \) and \( t : G \rightarrow H \) be objects of variable type; let us prove that \( !(t \circ s) = t ! s \) (we will use lemma 1.9).

First one easily checks that \( (t ! s)_X \subseteq (t \circ s)_Y \). Conversely, let \( \alpha, \gamma \) be an element of \( (t \circ s)_Y \). Then \( \alpha = [a_1, \ldots, a_m] \) and \( \gamma = [c_1, \ldots, c_m] \) with for any \( i \in \{1, \ldots, m\} \) \( (a_i, c_i) \in (t \circ s)_Y \). Therefore for any \( i \in \{1, \ldots, m\} \) there exists a family of sets \( Y^1_{i1}, \ldots, Y^r_{in} \) such that for any \( j \in \{1, \ldots, n\} \), we have \( X_j \subseteq Y^j_{ij} \) and \( (a_i, c_i) \in t_{Y^j_{ij}} \circ s \). For any \( j \in \{1, \ldots, n\} \), set \( Y_j = \bigcup_{i \in \{1, \ldots, m\}} Y^j_{ij} \). Then by pullback we have \( (a_i, c_i) \in t_{Y} \circ s \). Therefore there exists an element \( b_j \in G(Y) \) such that \( (a_i, b_j) \in s \) and \( (b_j, c_i) \in t_Y \). Now, set \( \beta = [b_1, \ldots, b_m] \), we have \( (a, \beta) \in (s)_Y \) and \( (\beta, \gamma) \in (t)_Y \). As a consequence, \((a, \gamma) \in (t \circ s)_Y \) and we are done by lemma 1.9. □

Therefore by [Bt,95], \( \text{VRel}[n] \) is a categorical model of LL.

2.4. Second-order.
In this section we address the second-order part of the model. The problem consists in showing that the “useless variable functor” \( \Psi : \text{VRel}[n] \rightarrow \text{VRel}[n+1] \) defined by \( \Psi F(X_1, \ldots, X_{n+1}) = F(X_1, \ldots, X_n) \) (and similarly for morphisms) admits a right adjoint \( \mathcal{T} : \text{VRel}[n+1] \rightarrow \text{VRel}[n] \).
2.4.1. The \( \mathcal{T} \) operation.

Given \( F : \mathcal{I}^{n+1} \rightarrow \mathcal{I} \) a stable functor and \( X_1, \ldots, X_n \) a family of sets, let us write
\[
\mathcal{T}F(X_1, \ldots, X_n) = \text{Tr}(Y \mapsto F(X, Y)).
\]

**Proposition 2.8.** Let \( F : \mathcal{I}^{n+1} \rightarrow \mathcal{I} \) be a stable functor, \( X_1, \ldots, X_n \) be sets, \( \langle n, a \rangle \in \mathcal{T}F(X) \) and for any \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \hookrightarrow Y_i \) be an injection, then \( \langle n, F(f, \pi)(a) \rangle \in \mathcal{T}F(Y) \).

**Proof.**

For notational convenience, we will take \( m = 1 \). Let \( Y \) be a set and \( f : X \hookrightarrow Y \) be an injection. Let \( \langle n, a \rangle \in \mathcal{T}F(X) \), then \( a \in F(X, \pi) \) and thus by mutilation \( F(f, \pi)(a) \in F(Y, \pi) \). Let \( m \leq n \), let \( b \in F(Y, \pi) \) and let \( g : \pi \hookrightarrow \pi \) be such that \( F(f, \pi)(a) = F(Y, g)(b) = d \). Then the following diagram is a pullback,
\[
\begin{array}{ccc}
(X, \pi) & \xrightarrow{(X, g)} & (X, \pi) \\
(f, \pi) \downarrow & & \downarrow (f, \pi) \\
(Y, \pi) & \xleftarrow{(Y, g)} & (Y, \pi)
\end{array}
\]

Since \( F \) is a stable functor, we get
\[
\begin{array}{ccc}
F(X, \pi) & \xrightarrow{F(X, g)} & F(X, \pi) \\
\downarrow F(f, \pi) & & \downarrow F(f, \pi) \\
F(Y, \pi) & \xleftarrow{F(Y, g)} & F(Y, \pi)
\end{array}
\]

and since this diagram is a pullback, there exists \( c \in F(X, \pi) \) such that \( a = F(X, g)(c) \). Therefore by minimality of \( n \), we have \( n = m \) and as a consequence \( \langle n, F(f, \pi)(a) \rangle \in \mathcal{T}F(Y) \). \( \square \)

**Definition 2.9.** Given \( F : \mathcal{I}^{m+1} \rightarrow \mathcal{I} \) a stable functor, the stable functor \( \mathcal{T}F : \mathcal{I}^m \rightarrow \mathcal{I} \) is defined by

- \( \mathcal{T}F(X) = \text{Tr}(Y \mapsto F(X, Y)) \) for any sets \( X_1, \ldots, X_m \)
- For any family of injections \( \langle f_i : X_i \hookrightarrow Y_i \rangle_{i \in [1 \ldots n]} \),
  \[
  \mathcal{T}F(f) : \mathcal{T}F(X) \quad \mapsto \quad \mathcal{T}F(Y)
  \]
  \[\langle n, a \rangle \quad \mapsto \quad \langle n, F(f, \pi)(a) \rangle\]

Notice that, by proposition 2.8, \( \mathcal{T}F \) is well defined and one easily checks that it is a functor. Moreover, one easily proves that \( \mathcal{T}F \) is stable; indeed, \( \mathcal{T}F \) obviously preserves inclusions and directed limits, as \( F \) does. Now, it is immediate that given two sets \( X \) and \( Y \), if an element \( \langle n, a \rangle \) belongs to both \( \mathcal{T}F(X) \) and \( \mathcal{T}F(Y) \), then by stability of \( F \), we have \( a \in F(X \cap Y, \pi) \) and we conclude that \( \langle n, a \rangle \in \mathcal{T}F(X \cap Y) \) (ie. \( \mathcal{T}F \) is stable).

2.4.2. \( V\text{Rel} \) is a model of second-order linear logic.

We are now about to show that the functor \( \Phi \) admits a right adjoint whose object part \( F \mapsto \mathcal{T}F \) has just been defined. First let us prove the following fundamental lemma,
Lemma 2.10. Let $F, H : T^n \rightarrow I$ and $G : T^{m+1} \rightarrow I$ be stable functors and let $t : \Phi F \rightarrow G$ and $s : H \rightarrow F$ be objects of variable type. Let us define:

$$\Lambda^{F,G}(t)_X = \{(b, \langle n, a \rangle); (n, a) \in \mathcal{S}G(X) \text{ and } (b, a) \in t_X, \pi\}$$

$$\varepsilon^{G}_{X,Y} = \{\langle (n, b), a \rangle; a \in G(X, Y), (n, b) \in \mathcal{S}G(X) \}$$

$$\exists f : \pi \mapsto Y, a = G(X, f)(b)\}$$

Then $\Lambda^{F,G}(t) : F \rightarrow \mathcal{S}G$ and $\varepsilon^{G} : \Phi \mathcal{S}G \rightarrow G$, and moreover the following equalities hold:

$$(\beta_2) \quad t = \varepsilon^{G} \circ \Phi \Lambda^{F,G}(t)$$

$$(\eta_2) \quad \text{Id}_{\mathcal{S}G} = \Lambda^{\mathcal{S}G,G}(\varepsilon^{G})$$

$$(\sigma) \quad \Lambda^{F,G}(t) \circ s = \Lambda^{H,G}(t \circ \Phi(s))$$

Proof.

For notational convenience, let us assume that $m = 1$. One easily proves that $\Lambda^{F,G}(t)$ and $\varepsilon^{G}$ are objects of variable type (using the normal form theorem and the hypothesis on $t$).

First, let us prove $(\beta_2)$. Given $X$ and $Y$ two sets, let us show that $t_{X,Y} \subseteq (\varepsilon^{G})_{X,Y} \circ (\Phi \Lambda^{F,G}(t))_{X,Y}$. Let $(a, c) \in t_{X,Y}$, by theorem 1.3 (and since $Y \mapsto G(X, Y)$ is stable), there exist an integer $n$, a point $b \in G(X, \pi)$ and an injection $f : \pi \mapsto Y$ such that $(n, b) \in \mathcal{S}G(X)$ and $c = G(X, g)(b)$. Then by mutilation $(a, b) \in t_{X, \pi}$ and thus $(a, (n, b)) \in \Lambda^{F,G}(t)_X$. Moreover $(n, b, c)$ belongs to $(\varepsilon^{G})_{X,Y}$. Therefore $(a, c) \in (\varepsilon^{G})_{X,Y} \circ (\Phi \Lambda^{F,G}(t))_{X,Y}$.

Conversely, one easily proves that $(\varepsilon^{G})_{X,Y} \circ (\Phi \Lambda^{F,G}(t))_{X,Y}$ is a subset of $t_{X,Y}$, and we are done by lemma 1.9.

Let us now prove $(\eta_2)$. Let $X$ be a set. Let us show first that $(\text{Id}_{\mathcal{S}G})_X \subseteq (\Lambda^{\mathcal{S}G,G}(\varepsilon^{G}))_X$. Let $(n, a, n, a) \in (\text{Id}_{\mathcal{S}G})_X$, we have $(n, a) \in (\varepsilon^{G})_X$ and thus $(n, a) \in \Lambda^{\mathcal{S}G,G}(\varepsilon^{G})$.

Conversely, let us prove that $(\Lambda^{\mathcal{S}G,G}(\varepsilon^{G}))_X \subseteq (\text{Id}_{\mathcal{S}G})_X$. Let $(a, c)$ be an element of $(\Lambda^{\mathcal{S}G,G}(\varepsilon^{G}))_X$ (a and $c$ belong to $\mathcal{S}G(X)$), then $c = (n, d', c) \in (\varepsilon^{G})_X$. Hence $(n, d', c) \in \mathcal{S}G(X)$ and there exists an injection $f : \pi \mapsto X$ such that $d' = G(X, f)(a)$. Therefore by minimality of $n$, we have $m = n$, and consequently $f$ is a bijection. Therefore $(n, d', c) \sim_{G} (n, c')$ (or in other words, $(n, c') = (n, c)$).

