On Köthe sequence spaces and linear logic

Thomas Ehrhard
Institut de Mathématiques de Luminy,
C.N.R.S. U.P.R. 9016
ehrhard@iml.univ-mrs.fr

October 18, 2000

Abstract

We present a category of locally convex topological vector spaces which is a model of propositional classical linear logic, based on the standard concept of Köthe sequence spaces. In this setting, the spaces interpreting the exponential have a quite simple structure of commutative Hopf algebra. The co-Kleisli category of this linear category is a cartesian closed category of entire mappings. This work provides a simple setting where typed \(\lambda\)-calculus and differential calculus can be combined; we give a few examples of computations.

1 Introduction

It has been clear from the very beginning that linear logic has something to do with linear algebra, and thus that, at some point, some real or complex coefficients should naturally appear. Nevertheless most concrete models discovered so far are essentially discrete (coherence spaces, hypercoherences, games ...). Interpreting formulae of linear logic as vector spaces is not easy because to formulae containing exponentials one must associate infinite dimensional spaces and for that reason, topological vector spaces must be considered. Several models based on topological vector spaces have already been presented, by Girard [Gir99] and Blute [Blu96] (in a setting considered first by Barr in [Bar76]).

We propose here a quite simple approach based on a standard notion in the theory of locally convex topological vector spaces (lcs for short), the notion of Köthe sequence spaces. These spaces constitute a well-behaved class of Hausdorff and complete lcs which are usually not Banach spaces and, with this respect, our approach will radically differ from Girard’s model of coherence Banach spaces. Another difference is the absence of any notion of coherence in our model, whereas in Girard’s setting, “coherent” vectors are those whose norm is less than 1. Our semantics will also be quite different from Blute’s model as the topology we shall consider on the underlying field (the field of real numbers, or the field of complex numbers) will be the standard topology, not the discrete one.

The central notion of the model under consideration is the concept of an absolutely converging sum. For understanding the relevance of this concept to denotational semantics, one must consider settings where the exponentials of linear logic are interpreted in a non-uniform way, like in the models induced by the symmetric product phase spaces of [BE99b]. The usual coherence space semantics indeed is uniform: this means that the elements of the web of the coherence space \(!E\) are the finite cliques (or multi-cliques) of the coherence space \(E\). For that reason, this semantics has the essential property that the intersection of a clique and an anti-clique contains at most one
element. Intuitively, in a language inspired by game models, the interaction between a strategy and a counter-strategy yields at most one result: this is a property of determinism. As explained in [BE99b], in a non-uniform version of coherence spaces, the intersection of a clique and an anti-clique can have more than one element. This is due to the fact that, in this setting, two distinct points of a web can be neutral (that is, neither coherent nor incoherent) without being equal.

The basic “algebraic” intuition behind coherence spaces is that cliques of $E$ should be considered as “vectors” and that anti-cliques of $E$ (that is, cliques of $E^\perp$) should be considered as “linear forms” on $E$; indeed, $E^\perp$ is isomorphic to $E \cong \perp$, the space of linear maps from $E$ to $\perp$ (the one point coherence space, which plays the rôle of the “field” considered as a space). If we consider a subset of $|E|$ as a $|E|$-indexed family of elements of $\{0, 1\}$ (the “field” considered as the set of scalars1), then given a clique $x$ of $E$ and an anti-clique $x'$ of $E$, their intersection has 0 or 1 element, and its cardinality is trivially given by

$$\#(x \cap x') = \sum_{a \in |E|} x_a x'_a,$$

this sum converging because at most one of its terms is different from 0(!). In a non-uniform setting, there is no reason why this sum should converge: in the non-uniform coherence spaces mentioned above, we just know that if $x_a x'_a = 1$ and $x_b x'_b = 1$, then $a$ and $b$ are neutral, but they can be different. It is here that Köthe spaces come in: let us replace the “field” $\{0, 1\}$ by one of the two fields $\mathbf{R}$ or $\mathbf{C}$ (denoted by $\mathbf{K}$), and let us take as basic requirement for building our spaces the convergence of the sum above. Another solution consists in staying in the discrete setting (still considering subsets of the webs) and saying that two subsets are orthogonal if their intersection is finite; this leads to the notion of finiteness spaces, see [Ehr00].

More precisely, an at most countable set $I$ being given (the web), we replace the notion of “subset of $I$” by the notion of “element of $\mathbf{K}^I$”. Then, given $x, x' \in \mathbf{K}^I$ we replace the notion of “cardinality of the intersection of $x$ and $x'$” by the notion of “value of the series $\sum_{i \in I} x_i x'_i$”. This latter value of course is not always well defined; since there is a priori no order relation on $I$, the only reasonable condition to ask on $x$ and $x'$ for this series to converge is to require its absolute convergence2. So, following Girard’s terminology (which, in the present context, could be misleading, and therefore will be avoided later on), let us say that $x, x' \in \mathbf{K}^I$ are orthogonal if the series $\sum_{i \in I} |x_i x'_i|$ converges, that is, if $xx' \in \ell^1(I)$. Then the notion of “coherence space of web $I$” will be replaced by the notion of subset $E$ of $\mathbf{K}^I$ such that $E = E^\perp E^\perp$ for this notion of orthogonality, where $E^\perp$ denotes the set of all the elements of $\mathbf{K}^I$ which are orthogonal to all the elements of $E^3$.

It turns out that such a subset $E$ of $\mathbf{K}^I$ is always a vector space (with operations defined in the obvious componentwise manner), and even a locally convex topological vector space which is Hausdorff and complete (its topology is defined by a family of semi-norms induced by the elements of $E^\perp$): these spaces belong to the class of Köthe sequence spaces (which are classically defined as the orthogonals of certain subsets of $\mathbf{K}^I$ called Köthe sets); see for instance [Jar81], which will be our basic reference for locally convex spaces. The topological vector spaces so defined are not always (more precisely, they are almost never) Banach spaces, they can even not be metrizable, but they behave remarkably well with respect to the operations needed for interpreting linear logic.

1It is definitely not a field in the usual sense, we just take the word “field” for developing an analogy.
2A classical result due to Riemann asserts that if a family $(x_i)_{i \in \mathbb{N}}$ is such that $\sum_{i \in \mathbb{N}} x_i$ converges for each permutation $\sigma$ of the natural numbers, then the series $\sum_{i \in \mathbb{N}} x_{\sigma(i)}$ converges absolutely.
3It is interesting to quote that coherence spaces can also be defined in this way: say that $x, x' \subseteq I$ are orthogonal if $\#(x \cap x') \leq 1$, then a coherence space of web $I$ can equivalently be defined as a subset of $\mathcal{P}(I)$ which is equal to its bi-orthogonal.
For us, a Kőthe space is a pair $X = ([X], E_X)$ where $[X]$ is an at most denumerable set and $E_X$ is a subset of $K^{[X]}$ such that $E_X \perp = E_X$. So the space is given together with $[X]$ (its web; the word has not to be taken in the sense it has in the theory of locally convex spaces), that is, with an explicit topological (actually, absolute, see [Jar81]) basis, and we shall heavily use this basis in our constructions. A morphism from $X$ to $Y$ is a linear continuous function from $E_X$ to $E_Y$, and we show that these functions are in bijective correspondence with the elements of $E_X \ast Y$ where $X \ast Y$ is a Kőthe space whose web is $[X] \times [Y]$; the element of $E_X \ast Y$ corresponding to a linear continuous function is its (infinite-dimensional) matrix. We also define a tensor product of Kőthe spaces and show that the category of Kőthe spaces and continuous linear maps is a $\ast$-autonomous category with (at most countable) sums and products, and we obtain in that way a model of multiplicative additive linear logic. One of the most striking features of this semantics is non-determinism. Typically, a boolean value will not simply be $t$ (true) or $f$ (false), but an arbitrary linear combination $\lambda t + \mu f$ of these two values, with coefficients $\lambda, \mu \in K$.

Next we introduce the exponential $!X$ of a Kőthe space $X$, which turns out to have a quite rich algebraic structure. As in [Gir99], the vector space $E_{!X} \ast Y$ will be isomorphic to a space of analytic mappings taking their values in $E_Y$, but here, these mappings will be defined on the whole space $E_X$ (and will actually be entire functions). In [Gir99] indeed, the analytic functions where defined only on the open unit ball, that is, on a set of "coherent" vectors: in that sense, the semantics of coherence Banach spaces is uniform whereas ours is not. Take $Y = \perp$ (that is, $E_Y = K$) for simplicity. The web of $!X$ will be, as it is standard in the semantics of linear logic, the set of all finite multi-sets on $[X]$ (a multi-set on a set $I$ is a map from $I$ to $N$, it is finite if it vanishes almost everywhere). Given $x \in E_X$ and $\mu \in ![X]$, we define $x^\mu$ as the finite product $\prod_{a \in [X]} x^\mu_a$; this is just the standard notion of a multi-exponent used in the theory of polynomials (or entire functions) of several variables (the rôle of the variables being played by the elements of the web), and we define $E_X$ simply as the biorthogonal of the set of all families $(x^\mu)_{\mu \in ![X]}$ when $x \in E_X$. Equivalently, $E_{|[X]|}$ is the set of all families $(u_\mu)_{\mu \in ![X]}$ of scalars such that the series $\sum_{\mu \in ![X]} u_\mu x^\mu$ converges absolutely, for all $x \in E_X$, that is, the space of all power series defined on the whole space $E_X$. To give a concrete example, when $X = 1$ (again, $[X]$ is a singleton, so $E_X = K$), $![X] = N$ and $E_{|[X]|}$ is the space of all power series of infinite convergence radius on $K$. Coming back to the general situation, an entire map from $X$ to $Y$ (Kőthe spaces) is then a function from $E_X$ to $E_Y$ which is definable by such a series (which is then necessarily unique). This interpretation of intuitionistic proofs as entire maps is of course completely in the spirit of Girard’s quantitive semantics (see [Gir88, BE99a]).