Last, let us prove $(\sigma)$. Let $X$ be a set. Let us prove first that $(\Lambda^{F,G}(t))_X \circ s_X \subseteq (\Lambda^{H,G}(t \circ \Phi(s)))_X$. Let $(a, (n, c)) \in (\Lambda^{F,G}(t))_X \circ s_X$. There exists a point $b \in F(X)$ such that $(a, b) \in s_X$ and $(b, (n, c)) \in (\Lambda^{F,G}(t))_X$. Thus $(b, c) \in t_{X, \pi}$ and as a consequence $(a, c) \in t_{X, \pi, \Phi(s)}(\Phi(s))_X$. Moreover $(n, c) \in \mathcal{S}G(X)$, hence $(a, (n, c)) \in \Lambda^{H,G}(t \circ \Phi(s))_X$.

Conversely, let $(a, (n, c))$ be an element of $\Lambda^{H,G}(t \circ \Phi(s))_X$ (that is $(a, c) \in (t \circ \Phi(s))_X$). Therefore by proposition 1.8, there exist two supersets $Y \supseteq X$ and $Z \supseteq \pi$ (up to renaming, we can choose $Z = \bar{m}$ for $m \geq n$) and a point $b \in \Phi F(Y, \bar{m})$ such that $(a, b) \in \Phi(s)_{Y, \bar{m}}$ and $(b, c) \in t_{Y, \pi}$. But as $b \in \Phi F(Y, \bar{m}) = \Phi F(Y, \bar{m}) = \Phi F(Y, \bar{m})$ and $c \in G(X, \bar{m}) \subseteq G(Y, \bar{m})$, by mutilation we have $(b, c) \in t_{Y, \pi}$. As a consequence, $(a, b) \in s_Y$ and $(b, (n, c)) \in (\Lambda^{F,G}(t))_Y$. Hence $(a, (n, c)) \in (\Lambda^{F,G}(t)_Y \circ s_Y$ and we are done by lemma 1.9.

In the previous lemma, we have proved all we needed in order to show that $VRel$ is a model of second-order linear logic. What follows are general categorical
considerations (or categorical abstract nonsense as some may say). First, let us define the morphism part of the functor $\mathcal{I}$.

**Proposition 2.11.** Let $F$ and $G$ be stable $n + 1$-ary functors, let $s$ be an object of variable type $F \rightarrow G$ and define $\mathcal{I} s = \Lambda^{F,G}(s \circ \varepsilon^F)$. Then $\mathcal{I}$ so defined is a functor from $\mathbf{VRel}[n + 1]$ into $\mathbf{VRel}[n]$.

**Proof.**

Let $F$ be a $n + 1$-ary stable functor, then we have $\mathcal{I} (\text{Id}_F) = \Lambda^{F,F}(\text{Id}_F \circ \varepsilon^F) = \Lambda^{F,F}(\varepsilon^F) = \text{Id}_{F,F}$ (by the $(\eta_2)$ relation).

Now, let $F, F'$ and $F''$ be stable $n + 1$-ary functors and let $f : F \rightarrow F'$ and $g : F'' \rightarrow F'$ be objects of variable type. We have

$$(\mathcal{I} g) \circ (\mathcal{I} f) = \Lambda^{F',F''} (g \circ \varepsilon^F') \circ \Lambda^{F,F'} (f \circ \varepsilon^F)$$

$$= \Lambda^{F,F'}(g \circ \varepsilon^F' \circ \Phi(\Lambda^{F,F'} (f \circ \varepsilon^F))) \quad \text{(by } \sigma)$$

$$= \Lambda^{F,F'}(g \circ f \circ \varepsilon^F) \quad \text{(by } \beta)$$

$$= \mathcal{I} (g \circ f)$$

$\square$

**Proposition 2.12.** For any integer $n$, we have

$$\mathbf{VRel}[n] \xrightarrow{\Phi} \mathbf{VRel}[n + 1]$$

and $\mathcal{I}$ is the right adjoint of $\Phi$.

See [ML,71] for a definition and more details on adjunction.

**Proof.**

First, for any stable functors $F$ and $G$ respectively in $\mathbf{VRel}[n]$ and $\mathbf{VRel}[n + 1]$, the application $\Lambda^{F,G}$ is bijective. Indeed, by $(\beta)$ for any object $t$ of variable type $\Phi F \rightarrow G$, we have $t = \varepsilon^G \circ \Phi \Lambda^{F,G} (t)$. Moreover if $s$ is an object of variable type $F \rightarrow G$, we have $\Lambda^{F,G}(\varepsilon^G \circ \Phi (s)) = \Lambda^{G,G}(\varepsilon^G) \circ s \equiv (\eta_2) \text{Id}_{\mathbf{VRel}[n]} \circ s = s$.

Now, let $F, F'$ be $n$-ary stable functors, let $G, G'$ be $n + 1$-ary stable functors and let $f : F' \rightarrow F$ and $g : G \rightarrow G'$ be objects of variable type. We want to prove that the following diagrams are commutative,

$$\xymatrix{ \mathbf{VRel}[n + 1](\Phi F, G) \ar[r]^{\Lambda^{F,G}} \ar[d] & \mathbf{VRel}[n](F, \mathcal{I} G) \ar[d] \\
\mathbf{VRel}[n + 1](\Phi F, G) \ar[r]^{\Lambda^{F,G}} & \mathbf{VRel}[n](F, \mathcal{I} G) \\
\mathbf{VRel}[n + 1](\Phi F', G) \ar[r]^{\Lambda^{F,G}} & \mathbf{VRel}[n](F', \mathcal{I} G) \\
\mathbf{VRel}[n + 1](\Phi F, G) \ar[r]^{\Lambda^{F,G}} \ar[d] & \mathbf{VRel}[n](F, \mathcal{I} G) \ar[d] \\
\mathbf{VRel}[n + 1](\Phi F, g) & \mathbf{VRel}[n](F, \mathcal{I} G) \\
\mathbf{VRel}[n + 1](\Phi F, G') \ar[r]^{\Lambda^{F,G'}} & \mathbf{VRel}[n](F, \mathcal{I} G') }
$$

Let $k : \Phi F \rightarrow G$ be an object of variable type. If $f$ is an object of variable type $F' \rightarrow F$, by $(\sigma)$ we have $\Lambda^{F',G}(k \circ \Phi f) = \Lambda^{F,G}(k) \circ f$. Now, if $g$ is an object of
variable type $G \rightarrow \rightarrow G'$ we have

$$
\Xi g \circ \Lambda^G G (k) = \Lambda^{G G'} (g \circ \varepsilon^G) \circ \Lambda^G G (k) \\
= \Lambda^G G (g \circ \varepsilon^G \circ \Phi G G (k)) (\text{by } \sigma) \\
= \Lambda^G G (g \circ k) (\text{by } \beta) 
$$

As a consequence, $\Lambda : F, G \mapsto \Lambda^G G$ is natural in $F$ and $G$ and $\langle \Phi, \Xi, \Lambda \rangle$ is an adjunction, indeed.

Now, as the relational model identifies variable types and their orthogonal, one easily proves that $\Phi$ is a right adjoint of $\Xi$. Hence we conclude that

$$
\Xi \models \Phi \vdash \Xi.
$$

As a consequence $\Xi$ will be the interpretation of both the $\forall$ and the $\exists$ connectives.

**Notation.** In the sequel, $X_1, \ldots, X_n \ldots$ will denote second-order variables and $X_1, \ldots, X_n \ldots$ will denote sets.

Let $X_1, \ldots, X_n$ be a list of variables without repetition. We will denote by $[\Gamma]X_1, \ldots, X_n$ the interpretation in $\text{VRel}[n]$ of a sequence of formulae $\Gamma$ whose free variables are among $X_1, \ldots, X_n$ and given a proof $\pi$ of $\Gamma$, we denote by $[\pi]X_1, \ldots, X_n$ the interpretation of $\pi$. When the free variables are clear from the context, we will simply denote by $\Gamma^*$ the relational interpretation of a sequence of formulae $\Gamma$ and similarly, we will write $\pi^*$ the interpretation $[\pi]X_1, \ldots, X_n$.

**Lemma 2.13.** Let $\Gamma$ be a sequence of formulae whose free variables belong to $X_1, \ldots, X_n$ and let $X$ be a second-order variable that does not belong to $X_1, \ldots, X_n$, then

$$
[\Gamma]X_1, \ldots, X_n, X = \Phi ([\Gamma]X_1, \ldots, X_n)
$$

The proof is a straightforward induction.

Now, let us give the interpretation of second-order rules. Let $\Gamma$ be a sequence of formulae and let $A$ be a formula. Let $X_1, \ldots, X_n$ be a list of second-order variables without repetitions that do not contain $X$ and let us assume that the free variables of $\forall X.A, \exists X.A$ and $\Gamma$ are among $X_1, \ldots, X_n$. First, let $\pi$ be a proof of $\vdash \Gamma^* \vdash \forall X.A$ of the shape

$$
\vdash \rho \\
\vdash \Gamma^*, \forall X.A (\text{where } X \text{ is not free in } \Gamma)
$$

The second-order variable $X$ is not free in $\Gamma$, thus we have

$$
[\rho]X_1, \ldots, X_n, X : (\Gamma^*) X_1, \ldots, X_n, X \rightarrow [A]X_1, \ldots, X_n, X \\
= \Phi ([\Gamma]X_1, \ldots, X_n) \rightarrow [A]X_1, \ldots, X_n, X
$$

(in the sequel, we will simply write $[\rho]X_1, \ldots, X_n, X : \Phi \Gamma^* \rightarrow A^*$). Now, the interpretation of $\pi$ is defined by

$$
[\pi]X_1, \ldots, X_n = \Lambda^* \rightarrow \Phi^* ([\rho]X_1, \ldots, X_n, X)
$$

and we have $[\pi]X_1, \ldots, X_n : (\Gamma \rightarrow \forall X.A)^*$.  

Then let $\pi$ be a proof of $\vdash \Gamma \vdash A[B/X]$. Thus $\Gamma \vdash A[B/X]$ for the free variables of $B$ are among $X_1, \ldots, X_n$. Thus $[\rho]^X\bar{X}$ is an object of variable type $\Gamma^* \vdash (A[B/X])^*$. The interpretation of $\pi$ is defined by

$$\left(\pi^X\bar{X}\right)^X = \varepsilon_{\bar{X}}(\pi^X) \circ \left([\rho]^X\bar{X}\right)^X$$

and we have $[\pi]^X : (\Gamma \vdash \exists X. A)^*$.

We are now about to prove that $\text{VRel}$ is a model of $\text{LL}^2$, but beforehand let us give three preliminary lemmas.

**Lemma 2.14.** Let $A$ be a formula whose free variables are among $X_1, \ldots, X_n$, let $\pi$ be the $\eta$-expanded proof of $\vdash A, A^\perp$, then $\pi^* = \text{Id}_A$.

Let us assume that $A$ is a negative formula. We recall that the $\eta$-expanded proof of $\vdash A, A^\perp$ is the proof obtained by applying successively the negative rule for $A$ followed by the positive rule for $A^\perp$, etc.. For instance, if $A = X \varphi(Y \oplus Z)$ the $\eta$-expanded proof of $\vdash A, A^\perp$ is:

$$\vdash X, X^\perp \quad \vdash Y, Y^\perp \quad \vdash Z, Z^\perp$$

$$\vdash X, Y \oplus Z, Y^\perp \quad \vdash Y \oplus Z, Z^\perp$$

$$\vdash X, X^\perp \quad \vdash Y \oplus Z, X^\perp \otimes (Y^\perp \otimes Z^\perp)$$

$$\vdash Y \varphi(Y \oplus Z), X^\perp \otimes (Y^\perp \otimes Z^\perp)$$

**Proof.** By induction on $A$.

In the multiplicative, additive, and exponential cases, one concludes by induction using that the relational interpretation is respectively a categorical model of $\text{MLL}$, $\text{MALL}$ and $\text{LL}$.

Therefore let us concentrate on the second-order case. Let us assume that $\pi$ is the following proof,

$$\vdash \rho \quad \vdash A, A^\perp \quad \vdash \forall X. A, A^\perp \forall$$

By induction: $\rho^* = \text{Id}_{A^*}$, therefore $\pi^* = \Lambda^\forall A^* \cdot A^* (\varepsilon A^* \circ \text{Id}_{A^*}) = \Lambda^\forall A^* \cdot A^* (\varepsilon A^*) = \text{Id}_{\forall A^*}$, (by (\eta_2)).

**Lemma 2.15.** Substitution in formulæ

Let $X_1, \ldots, X_n, X$ be second-order variables and let $X_1, \ldots, X_n$ be a family of sets. Let $A$ be a formula whose free variables are among $X_1, \ldots, X_n, X$ and let $B$ be a formula whose free variables are among $X_1, \ldots, X_n$ then

$$(A[B/X])^*(\bar{X}) = A^*(\bar{X}, B^*(\bar{X})).$$

The proof is straightforward.

**Lemma 2.16.** Substitution lemma

Let $A$ be a formula of $\text{LL}^2$ whose free variables are among $X_1, \ldots, X_n, X$. Let $\pi$ be a proof of $A$ and let $B$ be a formula whose free variables are among $X_1, \ldots, X_n$. Then for any family of sets $X_1, \ldots, X_n$ we have

$$\left([\pi]^X_{X_1, \ldots, X_n, X}\right)_{X_1, \ldots, X_n, B^*(X_1, \ldots, X_n)} \equiv \left([\pi[B/X]]_{X_1, \ldots, X_n}\right)_{X_1, \ldots, X_n}$$
(or, with simpler notations, \( \pi^*_{X, B^*(\vec{x})} = (\pi[B/X])^*_{X} \)).

**Proof.** By induction on \( \pi \).

First, when the last rule is an axiom rule, one concludes by lemma 2.14. Now, when the last rule is a multiplicative, additive, exponential or cut rule, the conclusion is immediate by induction.

For instance, if \( \pi \) is of the shape

\[
\vdash \Gamma, A_1 \vdash \Gamma', A_2 \vdash \Gamma, \Gamma', A_1 \otimes A_2 \tag{1}
\]

then for any family of sets \( X_1, \ldots, X_n, X \) we have

\[
\pi^*_{X_1,\ldots,X_n,X} = \{(a, a'), (b, b')\} : (a, b) \in (\pi^*_{1})_{X_1,\ldots,X_n,X}, (a', b') \in (\pi^*_{2})_{X_1,\ldots,X_n,X} \}
\]

And one concludes immediately using the inductive hypothesis.

Let us now assume that \( \pi \) is a proof of \( \vdash \Gamma, \forall Y.A \) (where \( \Gamma \) is a sequence of formulae) of the shape

\[
\vdash \pi_1 \vdash \Gamma, A \\
\vdash \Gamma, \forall Y.A
\]

where the second-order variable \( Y \) is not free in \( \Gamma \). Then \( \pi^* = \Lambda^{[\Gamma]} \cdot A^*(\pi^*_1) \) and one easily concludes by the inductive hypothesis. Now, let us assume that \( \pi \) is a proof of \( \vdash \Gamma, \exists Y.A \) of the shape

\[
\vdash \pi_1 \\
\vdash \Gamma, A^/[C/Y] \\
\vdash \Gamma, \exists Y.A
\]

Then for any family of sets \( X_1, \ldots, X_n, X \) we get \( \pi^*_{X, X, X, X} = \varepsilon^{(A')}_{X, X, X, X} \circ (\pi^*_1)_{X, X, X, X} \).

By induction, we have \( (\pi_1[B/X])^*_{X} = (\pi^*_1)_{X, B^*(\vec{x})} \) and moreover \( (\pi[B/X])^*_{X} = \varepsilon^{(A')}_{X, (C[B/X])^*_{X}}(\sigma^*)_{X} \). Therefore as by lemma 2.15 we have \( (C[B/X])^*_{X} = C^*(X, B^*(\vec{x})) \), we conclude that \( (\pi[B/X])^*_{X} = \pi^*_{X, B^*(\vec{x})} \). \( \square \)

**Theorem 2.17.**

\( \text{VRel} \) is a model of LL

**Proof.**

By proposition 2.7, \( \text{VRel}[\alpha] \) is a categorical model of LL. Let us now show that the interpretation of proofs in \( \text{VRel} \) is invariant by second-order cut-elimination.

Let us first consider a proof ending with a \( \forall \) commutative cut:

\[
\vdash A, B^\perp \vdash \pi \ \vdash \rho \\
\vdash A, B^\perp \vdash B, \Delta^\perp \\
\vdash A, \Delta^\perp \\
\vdash \forall Y \cdot A, \Delta^\perp
\]

Reducing to \( \vdash A, \Delta^\perp \\
\vdash \forall Y \cdot A, \Delta^\perp \)

The first proof is interpreted as \( \Lambda^{B^\perp \cdot A^*} (\pi^*) \circ \rho^* \) whereas the second is interpreted as \( \Lambda^{\Delta^\perp \cdot A^*} (\pi^* \circ \Phi(\rho^*)) \) which are equal by \( \sigma \).

Then let us consider a proof ending with a \( \exists \) commutative cut,
The first proof is interpreted as \((\varepsilon^{(A^*)}_{\overline{X},B^*(\overline{X})} \circ \pi^*_{\overline{X}})_{\overline{X}}\), which is also the interpretation of the second proof by associativity of composition.

Finally, let us consider a proof ending with a \(\forall \exists\) cut,

\[
\vdash A, \Gamma \vdash A \vdash [B/Y], \Delta
\]

reduces to

\[
\vdash A[B/Y], \Gamma \vdash A[B/Y], \Delta
\]

Since \(Y\) does not occur in \(\Gamma\), we have \((\Gamma^\perp)[B/Y] = \Gamma^\perp\).

The first proof is interpreted as \((\rho^*_{\overline{X}} \circ \varepsilon^{(A^*)}_{\overline{X},B^*(\overline{X})})_{\overline{X}}\) whereas the substitution lemma, the second is interpreted as \(\rho^*_{\overline{X}} \circ \pi^*_{\overline{X}}\), which are equal by \((\beta_2)\).

From now on, for notational convenience, we will identify formulae and their relational interpretation. In other words, depending on the context, \(A\) will denote either a formula or the relational interpretation of a formula \(A\) (it was previously written \(A^*)\).

### 2.4.3. An intuitive approach of stable functors.

The notion of stable functor is in fact very close to syntax. Stable functors can be considered as a formal way of describing a syntactical tree. Let us illustrate this idea on the following formula \(A = \mathbb{X} \rightarrow \mathbb{X}\) where \(\mathbb{X}\) is a second-order variable. Let \(X, Y\) be sets and let \(a, b, c \in X\) and \(a', b', c' \in Y\). By definition, \(A(X) = \mathcal{M}_{\text{fin}}(X) \times X\), therefore we have, for instance, \(\langle a, b, c \rangle \in A(X)\). Let \(f : X \leftrightarrow Y\) be an injection such that \(f(a) = a', f(b) = b'\) and \(f(c) = c'\). The functorial part of \(A\) behaves like a “renaming machine”, that is \(A(f)(\langle a, b, c \rangle, c) = \langle a', b', c' \rangle\). Now, the point \((\langle a, b, c \rangle, c)\) depends only on three points in \(X\), namely \(a, b, c\), but the very names are not important for the renaming machine; the only important thing is whether two points bear the same name or distinct names. This is the intuition behind the equivalence class of \(\langle (a, b, c), (a, b, c) \rangle\), denoted by \(\langle 3, ([0, 1, 2], 2) \rangle\). In fact, this equivalence class could be represented by

![diagram](image_url)

**Figure 1 - The point \(\langle 3, ([0, 1, 2], 2) \rangle\) in \(\text{Tr}(\mathbb{X} \rightarrow \mathbb{X})\)**

Actually, for any stable functor \(F\), the trace of \(F\) contains the various syntactical trees that, once decorated with names taken in \(X\), can occur in \(F(X)\). Names can be seen as input location connected to the leaves of the tree. In order to be able to address them, they must be given a specific label. But this label is not important in itself. All the same for a computer; it is not important whether your data has been written at 0x00ab or at 0x01e. However, if you ask for the wrong address...
when reading, you might be surprised! From this point of view, the notion of stable
functor is very close to the idea of \( \alpha \)-conversion for second-order variables. Distinct
names can be renamed into other distinct names: the operation is invisible.