The space $!X$ has a co-algebraic structure, as it is standard in the semantics of linear logic (this structure is used for interpreting contraction and weakening), but also an algebraic structure, and is actually a quite simple commutative Hopf algebra. The algebraic structure is used e.g. for computing the derivatives of entire maps. The operation $X \mapsto !X$ is functorial, and defines as usual a comonad on the category of Kőthe spaces and continuous linear maps satisfying the properties required for interpreting the ! modality of linear logic. The co-Kleisli category of this co-monad is therefore cartesian closed, and is isomorphic to the category of Kőthe spaces and entire maps. This may seem surprising a priori as one is used to the idea that continuity on Hausdorff spaces, and so a fortiori analyticity, is incompatible with cartesian closeness\footnote{See [Lan71] and also the introduction of [Gir99]}. The point is that our entire maps are not continuous with respect to the native topology of the Kőthe spaces on which they are defined; continuity with respect to this topology is relevant only for linear maps. Typically, the linear evaluation function $E_X \times E_{X^\perp}$ which maps $(x, x')$ to $\langle x, x' \rangle = \sum_{a \in [X]} x_a x'_a$ is not continuous.
with respect to the product topology, as soon as the set $|X|$ is infinite. Although this phenomenon may seem weird, it is completely standard in the theory of locally convex spaces. For instance, all
the developments of [KM97] are based on the idea that smoothness is not a topological notion, but a “bornological” notion.

At the end of the paper, we sketch a theory of intrinsic Köthe spaces for advocating the fact that, although we used intensively the webs in our space constructions, the spaces obtained do not really depend on them. We define an intrinsic Köthe space as a topological vector space $E$
which is linearly homeomorphic to some Köthe space: this is a property of the topology of $E$. Using the functoriality of the operations defined on Köthe spaces, we show how each operation on Köthe spaces has a corresponding operation on intrinsic Köthe spaces. For instance, given $E$ and $F$ intrinsic Köthe spaces, we endow $\mathcal{L}(E, F)$ (the linear space of linear continuous maps from $E$ to $F$) with a topology such that this space becomes an intrinsic Köthe space. For defining this topology, we use linear homeomorphisms of $E$ and $F$ to Köthe spaces, but the resulting topology does not depend on this choice. Of course, it would be much more satisfactory to directly define this topology in terms of the topologies of $E$ and $F$, but we do not know how to do that in general yet (we give a negative result which closes the most natural track). Note that an isomorphism between $E$ and some Köthe space $E_X$ can also be seen as a choice in $E$ of an absolute basis satisfying some additional requirement (corresponding to the fact that $E_X = E_X^{\perp\perp}$); we choose such bases in $E$ and $F$ for defining the topology of $\mathcal{L}(E, F)$, although this topology does not depend on this choice.

Of course, the present work does not pretend to bring any new idea to the rich theory of functional analysis and topological vector spaces. Our approach is intentionally quite concrete (bases are used everywhere), we do not seek for generality; for instance in our spaces of analytic mappings, we only consider mappings defined on the whole space by a power series having an “infinite radius of convergence” (entire functions), although Köthe spaces are probably more expressive than that. We just want to illustrate the fact that rather simple topological vector spaces can quite easily be used for modeling standard linear logic and typed $\lambda$-calculus\footnote{In [Gir99], linear logic is modified so as to make its interpretation in the category of coherence Banach spaces possible, by the introduction of weighted sequents. This is due to the fact that Girard's analytic functions are defined on the open unit ball whereas proofs are interpreted in the closed unit ball; weights are used for enabling the application of an analytic function to the interpretation of a proof. This discrepancy seems essentially due to the fact that his analytic maps are required to be continuous with respect to the topology of the Banach spaces on which they are defined.}. The paper remains at an elementary level, and no deep knowledge of the theory of topological vector spaces is required; this is due to the omni-presence of webs (canonical bases). Many important questions remain unanswered among which we can mention: characterization of the topology in linear and entire function spaces, characterization of the tensor product and comparison with the standard Grothendieck's tensor products on lcs. They are postponed to future work.
Contents

1 Introduction 1

2 Köthe spaces 5
   2.1 Köthe sequence spaces ........................................ 8
   2.2 Köthe spaces and linear continuous maps ...................... 9
   2.3 Equicontinuous sets ........................................... 11
   2.4 Direct sums and products ..................................... 13
   2.5 Tensor product ................................................ 13
   2.6 A negative result about the topology of linear function spaces .... 16

3 Exponentials and entire functions 18
   3.1 Functorial action of the exponentials ......................... 20
   3.2 The co-monicadic structure of the exponential ............... 23
   3.3 The co-algebraic structure of the exponential ............... 25
   3.4 The Hopf algebra structure of the exponential ............... 25

4 A cartesian closed category of entire mappings 26
   4.1 Computing derivatives ......................................... 27

5 Towards an intrinsic theory 30
   5.1 Topological dual ............................................... 31
   5.2 Linear function space ........................................ 32
   5.3 Tensor product ................................................ 33
   5.4 Exponentials .................................................. 33

6 Some examples 37

2 Köthe spaces

We denote by $\mathbf{K}$ the field of real or complex numbers endowed with its usual topology. Let $I$ be an at most denumerable set. If $u,v \in \mathbf{K}^I$, we shall say that $u$ and $v$ are in duality, if the sum $\sum_{i \in I} |u_i v_i|$ converges, that is, if $uv \in \ell^1(I)$. If $E$ is any subset of $\mathbf{K}^I$, we denote $E^\perp$ the set of all the elements of $\mathbf{K}^I$ which are in duality with all the elements of $E$. This set $E^\perp$ (called dual of $E$) is obviously a sub-vector space of $\mathbf{K}^I$ (over the field $\mathbf{K}$). If $E \subseteq \mathbf{K}^I$ satisfies $E = E^\perp$, then $E$ is a Hausdorff locally convex topological vector space, with topology defined by the following seminorms $(N_{x'})_{x' \in E^\perp}$:

$$N_{x'}(x) = \sum_{i \in I} |x'_i x_i|$$

Since this collection $\mathcal{N}$ of seminorms is directed (that is, if $N, N' \in \mathcal{N}$ there exists $N'' \in \mathcal{N}$ such that $N(x), N'(x) \leq N''(x)$ for all $x$), and satisfies also, for each $\lambda \in \mathbf{R}^+$, $N \in \mathcal{N} \Rightarrow \lambda N \in \mathcal{N}$, the induced locally convex topology can be described as follows: a subset $U$ of $E$ is a neighborhood of 0 in $E$ iff there is an element $x'$ of $E^\perp$ such that

$$\forall x \in E \quad N_{x'}(x) < 1 \Rightarrow x \in U.$$
When $E \subseteq \mathbf{K}^I$ satisfies $E^\perp = E$, we denote by $\varepsilon(E)$ the locally convex topology just described above. Unless otherwise specified, such a subspace $E$ of $\mathbf{K}^I$ will always be considered as equipped with the topology $\varepsilon(E)$. If $E$ is any subset of $\mathbf{K}^I$, then $E^\perp = E^\perp$, and so the dual $E$ of any subset of $\mathbf{K}^I$ is a subset of $\mathbf{K}^I$ which satisfies $E^\perp = E$.

If $x \in \mathbf{K}^I$, we denote by $|x|$ the family $(|x_i|)_{i \in I} \in (\mathbf{R}^+)^I \subseteq \mathbf{K}^I$. The next property (sometimes called solidity in the literature on sequence spaces) is obvious in the present setting, but will be intensively used in the sequel.

**Lemma 1** If $E \subseteq \mathbf{K}^I$ satisfies $E^\perp = E$, then

$$x \in E \iff \exists y \in E \quad |x| \leq |y|.$$ 

The next lemma (as part of the material in this section) is completely standard, we give its proof because it is simple.

**Lemma 2** Let $E \subseteq \mathbf{K}^I$ be such that $E = E^\perp$. Then $E$ is complete.

**Proof:** This actually results from the completeness of the Banach space $\ell^1(I)$. We give a direct proof. Let $(x(\gamma))_{\gamma \in \Gamma}$ be a Cauchy net in $E$ (since $I$ is at most denumerable, we could as well restrict our attention to sequences). So $(\Gamma, \leq)$ is a directed set and for all $x' \in E^\perp$ there exists $\gamma \in \Gamma$ such that, for $\delta, \delta' \geq \gamma$, one has $N_{x'}(x(\delta) - x(\delta')) < 1$. Then for each $i \in I$, the net $(x(\gamma)_i)_{\gamma \in \Gamma}$ is Cauchy in $\mathbf{K}$ and hence has a limit $x_i$. We show first that the family $x = (x_i)_{i \in I}$ belongs to $E$. So let $x' \in E^\perp$, we show that $N_{x'}(x) < \infty$. For each $n \in \mathbf{N}$, choose $\gamma_n \in \Gamma$ such that $N_{x'}(x(\delta) - x(\delta')) < 2^{-n}$ for $\delta, \delta' \geq \gamma_n$. We may assume the sequence $(\gamma_n)$ to be monotone, and we set $x(n) = x(\gamma_n)$ and assume moreover for convenience that $x(0) = 0$. For each $i \in I$, the sequence $(x'_i x(n))_{n \in \mathbf{N}}$ clearly converges to $x'_i x_i$ in $\mathbf{K}$. Let $J \subseteq I$ be finite, we have

$$\sum_{n=0}^{\infty} N_{x'}(x(n + 1) - x(n)) \leq 2$$

and we conclude since this holds for all finite $J \subseteq I$. One shows exactly in the same way that $N_{x'}(x - x(n)) \leq 2^{-n}$, and from this it follows that the net $(x_i)$ converges to $x$ in $E$. \hfill \blacksquare

We denote by $\mathbf{K}^{(I)}$ the subspace of $\mathbf{K}^I$ whose elements are the families which vanish almost everywhere.