Let us focus on another simpler example, namely \( A = X \rightarrow X \). The trace of
\( A \) contains only two points, namely \( \langle 2, (0, 1) \rangle \) and \( \langle 1, (0, 0) \rangle \). In other words, the
points of \( A(X) \) have either one or two input links and are of one of the following
shapes:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\quad
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\]

\textit{Figure 2 - The type } \( X \rightarrow X \)

Now stability, through the normal form theorem, asserts that such a syntactical
representation is possible.

3. Reminder: Product Phase Semantics and M-spaces

In this section, we recall part of the material developed by Bucciarelli and
Ehrhard in [Bucciarelli, 1999] and [Bucciarelli, 2000].

3.1. Preliminary definitions.

Let us first recall a few notations related to phase semantics (we will use multipli-
cative notations for monoids): if \( U \) and \( V \) are subsets of a monoid \( P \), then \( UV \)
denotes the set \( \{pq; p \in U, q \in V\} \).

From now on, we will denote by \( P_0 \) any monoid containing an absorbing element
\( 0 \). Moreover \( I \) will denote an infinite denumerable set. Let us call \( I(I) \) the category
whose objects are the subsets of \( I \) and whose morphisms are the injective functions
between them.

Let \( \text{Fam}_E \) denote the contravariant functor from \( I(I) \) to \( \text{Set} \) which associates to
\( J \) the set of all \( J \)-indexed families of elements of \( E \). Given \( u : J \rightarrow K \) an injection,
for any \( \alpha \in E^K \), we define \( \text{Fam}_E(u)(\alpha) = \alpha \circ u \in E^J \). We will denote \( \text{Fam}_E(u) \) by
\( u^* \).

\textbf{Definition 3.1.} Let \( J, K \in \text{Set} \). A function \( u : J \rightarrow K \) is called \textit{almost injective}
when for any \( k \in K \), the set \( u^{-1}(k) \) is finite.

For any \( J \in \text{Set} \), \( P_0^J \) is equipped with the product structure of monoid. Let
\( J, K \) be sets and \( u : J \rightarrow K \) be an almost injective function, then we define the
following monoid morphism:

\[
u_* : P_0^J \rightarrow P_0^K \\
(p_j)_{j \in J} \mapsto \prod_{j \in u^{-1}(k)} p_j \in K
\]

For any \( J \subseteq I \), we will denote by \( \varepsilon_J \in P_0^I \) the characteristic function of \( J \).

3.2. Product phase spaces.

\textbf{Definition 3.2.} A \textit{product phase space} is a pair \( (P_0^I, \bot) \) where \( P_0 \) is a monoid
containing an absorbing element \( 0 \) and \( \bot \) is a subset of \( P_0^I \) such that

(i) \( \bot \) is not empty
(ii) for any \( J \subseteq I \), we have \( \{ \varepsilon_J \} \bot \subseteq \bot \) (this condition is called \textit{closure under
restriction}).
Let \( M = (P_0^I, \perp) \) be a product phase space. Let \( J \) be a subset of \( I \), we denote by \( \pi_J \) the canonical projection onto \( P_0^J \) and by \( \perp_J \) the projection of \( \perp \) on \( P_0^J \). The couple \( (P_0^J, \perp_J) \) is a product phase space that we will call the \textit{local product phase space at level} \( J \) and denote by \( M(J) \). Let us write \( 1^J \) the unit of \( P_0^J \). When we use the notation \( U^\perp \) for a subset \( U \) of \( P_0^I \) we will always mean that the orthogonal is taken inside \( M(J) \).

Facts are subsets of \( M \) equal to their \textit{adjoint}. We will denote by \( \mathcal{F}_M(J) \) the set of all the facts of the local model \( M(J) \) and we will call them \textit{facts of level} \( J \).

A corollary of the closure condition is:

**Lemma 3.3.** If \( K \subseteq J \subseteq I \), for any \( F \in \mathcal{F}_M(J) \), we have \( \pi_K(F) \in \mathcal{F}_M(K) \).

We now define distinguished facts and operations on facts:

- For any \( J \subseteq I \), we define \( 1_J = \perp_J = \{1^J\}^{\perp_J} \)
- Given \( F,G \) two facts of level \( J \), we define the following facts of level \( J \),
  \[
  F \otimes G = (FG)^{\perp_J} \\
  F \circ G = (F^{\perp_J}G)^{\perp_J}
  \]

- If \( F \in \mathcal{F}_M(L) \) and \( G \in \mathcal{F}_M(R) \) with \( L + R = J \) (that is \( J = L \cup R \) and \( L \cap R = \emptyset \)), we define the following facts of level \( J \),
  \[
  F \& G = \{p \in P_0^J : \pi_L(p) \in F, \pi_R(p) \in G\} \\
  = \pi_L^{-1}(F) \cap \pi_R^{-1}(G) \in \mathcal{F}_M(J) \\
  F \oplus G = (F^{\perp_J} \& G)^{\perp_J}
  \]

- If \( u : K \to J \) is almost injective and \( F \in \mathcal{F}_M(K) \), we define the following facts of level \( J \)
  \[
  ^1u F = (u,F)^{\perp_J} \\
  ^?u F = (u,F^{\perp_J})
  \]

**Lemma 3.4.** Let \( J = L + R \subseteq I \). Given \( K \) a subset of \( J \), let \( \zeta_K : P_0^K \to P_0^J \) denote the function defined by
  \[
  (\zeta_K(p))_J = \begin{cases} 
  p_j & \text{if } j \in K \\
  0 & \text{otherwise}
  \end{cases}
  \]

Let \( F \in \mathcal{F}_M(L) \) and \( G \in \mathcal{F}_M(R) \), then
  \[
  F \oplus G = (\zeta_L(F) \cup \zeta_R(G))^{\perp_J}
  \]

Notice that projections commute to all of this constructions.

Let us now define the notion of \textit{symmetric product phase model}. In order to interpret exponentials, multisets occurring in the web of formulae must be represented in terms of indexed families. Therefore, in order to get a correctness theorem for the denotational semantics of \( \mathsf{LL} \), the model should not distinguish between the various ways of indexing a given multiset. This requirement is precisely reached in the symmetric product phase models.

**Definition 3.5.** A product phase model \( M \) is \textit{symmetric} when for any \( J,K \subseteq I \) and any bijection \( u : J \to K \), one has \( u_* (\perp_J) = \perp_K \).

An equivalent definition is that for any two injections \( u,v : J \to I \), one has \( u^* \perp = v^* \perp \).

This condition is stronger than simply requiring that for any bijection \( u \) from \( I \) into itself \( u_* \perp = \perp \) (see [BuEt,00] for a counter-example).

From now on, all product phase models will be symmetric.
Lemma 3.6. Let \( J, K \subseteq I \) and let \( u : J \leftrightarrow K \) be an injection,

(i) For any \( U \subseteq P^J_0 \),

\[
(u^* U)^\perp = u^*(U^\perp)
\]

(ii) Moreover \( u^* \) commutes to all of the previous multiplicative-additive constructions on facts.

As for exponential constructions, let \( L \) be a subset of \( I \), \( v : L \to K \) be an almost injective function and \( F \) be a fact of level \( L \). If \( u' \) and \( v' \) are such that the following diagram is a pullback,

\[
\begin{array}{ccc}
R & \xrightarrow{v'} & J \\
\downarrow{u'} & & \downarrow{u} \\
L & \xrightarrow{v} & K
\end{array}
\]

then \( u' \) is injective, \( v' \) is almost injective, and we have \( u^*(F) = !u'^*F \) and \( u^*(?F) = ?u'^*F \).

By lemma 3.6, when \( M \) is a symmetric product phase space, \( F_M \) is a contravariant functor from \( I(I) \) to \( \text{Set} \) (with \( F_M(u) = u^* \) for \( u : J \leftrightarrow K \)).

3.3. \( M \)-spaces for LL.

Given a symmetric product phase model \( M \), we define a category \( \mathcal{C}(M) \) called the category of \( M \)-spaces.

Definition 3.7. Let \( M = (P^I_0, \perp) \) be a symmetric product phase model. A \( M \)-space is a pair \( X = ([X], \bar{X}) \) where \([X]\) is a finite denumerable set and \( \bar{X} \) is a natural transformation from the contravariant functor \( \text{Fam}_{[X]} \) to the contravariant functor \( F_M \).

In other terms, given \( J, K \subseteq I \), \( \alpha \in [X]^J \) and \( u : K \leftrightarrow J \), we require that \( \bar{X}_K(u^* \alpha) = u^* \bar{X}_J(\alpha) \).

Definition 3.8. Let \( X \) be a \( M \)-space. A clique of \( X \) is a subset \( x \) of \([X]\) such that for any \( J \subseteq I \) and \( \alpha \in x^J \), we have \( 1^J \in \bar{X}_J(\alpha) \). We will denote by \( \text{Cl}_M(X) \) the set of all cliques of \( X \).

In order to interpret exponentials, we will also need the following definition:

Definition 3.9. Let \( X \) be a \( M \)-space, \( J \subseteq I \) and \( \alpha \in \mathcal{M}_{\text{fin}}([X])^J \). Then, let \( K \) be a subset of \( I \), \( \xi \) be a \( K \)-indexed family of \([X]\) and \( u : K \leftrightarrow J \) be an injection such that \( \alpha_J = \sum_{k \in u^{-1}(j)} [\xi_k] \). Then \( (\xi, u) \) is called a representative of \( \alpha \).

In the sequel, given \( J \) a subset of \( I \) and a point \( \alpha \), we will denote by \( \alpha^J \) the constant \( J \)-indexed family.

Let us now define for each logical connective of LL a corresponding construction on \( M \)-spaces:

- Additive constants are interpreted as \( 0 = (\emptyset, \hat{0}) = \top \), where \( \hat{0} \) is the following function: \( \hat{0}_0(\emptyset) = \{\emptyset\} \).
- Multiplicative constants are interpreted as \( \bot = [1] = \{\ast\} \), with \( \hat{\bot}_J(\ast^J) = \bot_J \) and \( \hat{1}_J(\ast^J) = 1_J \).
Given $X$ and $Y$ two $M$-spaces, $\alpha \in |X|^I$ and $\beta \in |Y|^R$ with $L + R = J \subseteq I$, one defines $M$-spaces $X \oplus Y$ and $X \& Y$ by $|X \oplus Y| = |X \& Y| = |X| + |Y|$, with $\tilde{X} \oplus Y_J(\alpha + \beta) = \tilde{X}_L(\alpha) \oplus \tilde{Y}_R(\beta)$ and $\tilde{X} \& Y_J(\alpha + \beta) = \tilde{X}_L(\alpha) \& \tilde{Y}_R(\beta)$.

Given $X$ and $Y$ two $M$-spaces, $J \subseteq I$ and $\gamma = (\alpha, \beta) \in |X|^J \times |Y|^J$, one defines $M$-spaces $X \otimes Y$ and $X \circ Y$ by $|X \otimes Y| = |X \circ Y| = |X| \times |Y|$, with $\tilde{X} \otimes Y_J(\gamma) = \tilde{X}_J(\alpha) \otimes \tilde{Y}_J(\beta)$ and $\tilde{X} \circ Y_J(\gamma) = \tilde{X}_J(\alpha) \circ \tilde{Y}_J(\beta)$.

Let $X$ be a $M$-space, $J \subseteq I$ and $\alpha \in M_{fin}(|X|)^J$. Let $K$ be a subset of $I$, $u : K \leftrightarrow J$ be an injection and $\xi$ be a $K$-indexed family of $|X|$ such that $(\xi, u)$ is a representative of $\alpha$. Then one defines $M$-spaces $\lambda X$ and $\lambda Y$ by $|\lambda X| = |\lambda Y| = M_{fin}(|X|)$, with $\tilde{\lambda X}_J(\alpha) = X_K(\xi)$ and $\tilde{\lambda Y}_J(\beta) = Y_K(\xi)$.

By naturality of $\tilde{X}$, this definition does not depend on the choice of the representative of $\alpha$.

In the definition of the exponential constructions, non-uniformity is crucial. In fact, in the multiset version of coherence spaces, the web of $\lambda F$ is the set of all the multisets on $|F|$ whose support is a clique. In the present model, the web of the $M$-space $\lambda X$ consists of the set of all finite multisets on $|X|$ and as we shall see, this non-uniform model is therefore closer to syntax. For instance, let us consider the following proof $\pi$

$$\begin{array}{c}
1_T, 1_T \vdash 1_T \\
1_T, 1_F \vdash 1_T \\
1_F, 1_T \vdash 1_F \\
1_F, 1_F \vdash 1_F \\
\hline
1_T \oplus 1_F, 1_T \oplus 1_F \vdash 1_T \oplus 1_F \\
\hline
1_T \oplus 1_F \vdash 1_T \oplus 1_F \\
\hline
\end{array}$$

(we have denoted by $1_T$ and $1_F$ the first and second component of $1 \oplus 1$; remember that $1 \oplus 1$ is isomorphic to $\text{Bool}$). The multiset coherent interpretation of this proof is $\{([T, T], T), ([F, F], F)\}$ which completely forgets two “leaves” of the proof, whereas the non-uniform interpretation of this proof, namely the set $\{([T, T], T), ([F, F], T), ([F, T], F), ([F, F], F)\}$ does not.

Now, define $C(M)$ as the category whose objects are the $M$-spaces and whose morphisms from $X$ to $Y$ are the cliques of $X \cong Y$.

In [BuEih,00], it is proved that $C(M)$ is a model of LL.

### 4. $M$-SPACES AND THE SECOND ORDER

In this section, we extend the non-uniform denotational semantics of LL described in section 3 to the second-order. For this purpose, let us first define the notion of variable $M$-space.

**Notation.** Let $U, V$ be sets, $J \subseteq I$ and $f$ be an application from $U$ to $V$. Let $\alpha \in U^J$; in the sequel we will often write $f(\alpha)$ instead of $(f(\alpha_j))_{j \in J}$.

**Definition 4.1.** Let $X$ and $Y$ be $M$-spaces. A $M$-space embedding of $X$ into $Y$ is an injection $f : |X| \hookrightarrow |Y|$ such that for any $J \subseteq I$ and any $\alpha \in |X|^J$, we have $\tilde{X}_J(\alpha) = \tilde{Y}_J(f(\alpha))$.

We will write $f : X \hookrightarrow Y$.

**Definition 4.2.** Let $M = (P_0^I, \perp)$ be a symmetric product phase space. A $n$-ary variable $M$-space structure is a pair $F = ([F], \tilde{F})$ where:

1. $[F]$ is a $n$-ary stable functor (that is $[F] \in \text{VRel}[n]$) called the variable web of $F$.
2. Given $X_1, \ldots, X_n$ a family of $M$-spaces, the pair $F(X_1, \ldots, X_n) = ([F][|X_1|, \ldots, |X_n|], \tilde{F}(X_1, \ldots, X_n))$
is an $M$-space and $\tilde{F}(X_1, \ldots , X_n)$ is called the $M$-coherence of $F(\tilde{X})$.

(iii) $|F|$ preserves $M$-space embeddings.

In other terms, given $f_1 : X_1 \leftrightarrow Y_1, \ldots , f_n : X_n \leftrightarrow Y_n$ a family of $M$-space embeddings, $|F|(f_1, \ldots , f_n)$ is a $M$-space embedding of $|F|([X_1], \ldots , [X_n])$ into $|F|([Y_1], \ldots , [Y_n])$.

In the sequel, we will write $X, Y$ for $M$-spaces and $U, V$ for sets.

Rather than defining the interpretation of formulae by induction, we will rather define the logical operations on variable $M$-spaces.

4.1. Additives.

Given a family $U_1, \ldots , U_n$ of sets, one sets $|\emptyset|(\emptyset) = |\top|(\top) = 0$ and given $X_1, \ldots , X_n$ a family of $M$-spaces, $|\emptyset|(X_1, \ldots , X_n) = 0$ (defined in section 3) and $|\top|(X_1, \ldots , X_n) = \top$.

Now, let $F$ and $G$ be $n$-ary variable $M$-spaces. One defines $|F \oplus G| = |F| \& |G|$ as the relational interpretation of additivities, that is as $|F| \& |G|$. Then given $X_1, \ldots , X_n$ a family of $M$-spaces, the $M$-coherence $\tilde{F} \oplus \tilde{G}(X_1, \ldots , X_n)$ over the web $|F \oplus G|([X_1], \ldots , [X_n])$ is defined by $\tilde{F}(\tilde{X}) \oplus \tilde{G}(\tilde{X})$ and similarly one sets $\tilde{F} & \tilde{G}(\tilde{X}) = F(\tilde{X}) \& G(\tilde{X})$.

One easily checks that $|F \oplus G|$ and $|F \& G|$ preserve $M$-space embeddings.

4.2. Multiplicatives.

Given a family $U_1, \ldots , U_n$ of sets, one sets $\|\top|(\top) = |\bot|(\bot) = \{\ast\}$ and for any family of $M$-spaces $X_1, \ldots , X_n$, one defines $\downarrow(\tilde{X}) = \bot$ and $\uparrow(\tilde{X}) = \top$ (where $\downarrow$ and $\uparrow$ denote the $M$-spaces defined in section 3).

Now, let $F$ and $G$ be two $n$-ary variable $M$-spaces. One defines $|F \otimes G| = |F| \vee |G|$ as the relational interpretation of multiplicatives, that is as $|F| \vee |G|$. Then given $X_1, \ldots , X_n$ a family of $M$-spaces, one sets $\hat{F} \otimes \hat{G}(\hat{X}) = F(\tilde{X}) \otimes G(\tilde{X})$ and $\tilde{F} \otimes \tilde{G}(\tilde{X}) = \hat{F}(\hat{X}) \otimes \hat{G}(\hat{X})$.

One easily checks that $|F \otimes G|$ and $|F \vee G|$ preserve $M$-space embeddings.

4.3. Exponentials.

Let $F$ be a $n$-ary variable $M$-space and let $X_1, \ldots , X_n$ be a family of $M$-spaces, then one sets $|\forall X_1, \ldots , X_n| = |\forall X_1| \& \cdots \& |\forall X_n|$. Moreover the $M$-coherence $\hat{F}(X_1, \ldots , X_n)$ is defined as $|\forall \hat{F}(\tilde{X})|$ and similarly $|\exists \hat{F}(X_1, \ldots , X_n)| = |\exists \hat{F}(\tilde{X})|$.

Now, one easily checks that $|\forall F|$ and $|\exists F|$ preserve $M$-space embeddings.

4.4. Second-order.

Let $F$ be a $n + 1$-ary variable $M$-space, one sets $|\forall X. F| = |\exists X. F| = \exists \forall X. F$.

Observe that at this point we make a strong non-uniformity hypothesis. Indeed in coherence spaces, the web of $\forall X. F$ consists only of the equivalence classes $\langle X, a \rangle$ such that for any two embeddings $f, g : X \leftrightarrow Y$, we have $F(f)(a) \equiv_Y F(g)(a)$. In other terms, in coherence spaces, the web of $\forall X. F$ consists only in the equivalence classes that are coherent with themselves. In the non-uniform model, the web of $\forall X. F$ consists of the entire trace of $F$.