**Lemma 3** Let $E \subseteq \mathbf{K}^I$ be such that $E = E^\perp$. Then $\mathbf{K}^{(I)} \subseteq E$ and $\mathbf{K}^{(I)}$ is dense in $E$. Therefore, $E$ is always separable.

So $E$ can also be considered as the completion of $\mathbf{K}^{(I)}$ for the topology $\varepsilon(E)$. The proof is trivial.
Remark: Such a space $E$ satisfying $E^{1+} = E$ is not necessarily a Banach space. Take for $E$ the space $\ell^q$ of all the $x \in K^\prime$ such that $\sum_{i \in I} |x_i|$ converges. Then $E^1$ is the space of all bounded families. Indeed, if $x' \in K^\prime$ is unbounded, one can find a sequence $(i_n)_{n \in N}$ of pairwise distinct elements of $I$ such that $|x_{i_n}^\prime| > 2^n$. If we define $x \in K^\prime$ by $x_i = 0$ if $i \notin \{i_n \mid n \in N\}$ and $x_i^\prime = 2^n$ if $i = i_n$, then $x \in E$ and $x$ and $x'$ are not in duality, so $x' \notin E^1$. Therefore $E = E^{1+}$, and as a topological vector space, $E$ is easily seen to be the usual Banach space $\ell^1$. But $E^1$ is identical to $\ell^\infty$ only as a vector space; its topology (induced by $E$) is strictly coarser than the topology of the Banach space $\ell^\infty$. Indeed, if $x \in E$, there is $i \in I$ such that $|x_i| \leq 1/4$ and for such an $i$, we have $N_x(x') = 1/2$ for $x' \in E^1$ defined by $x_j^\prime = 0$ if $j \neq i$ and $x_i^\prime = 2$. So the unit ball of the semi-norm $N_x$ is not contained in the unit ball of the norm of $\ell^\infty$, and therefore, this latter unit ball is not a neighborhood of 0 in $E^1$. Observe by the way that $E^1$ is separable, whereas $\ell^\infty$ is not.

More generally, as mentioned to us by Alexandra Bruasse, given $p, q \in ]1, \infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$, one can establish a linear isomorphism between $(\ell^p)^{\perp}$ and $\ell^q$, and therefore $(\ell^p)^{1+} = \ell^p$. When $p > 1$, neither $\ell^p$ nor $\ell^q$ can be a Banach space, when equipped with their topologies $\varepsilon(\ell^p)$ and $\varepsilon(\ell^q)$ (it is proved in [Jar81] that any Köthe sequence space – see below the definition of this notion – which is a Banach space is linearly homeomorphic to $\ell^1$). An easy direct reasoning (similar to the one above in the case $\ell^1/\ell^\infty$) shows that indeed the topology $\varepsilon(\ell^p)$ is strictly coarser than the topology of the Banach space $\ell^p$.

The duality between $E$ and $E^{1+}$ is determined by the following bilinear map, which is well-defined by definition (the series involved in this definition converges absolutely in $K$).

$$E \times E^{1+} \rightarrow K,$$

$$(x, x') \mapsto \langle x, x' \rangle = \sum_{i \in I} x_i x_i^\prime.$$  

Remark: This map is separately continuous, but not continuous at $(0, 0)$ (for the product topology), as soon as the set $I$ is infinite. Indeed, if it were continuous at $(0, 0)$, there would exist $y' \in E^{1+}$ and $y \in E$ such that, whenever $(x, x') \in E \times E^{1+}$ satisfy $N_{y'}(x) < 1$ and $N_y(x') < 1$ then $|\langle x, x' \rangle| < 1$. But since $\sum_{i \in I} |x_i y_i| < \infty$, there exists $i \in I$ such that $|x_i y_i| < \frac{1}{2}$. Then one can find $x, x' \in (\mathbb{R}^+)^I$ such that $x_j = x_j' = 0$ for $j \neq i$ and $x_i \leq \frac{1}{2|x_i|}$, $x_i' \leq \frac{1}{2|x_i|}$ and $x_i x_i' \geq 2$, and we have a contradiction.

**Lemma 4** A seminorm $N : E \rightarrow \mathbb{R}^+$ is continuous iff there exists $x' \in E^{1+}$ such that for all $x \in E$, $N(x) \leq N_{x'}(x)$.