Now, let $X_1, \ldots , X_n$ be a family of $M$-spaces, let $J$ be a subset of $I$ and let $\alpha = \langle \alpha_j \rangle_{j \in J}$ be a $J$-indexed family of $|\forall X. F|([X_1], \ldots , [X_n])$. One sets $\hat{\forall X. F}(X_1, \ldots , X_n)(\alpha) = \{ p \in P_J^I : \forall X \in \mathcal{C}(M), \forall f_j \in \mathcal{P}^I \mapsto \{ X_j \}_{j \in J}, p \in \hat{F}(X_1, \ldots , X_n, X)(\langle F(X_1, \ldots , X_n, f_j)(\alpha_j) \rangle_{j \in J}) \}$
Observe that this definition does not depend on the choice of the representatives of \((n_j, a_j)_{j \in J}\). Indeed, assume that \((n_j, a_j)_{j \in J} = (n_j, b_j)_{j \in J}\), then for any \(j \in J\) there exists \(a_j \in \Theta_{\pi_j}\) such that \(b_j = F(\bar{X}, \sigma_j)(a_j)\). Let \(p \in \sqrt{X}.\bar{F}(\bar{X})((n_j, a_j)_{j \in J})\). Let \(X\) be a \(M\)-space and let \((f_j : \pi_j \rightarrow |X|)_{j \in J}\) be a family of injections. For any \(j \in J\), we have \((f_j \circ \sigma_j) : \pi_j \rightarrow |X|\), therefore \(p \in F(\bar{X}, X)_J((F(\bar{X}, f_j \circ \sigma_j)(a_j))_{j \in J})\).

Hence, as for any \(j \in J\) we have \(F(\bar{X}, f_j \circ \sigma_j)(a_j) = F(\bar{X}, f_j(b_j))\) we get \(p \in F(\bar{X}, X)_J((F(\bar{X}, f_j)(b_j))_{j \in J})\) and as a consequence, \(p \in \sqrt{X}.\bar{F}(\bar{X})((n_j, b_j)_{j \in J})\).

The variable \(M\)-space structure \(\exists X.\bar{F}\) is defined by duality \((\exists X.\bar{F}) = (\forall X.\bar{Y})\).

Let us also define a \(n + 1\)-ary variable \(M\)-space \(\Phi F\) by \(|\Phi F| = |\Phi|\) and 
\[
\Phi F(X_1, \ldots, X_{n+1}) = \bar{F}(X_1, \ldots, X_n).
\]

**Proposition 4.3.** For any \(n\)-ary variable \(M\)-space \(F\) and for any family of \(M\)-spaces \(X_1, \ldots, X_n\),

\[
\forall X. F(X_1, \ldots, X_n)
\]

so defined is indeed a \(M\)-space

**Proof.**

For notational convenience, we will assume that \(n = 1\). As \(\sqrt{X}.\bar{F}(\bar{X})_J(\alpha)\) is by definition an intersection of facts, it is a fact. Let \(u : K \rightarrow J\) be an injection, \(X\) be a \(M\)-space and \(\alpha = (n_j, a_j)_{j \in J}\) be a \(J\)-indexed family of elements of \(\forall X. F([|X|])\).

Let us show that \(\exists X.\bar{F}(X)_K(u^* \alpha) = u^* \exists X.\bar{F}(X)_J(\alpha)\).

Let \(p \in \exists X.\bar{F}(X)_K(u^* \alpha)\). Let us define \(q = (q_j)_{j \in J}\) by \(q_j = 0\) if \(j \notin u(K)\) and \(q_j = p_u\) if \(j = u(k)\). Obviously, \(p = u^* q\) and we will now prove that \(q \in \exists X.\bar{F}(X)_J(\alpha)\).

Let \(Y\) be a \(M\)-space and for any \(j \in J\), let \(f_j : \pi_j \rightarrow |Y|\). Let us show that \(q \in \bar{F}(X, Y)_J((F(\bar{X}, f_j)(a_j))_{j \in J})\). We have

\[
u^* q = p \in \bar{F}(X, Y)_K(u^* (F(\bar{X}, f_j)(a_j))_{j \in J})\]

Therefore, there exists \(p' \in \bar{F}(X, Y)_J((F(\bar{X}, f_j)(a_j))_{j \in J})\) such that \(p = u^* p'\) (which implies that for any \(j \in u(K)\), we have \(p' = q_j\)). But then \(q = \varepsilon_j p'\) and as facts are closed under restriction, we get \(q \in \bar{F}(X, Y)_J((F(\bar{X}, f_j)(a_j))_{j \in J})\).

Conversely, let \(p \in u^* \exists X.\bar{F}(X)_J(\alpha)\). There exists \(q \in \exists X.\bar{F}(X)_J(\alpha)\) such that \(p = u^* q\). Then it is immediate (using that \(F(X, Y)\) is a \(M\)-space) that \(u^* q \in \exists X.\bar{F}(X)_K(u^* \alpha)\).

**Proposition 4.4.** Let \(F\) be a \(n + 1\)-ary variable \(M\)-space, let \(X, Y\) and \(\bar{Z}\) be \(M\)-spaces and let \(f : |X| \rightarrow |Y|\) be a \(M\)-space embedding.

Then \(\forall X. F(f,[\bar{Z}]) : \forall X. F([|X|],[\bar{Z}]) \rightarrow \forall X. F([|Y|],[\bar{Z}])\) is a \(M\)-space embedding.

**Proof.** For notational convenience, let us assume that \(\forall X. F\) has only one free variable. Let \(X, Y\) be \(M\)-spaces, \(f : |X| \rightarrow |Y|\) be a \(M\)-space embedding and \(\alpha = (n_j, a_j)_{j \in J}\) be a point of \(\forall X. F([|X|])^J\). Let us show that \(\exists X.\bar{F}(X)_J(\alpha) = \exists X.\bar{F}(Y)_J(\forall X. F(f)(\alpha))\).

Let \(Z\) be a \(M\)-space and for any \(j \in J\), let \(f_j : \pi_j \rightarrow |Z|\) be an injection. By hypothesis on \(F\) we have

\[
\bar{F}(X, Z)_J((|F|[|X|], f_j)(a_j))_{j \in J}) = \bar{F}(Y, Z)_J((|F|[|Y|][f_j](|F|[|X|], f_j)(a_j))_{j \in J})
\]
Moreover $\mathcal{F}$ preserves the following commutative diagram

\begin{align*}
(\langle X\rangle, \pi_J) &\xrightarrow{\langle \mathcal{F} \langle X\rangle, \mathcal{F} \pi_J \rangle} (\langle X\rangle, \mathcal{F} \mathcal{F}) \\
(f, \pi_J) &\xrightarrow{(f, \mathcal{F} \pi_J)} (f, \mathcal{F} \mathcal{F})
\end{align*}

Therefore (1) becomes

\begin{align*}
\widehat{\mathcal{F}}(X, Z)_J((\mathcal{F} \langle X\rangle, \mathcal{F} \pi_J)(\alpha_j))_{j \in J}
&= \widehat{\mathcal{F}}(Y, Z)_J((\mathcal{F} \langle Y\rangle, \mathcal{F} \pi_J)(\mathcal{F}(f, \pi_J)(\alpha_j))_{j \in J})
\end{align*}

But $\forall X. \mathcal{F}(f)(\langle n_j, \alpha_j \rangle)_{j \in J} = \langle n_j, \mathcal{F}(f, \pi_J)(\alpha_j) \rangle_{j \in J}$ and as a consequence, by (2) we have

\begin{align*}
\forall X. \mathcal{F}(X)_J(\alpha)
&= \forall X. \mathcal{F}(Y)_J \left( \langle n_j, \mathcal{F}(f, \pi_J)(\alpha_j) \rangle_{j \in J} \right)
\end{align*}

\begin{align*}
&= \forall X. \mathcal{F}(Y)_J(\forall X. \mathcal{F}(f)(\alpha))
\end{align*}

\[ \square \]

4.5. **Soundness.**

Proofs are interpreted as in the relational model.

**Definition 4.5.** We will call variable clique an object of variable type which is a clique on any $M$-space.

In other words, given a variable $M$-space $F$, an object $t$ of variable type $[F]$ is a variable clique if, for any family of $M$-spaces $X_1, \ldots, X_n$, we have $t_{[X_1], \ldots, [X_n]} \in \mathcal{C}_M(F(X_1, \ldots, X_n))$

We define the category of variable $M$-spaces as:

- **Objects:** variable $M$-spaces.
- **Morphisms:**
  
  given $F$ and $G$ two $n$-ary variable $M$-spaces, the morphisms between $F$ and $G$ are the variable cliques of $F \rightarrow G$.

We will denote by $\mathcal{C}_M(M)$ this category.

**Theorem 4.6.** Soundness

Let $F$ be a formula of $\mathbb{L}^2$ whose free variables are among $X_1, \ldots, X_n$ and $\pi$ be a proof of $F$, then $\pi^*$ is a variable clique.

In order to prove soundness, we show first in lemma 4.8 that the composite of two morphisms which are variable cliques is a variable clique, for this we need the following lemma.

**Lemma 4.7.** Let $X$ be a $M$-space, $U$ be a set and $f : \langle X\rangle \mapsto U$ be an injection. Then $U$ can be equipped with a variable $M$-space structure $U^+ = (U, \widetilde{U}^+)$ such that for any $\alpha \in \langle X\rangle$, we have $\widetilde{U}^+ J(f(\alpha)) = \widetilde{X}^+ J(\alpha)$ (ie. $f$ is a $M$-space embedding).

**Proof.**

Let $\alpha \in \langle X\rangle$ and $K = \{ k \in J ; \alpha_k \in f(\langle X\rangle) \}$. Then let us define $\widetilde{U}^+ J(\alpha) = (\zeta_K \widetilde{X}^+ K ((f^{-1}(\alpha_k))_{k \in K}))^{+}$ (where $\zeta_K^+$ is the zero-completion defined in lemma 3.4). Let us prove that $U^+$ so defined is a $M$-space.
Let \( u : J \hookrightarrow L \) and \( \alpha \in \{ U^+ \downarrow L = U^L \}. \) For any \( j \in J \), we have that \( (u^*\alpha)_j \in \operatorname{f}( |X| ) \) if and only if \( u(j) \in K. \) Now let \( S = u^{-1}(K) \), we get that
\[
\begin{align*}
\tilde{U}^+_J(u^*\alpha) &= (\zeta_S \tilde{X}_S \left( \left( f^{-1}(\alpha_{u(j)}) \right)_{u \in S} \right)) \upharpoonright_{\left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right)} \\
&= (\zeta_S u|_{u \in S} \left( \tilde{X}_K \left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right) \right)) \upharpoonright_{\left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right)} \\
&= (u^* \zeta_K \left( \tilde{X}_K \left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right) \right)) \upharpoonright_{\left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right)} \\
&= u^* \zeta_K \left( \tilde{X}_K \left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right) \right) \upharpoonright_{\left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right)} \quad \text{(by lemma \ref{lem:1})}
\end{align*}
\]
Observe that when defining \( \tilde{U}^+_J(\alpha) \), instead of completing facts of level \( K \) by zeros (using \( \tilde{X}_K \)), we could also have taken \( \tilde{U}^+_J(\alpha) = \pi_K^{-1} \left( \tilde{X}_K \left( \left( f^{-1}(\alpha_k) \right)_{k \in K} \right) \right) \) and the result would still hold. \( \square \)

**Lemma 4.8.** Let \( F,G,H \) be variable \( M \)-spaces and let \( t : |F \to G| \) be objects of variable type. Let us assume that for any \( M \)-space \( X \) we have \( t|_{|X|} \in \mathcal{C}_{M}((F \to G)(|X|)) \) and \( s|_{|X|} \in \mathcal{C}_{M}((G \to H)(|X|)) \). Then for any \( M \)-space \( X \),
\[
(s \circ t)|_{|X|} \in \mathcal{C}_{M}((\langle F \to H \rangle)(|X|))
\]
Remember that given \( X \) and \( Y \) two \( M \)-spaces, \( J \) a subset of \( I \) and \( \alpha, \beta \) two \( J \)-indexed families with elements respectively in \( |X| \) and \( |Y| \), we have
\[
1^J \subseteq \tilde{X}_J(\alpha, \beta)
\]
if and only if \( \tilde{X}_J(\alpha) \subseteq \tilde{Y}_J(\beta) \).

**Proof.**
Let \((\alpha, \gamma) \in (s \circ t)^2|_{|X|} \). For any \( j \in J \), we have \((\alpha_j, \gamma_j) \in (s \circ t)^2|_{|X|} \). Therefore by definition of composition, there exist a family of sets \((U_j)_{j \in J} \), a family of injections \((f_j : |X| \hookrightarrow U_j)_{j \in J} \) and a point \( \beta_j \in |G|(|U_j|) \) such that \((\langle F \rangle(f_j)(\alpha_j), \beta_j) \in t_U \) and \((\beta_j, |H|(|f_j|)(\gamma_j)) \in s_{U_j} \).

Let \( U = \bigcup_{j \in J} U_j / \equiv_i \) where the equivalence relation \( \equiv_i \) is defined by: \((j, a) \equiv_i (j', b) \) if there exists \( c \in |X| \) such that \( a = f_j(c) \) and \( b = f_{j'}(c) \) (observe that it is indeed an equivalence relation by injectivity of \( f \)). For any \( j \in J \), let \( g_j : U_j \hookrightarrow U \) be the canonical injection. Then, we have the following pullback.
\[
\begin{array}{ccc}
|X| & \xrightarrow{f_j} & U_j \\
\downarrow & \searrow & \uparrow \circ \downarrow \circ \uparrow \\
U_h & \xleftarrow{g_k} & U
\end{array}
\]

Let us denote by \( h \) the composite \( g_j \circ f_j : |X| \hookrightarrow U \) (this definition is independent of \( j \) since the diagram is commutative). As a consequence of mutilation, \((\langle F \rangle| \langle f_j \rangle| \langle \alpha_j \rangle, \langle g_j \rangle| \langle \beta_j \rangle) \) in \( t_U \) and 
\[(\langle G \rangle| \langle \beta_j \rangle, |H|(|g_j|)|H|(|f_j|)(\gamma_j)) \) in \( s_{U} \).

For any \( j \in J \), let us define \( \beta' \) \( = \langle G \rangle| \langle g_j \rangle| \langle f_j \rangle| \langle \beta_j \rangle \). By lemma 4.7, the set \( U \) can be equipped with a \( M \)-space structure \( U^+ \) for which \( h : X \hookrightarrow U^+ \) is a \( M \)-space embedding. By hypothesis, for any \( M \)-space \( X \), we have \( s_U \in \mathcal{C}_{M}((\langle F \to G \rangle(|X|))) \). In our particular case, we get \( s_U \in \mathcal{C}_{M}((\langle F \to G \rangle(U^+))) \) and \( t_U \in \mathcal{C}_{M}((\langle G \to H \rangle(U^+))) \). Therefore, since 
\[(\langle F \rangle| \langle g_j \rangle|\langle f_j \rangle|\langle \alpha_j \rangle, \beta') \) in \( t_U \) we have \( \tilde{F}(U^+)_J, \langle F \rangle| \langle g_j \circ f_j \rangle| \langle \alpha_j \rangle \subseteq \tilde{G}(U^+)_J, \beta' \) and since 
\[(\beta', |H|(|g_j|)|H|(|f_j|)(\gamma_j)) \) in \( s_U \) we have \( \tilde{G}(U^+)_J, \beta' \subseteq \tilde{H}(U^+)_J, |H|(|g_j \circ f_j|)(\gamma_j) \).
As a consequence, we get $\tilde{F}(U^+)_J(\langle F \mid (\alpha) \rangle) \subseteq \tilde{H}(U^+)_J(\langle H \mid (\gamma) \rangle)$. But by definition of variable $M$-spaces, $[F]$ preserves $M$-spaces embeddings and by construction of $U^+$, $h$ is a $M$-space embedding. Therefore, $\tilde{F}(U^+)_J(\langle F \mid (\alpha) \rangle) = \tilde{F}(X)_J(\alpha)$ and $\tilde{H}(U^+)_J(\langle H \mid (\gamma) \rangle) = \tilde{H}(X)_J(\gamma)$, from which we conclude that $\tilde{F}(X)_J(\alpha) \subseteq \tilde{H}(X)_J(\gamma)$, that is $(s \circ t)|_{X_1} \in \text{Cl}_{M} ((F \rightarrow H)(X))$. \hfill ∎

**Proof of theorem 4.6.** We check that all the structure morphisms introduced in section 2 are variable cliques. The case of $\text{MALL}$ is easy and left to the reader.

Let us focus on the exponential structures. One easily checks that $\delta^E$ and $p^E$ are variable cliques. Given $f : F \rightarrow G$ a variable clique, let us show that $[f]$ is a variable clique.

Let $X$ be a $M$-space, let $J$ be a subset of $I$ and let $\alpha$ be a $J$-indexed family of $\{ f \}_{X_1}$. For any $j \in J$ we have $\alpha_j = (\beta_j, \gamma_j)$ where $\beta_j = [b_1^j, \ldots, b_n^j]$ and $\gamma_j = [c_1^j, \ldots, c_n^j]$, with $(b_1^j, \ldots, b_n^j) \in f_1|_{X_1}$ for every $j \in J$ and $k \in \{1, \ldots, n_j\}$. Let $K$ be a subset of $I$, let $u : K \rightarrow J$ be an injection and let $\varphi$ be a $K$-indexed family whose elements are in $f_1|_{X_1}$, such that $(\varphi, u)$ is a representative of $\alpha$. For any $k \in K$, we have $\varphi_k \in (F \rightarrow G)(\langle X \rangle)$, hence $\varphi_k = (\xi_k, \zeta_k)$ with $\xi_k \in (F \mid (X))$ and $\zeta_k \in G(\langle X \rangle)$. One easily checks that $(\xi_k, \zeta_k)$ is a representative of $\beta$ and that $(\zeta_k, u)$ is a representative of $\gamma$. Furthermore, for any $k \in K$ we have $(\xi_k, \zeta_k) \in f_1|_{X_1}$ and as $f$ is a variable clique, we get $\tilde{F}(X)_K(\xi) \subseteq \tilde{G}(X)_K(\zeta)$. As a consequence, $(u, F(X)_K(\xi)) \subseteq (u, G(X)_K(\zeta))$ and hence $\tilde{F}(X)_J(\beta) \subseteq \tilde{G}(X)_J(\gamma)$.

Then, let us focus on the second-order structures. Let $F$ and $G$ be variable $M$-spaces (with respective arity 1 and 2) and let $t : \Phi F \rightarrow G$ be a variable clique, then let us prove that $\Lambda^{E,G}(t) : F \rightarrow \mathbb{Y}G$ and $\varepsilon^G : \Phi \mathbb{Y}G \rightarrow G$ are respectively variable cliques of $F \rightarrow \mathbb{Y}G$ and $\Phi \mathbb{Y}G \rightarrow G$.

Let $J$ be a subset of $I$, and let $\alpha = (a_j, (n_j, b_j))_{j \in J}$ be a $J$-indexed family of $\Lambda^{E,G}(t)|_{X_1}$. For any $j \in J$, $(a_j, b_j)$ belongs to $t|_{X_1}$. Let $Y$ be a $M$-space and for any $j \in J$, let $f_j : Y \rightarrow |\mathbb{Y}G|$ be an injection. By mutuality, we have $t|_{X_1}[\mathbb{Y}G] = [\Phi F \rightarrow G][|X_1], f_j \rangle|_{X_1}[\mathbb{Y}G]$. Therefore $[\Phi F \rightarrow G][|X_1], f_j \rangle(a_j, b_j) = (a_j, G(|X_1], f_j)\langle b_j \rangle) \in t|_{X_1}[\mathbb{Y}G]$. Moreover given that $t|_{X_1}[\mathbb{Y}G]$ belongs to $\text{Cl}_{M}((\Phi F \rightarrow G)(X,Y))$, we get $F(X)|_j(a_j, b_j) \subseteq \tilde{G}(X,Y)|_j[G(|X_1], f_j)\langle b_j \rangle]$, from which, $\tilde{F}(X)|_j(a_j, b_j) \subseteq \mathbb{Y}G(X,Y)|_j[b_j]$. (cf. section 4.4).

Let $\alpha = (a_j, (n_j, b_j))_{j \in J}$ be a $J$-indexed family of $\varepsilon^G|_{X_1}$. By definition of $\varepsilon^G$, for any $j \in J$ there exists an injection $f_j : Y \rightarrow |\mathbb{Y}G|$ such that $a_j = G(|X_1], f_j)\langle b_j \rangle$. Let $p \in \Phi \mathbb{Y}G(X,Y)|_j[b_j] \subseteq \mathbb{Y}G(X,Y)|_j[b_j]$. By definition of $\mathbb{Y}G$, $p \in G(X,Y)|_j[G(|X_1], f_j)\langle b_j \rangle \rightarrow [a_j] \equiv a_j$.