**Proof:** Assume first that $N$ is continuous. There is a neighborhood $V$ of 0 in $E$ such that $x \in V \Rightarrow N(x) < 1$. So, by the very definition of the topology of $E$, there exists $x' \in E^{1+}$ such that $N_{x'}(x) < 1 \Rightarrow N(x) < 1$, for all $x \in E$. Assume that, for some $x \in E$, $N(x) > N_{x'}(x)$. Then there exists $t \in [0, 1]$ such that $N(tx) \geq t > N_{x'}(tx)$ and we have a contradiction.

Conversely, let $N$ be a semi-norm on $E_X$ and let $x' \in E_X^{1+}$ be such that $N \leq N_{x'}$. Let $x \in E_X$ and let $\varepsilon > 0$. Let $U$ be the open unit ball of the semi-norm $\bar{N}x'$ and let $y \in E_X$ be such that $y - x \in U$. We have $N(y) = N(y - x + x) \leq N(y - x) + N(x)$ and so $|N(y) - N(x)| \leq N(y - x) \leq N_{x'}(y - x) < \varepsilon$. So $N$ is continuous.

Since in $K$ there is a basis of neighborhood of 0 consisting only of disks, we have:

**Lemma 5** A linear form $f : E \rightarrow K$ is continuous iff the semi-norm $|f| : E \rightarrow \mathbb{R}^+$ which maps $x$ to $|f(x)|$ is continuous.
For $i \in I$, we denote by $e_i$ the element of $K^t$ defined by $(e_i)_{j} = \delta_{i,j}$ (the Kronecker symbol).

**Lemma 6** If $E \subseteq K^t$ satisfies $E^{\perp \perp} = E$, then $E^\perp$ is linearly isomorphic to the topological dual $E'$ of $E$.

**Proof:** First, if $x' \in E^\perp$, then the map $x \mapsto \langle x, x' \rangle$ is well defined and linear from $E$ to $K$. It is continuous by lemma 4 since $|langle x, x' \rangle| \leq N_{x'}(x)$. We denote by $\hat{x}$ this continuous linear map. Clearly, the map $x' \mapsto \hat{x}$ is linear. Conversely, given a continuous linear map $f : E \rightarrow K$, the semi-norm $|f|$ is continuous and so there is $x' \in E^\perp$ such that $|f|(x) \leq N_{x'}(x)$ for all $x \in E$. Taking $x = e_i$, we get $|f(e_i)| \leq |x'|$. From this, and from lemma 1, it results that setting $y'_i = f(e_i)$ for each $i \in I$, one defines an element $y'$ of $E^\perp$ that we denote by $M(f)$. The map $f \mapsto M(f)$ is clearly linear. Lemma 3, together with the continuity and linearity of $f$, implies that $f(x) = \langle x, M(f) \rangle$ for each $x \in E$, that is $M(f) = f$. Conversely, for $x' \in E^\perp$, one has for each $i \in I$: $M(x'_i) = \hat{x}'(e_i) = \langle e_i, x' \rangle = x'_i$ and the announced linear bijective correspondence is established.

### 2.1 Köthe sequence spaces

A particularly important way of obtaining subspaces $E$ of $K^t$ such that $E^{\perp \perp} = E$ is to define them as the dual of a Köthe set, that is, a subset $P$ of $(\mathbb{R}^+)^t$ such that

- $\forall i \in I \exists p \in P$ $p_i \neq 0$

- and $P$ is directed for the pointwise order on functions, that is: $\forall p, q \in P \exists r \in P$ $p, q \leq r$.

In that situation, one writes traditionally $E = \Lambda(P)$ and $E$ is called a Köthe sequence space; $\Lambda(P)$ is always considered as a topological vector space for the topology defined by the semi-norms $CN_p$ (for $p \in P$ and $C \in \mathbb{R}^+$). Indeed, the first condition on $P$ ensures that this topology is Hausdorff, and the second one ensures that the absolutely convex sets $\{x \in \Lambda(P) \mid N_p(x) < \varepsilon\}$ constitute a filter basis at 0.

**Lemma 7** The topologies induced on $P^\perp$ by $P$ and by $P^{\perp \perp}$ are identical if and only if for any $x \in P^{\perp \perp}$ there exists $p \in P$ and $C \in \mathbb{R}^+$ such that $|x| \leq Cp$.

**Proof:** First, since $P \subseteq P^{\perp \perp}$, the topology $\tau'$ induced by $P^{\perp \perp}$ is always finer than the topology $\tau'$ induced by $P$. If the condition stated in the lemma holds, it is clear that $\tau'$ is finer than $\tau$. Conversely, assume that $\tau'$ is finer than $\tau$. Then for any $x \in P^{\perp \perp}$, the semi-norm $N_x$ is $\tau'$-continuous, since it is $\tau$-continuous, hence there exists $p \in P$ and $C \in \mathbb{R}^+$ such that $CN_p \geq N_x$ (see the proof of lemma 4), but this in turn implies $Cp \geq |x|$.

Consequently, the topology induced by $P^{\perp \perp}$ can be strictly finer than the topology induced by $P$. Consider indeed the following situation: let $D$ be a maximal ideal of $\mathcal{P}(\mathbb{N})$, that is a subset of $\mathcal{P}(\mathbb{N})$ which is closed under finite unions, downwards closed with respect to inclusion and which does not contain $\mathbb{N}$, and which also is maximal with all these properties. Assume moreover that all the finite subsets of $\mathbb{N}$ belong do $D$; such a maximal ideal exists by Zorn’s lemma. Then let $P$ be the set of all characteristic maps of elements of $D$, it is a Köthe set. The element $u$ of $K^\mathbb{N}$ which is constantly equal to 1 belongs to $P^{\perp \perp}$, and so by lemma 7 the topology induced by $P^{\perp \perp}$ on $P^\perp$ is strictly finer than the topology induced by $P$. That $u \in P^{\perp \perp}$ results from the fact that $P^\perp = \ell^1$ (as a vector space) as, if this were not the case, there would exist $p \in K^\mathbb{N}$ such that $p \notin P^\perp$ and $p \in P^\perp$. But for such a $p$, it is possible to find two disjoint infinite subsets of $\mathbb{N}$, $L$ and $R$, such that none of the restrictions of $p$ to $L$ and $R$ belong to $\ell^1$. Since $D$ is maximal, we must have $L \in D$ or $\mathbb{N} \setminus L \in D$, and hence $L \in D$ or $R \in D$, in contradiction with the fact that $p \in P^\perp$. 


2.2 Köthe spaces and linear continuous maps

We adopt the following slightly unusual definition of Köthe space. The discussion above shows that it is more restrictive than the traditional notion of Köthe sequence spaces.

**Definition 8** A Köthe space is a pair $X = (|X|, E_X)$ where $|X|$ is a set (the web) and $E_X \subseteq K^{|X|}$ satisfies $E_X^{\perp\perp} = E_X$. If $X$ is a Köthe space, its dual $X^\perp$ is defined by $|X^\perp| = |X|$ and $E_X^{\perp} = E_X^{\perp\perp}$. When $X$ is a Köthe space, $E_X$ will always be considered as topological vector space, with topology $\varepsilon (E_X)$.

Observe that when $|X|$ is finite, $E_X = K^{|X|}$, with the usual product topology.

**Lemma 9** A linear function $f : E_X \to E_Y$ is continuous iff for all $y^\prime \in E_Y^{\perp}$, the seminorm $N_{y^\prime} \circ f : E_X \to \mathbb{R}^+$ is continuous.

**Proof:** Assume that for all $y^\prime \in E_Y^{\perp}$ the semi-norm $N_{y^\prime} \circ f$ is continuous. It suffices to show that $f$ is continuous at 0. Let $V$ be a neighborhood of 0 in $E_Y$. Let $y^\prime \in E_Y^{\perp}$ be such that the open unit ball of the semi-norm $N_{y^\prime}$ is included in $V$. Since the semi-norm $N_{y^\prime} \circ f$ is continuous, there exists $x^\prime \in E_X^{\perp}$ such that $N_{y^\prime} \circ f \leq N_{y^\prime}$. Now if $x \in E_X$ satisfies $N_{y^\prime}(x) < 1$, we have $f(x) \in V$ and so $f^{-1}(V)$ is a 0-neighborhood in $E_X$, so $f$ is continuous. The converse implication is trivial. ■

Let $f : E_X \to E_Y$ be a linear function. We define its matrix $M(f) \in K^{|X| \times |Y|}$ by $M(f)_{(a,b)} = f(e_a)_b$.

**Lemma 10** Let $f : E_X \to E_Y$ be linear. If $f$ is continuous, then for all $x \in E_X$ and $y^\prime \in E_Y^{\perp}$, the double series

$$\sum_{a \in |X|, b \in |Y|} M(f)_{(a,b)} x_a y^\prime_b$$

converges absolutely.