5. Example: a non-uniform coherence semantics

We will now use this phase semantics and define a non-uniform version of the standard coherence semantics of linear logic.

5.1. Reminder: the product phase space $\text{Cob}_2$.

In this section, we recall the product phase space $\text{Cob}_2$ defined in [Bu-Eu,00]. Let $P_0$ be the monoid $\{0, 1, \tau\}$ defined by the following equation: $\tau \tau = \tau$. Now let us denote by $\perp$ the subset of $P_0^2$ defined by $\{p \in (p_1, p_2) \mid p_1 p_2 \in \{0, 1\}\}$. The product phase space $(P_0^2, \perp)$ has three facts, namely

- $\overline{0} = \{(0, 0), (1, 0), (0, 1), (0, \tau), (\tau, 0)\}$ that we will call incoherence,
- $N = \perp$ that we will call neutrality,
• C = \( P^2_0 \) that we will call coherence.

Let us now define the notion of coherence graph.

**Definition 5.1.** A coherence graph \( G \) on a set of vertices \( J \) is given by to disjoint subsets of the set of unordered distinct pairs of elements of \( J \) called coherences and incoherences of \( G \) (of course the edges which are neither coherent nor incoherent are called neutral).

Such a coherence graph will often be represented as a labelled graph.

Now let us define a product phase space on \( P^2_0 \) by:

\[ \perp = \{ p \in P^2_0 : \forall i \neq j \ (p_i, p_j) \in \perp^2 \} \]

**Proposition 5.2.**

(i) The product phase space \( M = (P^2_0, \perp) \) is symmetric.

(ii) Let \( J \) be a subset of \( I \). Any fact \( F \) of level \( J \) of \( M \) is completely determined by its binary projections. In other words, there is a bijective correspondence between \( F_M(J) \) and the coherence graphs on \( J \).

We will only precise this equivalence throughout an example (see [BuEtH.00] for more details). In the sequel, given \( i,j \) two distinct elements of \( J \), we will denote by \( \pi_{i,j} \) the function from \( P^2_0 \) onto \( P^2_0 \) defined by \( p \mapsto (p_i, p_j) \). Let us assume (for convenience) that \( I = 3 \). Then the following subset of \( P^2_0 \) is a fact:

\[
F = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1),
(\tau,0,0),(0,\tau,0),(0,0,\tau),
(\tau,\tau,0),(\tau,1,0),(1,\tau,0),(1,1,0)\}
\]

By proposition 5.2, all the binary projections of \( F \) are facts of \( (P^2_0, \perp^2) \) (actually, two facts are equal if and only if they have the same binary projections). Now, one easily computes that \( \pi_{1,2}(F) = C, \pi_{2,3}(F) = \overline{C} \) and \( \pi_{1,3}(F) = N \) and \( F \) is precisely the set \( \{ p \in P^2_0 : (p_1, p_2) \in C, (p_2, p_3) \in \overline{C}, (p_1, p_3) \in N \} \). Therefore, we will represent \( F \) by the following graph:

```
1 ------- 2
|        |
|        |
3 -------
```

---

**5.2. Non-uniform coherent spaces.**

We now give a direct description of the category \( \mathcal{C}_*(\text{Coh}^2) \) in term of non-uniform coherence spaces.

**Definition 5.3.** A non-uniform coherence space is a triple \( E = ([E], \sim_E, \omega_E) \) where \( |E| \) is a denumerable set called the web of \( E \), and \( \sim_E \) and \( \omega_E \) are two binary symmetric relations on \( |E| \) called respectively coherence and incoherence whose intersection is empty.

We will denote by \( \equiv_E \) the neutrality relation.

The idea of the non-uniform coherence semantics is that given a close formula \( A \), whose interpretation in \( \mathcal{C}_*(\text{Coh}^2) \) is \( ([A], \hat{A}) \), we define a non-uniform coherence space \( ([A], \sim_A, \omega_A) \) by:

- \( a \sim_A b \) if and only if \( \hat{A}_{[1,2]}(a, b) = C \)
- \( a \sim_A b \) if and only if \( \hat{A}_{[1,2]}(a, b) = \overline{C} \)

Let us now define by induction the interpretation of a formula of \( \text{LL}^2 \). Let \( A \) be a formula whose free variables are among \( X_1, \ldots, X_n \) and let \( E_1, \ldots, E_n \) be non-uniform coherence spaces. We will denote by \( A_{E_1,\ldots,E_n}^* \) the non-uniform coherent interpretation of \( A \) parametrized by \( E_1, \ldots, E_n \). The web of \( A_{E_1,\ldots,E_n}^* \) is
$|A|([E_1], \ldots, [E_n])$ (where $|A|$ denotes the stable functor interpreting $A$) and the coherence and incoherence relations are defined by

- **Units** are interpreted as in coherence spaces.
- If $A = X_i$ then we define $A^*_{E_1}, \ldots, E_n = E_i$.
- If $A = B^l$, let us write $A = A^*_{E_1}, \ldots, E_n$ and $B = B^*_{E_1}, \ldots, E_n$.
  
  For every $a, b \in |A^*_{E_1}, \ldots, E_n|$ we define: $a \sim_A b$ if and only if $a \sim_B b$ and $a \sim_A b$ if and only if $a \sim_B b$.
- If $A = B \oplus C$, let us write $A = A^*_{E_1}, \ldots, E_n$, $B = B^*_{E_1}, \ldots, E_n$ and $C = C^*_{E_1}, \ldots, E_n$.
  
  Then we define $\sim_A$ by:
  
  $$b \sim_B b' \Leftrightarrow (1, b) \sim_A (1, b')$$
  
  $$c \sim_C c' \Leftrightarrow (2, c) \sim_A (2, c')$$
  
  And we define $\sim_A$ by:
  
  $$(1, b) \sim_A (2, c)$$
  
  $$b \sim_B b' \Leftrightarrow (1, b) \sim_A (1, b')$$
  
  $$c \sim_C c' \Leftrightarrow (2, c) \sim_A (2, c')$$

- If $A = B \otimes C$, let us write $A = A^*_{E_1}, \ldots, E_n$, $B = B^*_{E_1}, \ldots, E_n$ and $C = C^*_{E_1}, \ldots, E_n$.
  
  Then $a \sim_A b$ if and only if $\pi_1 a \sim_B \pi_1 b$ and $\pi_2 a \sim_C \pi_2 b$ (one of these coherences being strict). And $a \sim_A b$ if and only if $\pi_1 a \sim_B \pi_1 b$ or $\pi_2 a \sim_C \pi_2 b$.

- If $A = [B]$, let us write $A = A^*_{E_1}, \ldots, E_n$ and $B = B^*_{E_1}, \ldots, E_n$.
  
  Let us write $\alpha = \sum_{k \in K} b_k$, then $\alpha \sim_A \beta$ if and only if $\alpha + \beta$ is a multidic of $B$ (i.e. for any $k \neq l \in K$ we have $b_k \cap b_l$) which is star-shaped for $\alpha \cap b_l$ (i.e. there exists $k \in K$ such that for any $l \in K$ with $l \neq k$ we have $b_k \cap b_l$). And $\alpha \sim_A \beta$ if and only if there exists $k \neq l \in K$ such that $b_k \cap b_l$.

- If $A = \forall X, B$, let us write $A = A^*_{E_1}, \ldots, E_n$.
  
  Then $(n, a) \sim_A (m, b)$ if and only if for any non-uniform coherence space $E$ (let us write $B_E = B^*_{E_1}, \ldots, E_n, E$) and any two injections $f : \pi \hookrightarrow [E]$ and $g : \overline{\pi} \hookrightarrow [E]$ we have
  
  $$|B|([E_1], \ldots, [E_n], f)(a) \sim_B g(|B|([E_1], \ldots, [E_n], g)(b)).$$
  
  And $(n, a) \sim_A (m, b)$ if and only if there exists a non-uniform coherence space $E$ (let us write $B_E = B^*_{E_1}, \ldots, E_n, E$) and two injections $f : \pi \hookrightarrow [E]$ and $g : \overline{\pi} \hookrightarrow [E]$ such that we have $|B|([E_1], \ldots, [E_n], f)(a) \sim_B g(|B|([E_1], \ldots, [E_n], g)(b))$.

5.3. **The example of $\text{Bool}$.**

Let us define $\text{Bool} = \forall X (X \Rightarrow (X \Rightarrow X))$, that is in terms of linear connectives $\text{Bool} = \forall X (X \Rightarrow (X \Rightarrow X))$.

First, one proves that the web of $\text{Bool}$ is as follows

$$|\text{Bool}| = \{ (n, (\mu, (\nu, p))) ; \mu, \nu \in \mathcal{M}_x, p \in \pi, \}$$

$$|\mu| \cup |\nu| \cup \{ p = \overline{\pi} \}$$

Now, one checks that the only points of $|\text{Bool}|$ which are coherent with themselves are of three types

- **True$_n$** = $(1, ([0]^n, ([], 0)))$ with $n \in \mathbb{N}^*$.
- **False$_n$** = $(1, ([], ([0]^n, 0)))$ with $n \in \mathbb{N}^*$.
- **Inter$_{m,n}$** = $(1, ([0]^n, ([0]^m, 0)))$ with $m, n \in \mathbb{N}^*$.

The coherence relations between these points are as follows: for any $n, m, n', m' \in \mathbb{N}^*$ we have
• True \equiv \text{Bool} \ True_{n'} ,
• False \equiv \text{Bool} \ False_{n'},
• \text{Inter}_{n,m} \equiv \text{Bool} \ \text{Inter}_{n',m'}
• \text{True, False and Inter} \text{ are pairwise inequivalent.}

As a conclusion, the uniform coherent interpretation of \text{Bool} can be retrieved as the uniform fragment of its interpretation in \mathcal{C}_0(\text{Coh}^2). However, the non-uniform web of \text{Bool} is far richer than its uniform web. Points such as \langle 2, (0, (1, 0)) \rangle belong to the web of \text{Bool} and are inequivalent with themselves.

References


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