**Proof:** Let $y^\prime \in E_Y^{\perp}$; the seminorm $N_{y^\prime} \circ f$ on $E_X$ being continuous, there exists $x^\prime \in E_X^{\perp}$ such that

$$\forall x \in E_X \quad N_{y^\prime}(f(x)) \leq N_{y^\prime}(x),$$

so that in particular (taking $x = e_a$)

$$\forall a \in |X| \quad \sum_{b \in |Y|} |y^\prime_b M(f)_{(a,b)}| \leq |x^\prime_a|.$$  \hspace{1cm} (1)

Now let $x \in E_X$, the series $\sum_{a \in |X|} |x_a x^\prime_a|$ converges, and so by (1) the series

$$\sum_{a \in |X|} |x_a| \left( \sum_{b \in |Y|} |y^\prime_b M(f)_{(a,b)}| \right)$$

converges and the lemma is proved, since we are dealing with series of positive terms. ■

**Lemma 11** Let $M \in K^{|X| \times |Y|}$ be a matrix such that the series

$$\sum_{a \in |X|, b \in |Y|} M_{(a,b)} x_a y^\prime_b$$

converges absolutely for all $x \in E_X$ and $y^\prime \in E_Y^{\perp}$. Then the linear function $f : E_X \to K^{|Y|}$ given by $f(x)_b = \sum_{a \in |X|} M_{(a,b)} x_a$ is well defined, takes its values in $E_Y$ and is continuous from $E_X$ to $E_Y$. 

Proof: By our assumption about $M$, taking $y' = e_b$, we see that $(M_{a,b})_{a \in |X|} \in E_{X^\perp}$ and so $f$ is well defined and is obviously linear $E_X \to K^{[Y]}$. Moreover, if $x \in E_X$, then $f(x) \in E_Y$ as indeed, if $y' \in E_{Y^\perp}$, one has
\[
N_{y'}(f(x)) = \sum_{b \in |Y|} |y'_b| \sum_{a \in |X|} |M_{a,b}| x_a \leq \sum_{b \in |Y|} |y'_b| \sum_{a \in |X|} |M_{a,b}| x_a < \infty
\]

by our assumption about $M$.

It remains to show that $f$ is continuous. Let $y' \in E_{Y^\perp}$ and let $x' \in E_{X^\perp}$ be given by $x'_a = \sum_{b \in |Y|} |M_{(a,b)} y'_b|$ (that $x'$ is well defined and belongs to $E_{X^\perp}$ is a consequence of our assumption about $M$). Then if $x \in E_X$ we have $N_{y'}(f(x)) \leq N_{x'}(x)$ and we conclude that $f$ is continuous by lemma 9.

We denote by $\widehat{M}$ the function $f$ defined above, and when $x \in E_X$, we denote sometimes by $M \cdot x$ the vector $\widehat{M}(x) \in E_Y$.

Given two Köthe spaces $X$ and $Y$, the set $E$ of all matrices $M \in K^{[X] \times [Y]}$ such that the series
\[
\sum_{a \in |X|, b \in |Y|} M_{(a,b)} x_a y'_b
\]
converges absolutely for all $x \in E_X$ and $y' \in E_{Y^\perp}$ is a subset of $K^{[X] \times [Y]}$ which satisfies $E_{X^\perp} = E$ (because it is defined as the dual of something), and so the pair $(|X| \times |Y|, E)$ is a Köthe space that we denote by $X \ll Y$.

Proposition 12 The linear maps $M \mapsto \widehat{M}$ from $E_X \ll Y$ to $\mathcal{L}(E_X, E_Y)$ and $f \mapsto M(f)$ from $\mathcal{L}(E_X, E_Y)$ to $E_X \ll Y$ define a linear isomorphism between these two spaces.

Proof: Given $M \in E_X \ll Y$, we have $M(\widehat{M}) = \widehat{M(e_a)} = M(a,b)$ and given $f \in \mathcal{L}(E_X, E_Y)$, for each $b \in |Y|$, the map $g : E_X \to K$ defined by $g(x) = f(x)_b$ is a continuous linear form, and so $g = \widehat{M(g)}$ by lemma 6. But $M(g)_a = g(e_a) = f(e_a)_b = M(f)_{a,b}$, and so we have, for each $x \in E_X$, $f(x)_b = \sum_{a \in |X|} M(f)_{a,b} x_a$, and since this holds for each $b \in |Y|$, finally we get that $f = \widehat{M(f)}$.

In that way, we have defined a category $\mathcal{K}_K$ whose objects are the Köthe spaces and whose morphisms are the linear continuous functions between them.

The identity morphism from $X$ to $X$ is of course the usual identity function, and its matrix $I$ is the diagonal matrix $I_{a,b} = \delta_{a,b}$. Corresponding to composition of functions, the product of matrices is defined as usual: let $A \in E_X \ll Y$ and $B \in E_Y \ll Z$, the matrix $BA \in E_X \ll Z$ is given by
\[
(BA)_{a,c} = \sum_{b \in |Y|} A_{a,b} B_{b,c}.
\]

This sum converges absolutely because $(A_{a,b})_{b \in |Y|} \in E_Y$ as $e_a \in E_X$ and $(B_{b,c})_{b \in |Y|} \in E_{Y^\perp}$ as $e_c \in E_{Z^\perp}$. Moreover, if $f : E_X \to E_Y$ and $g : E_Y \to E_Z$ are the linear continuous maps defined by $A$ and $B$ respectively, then $M(g \circ f)_{a,c} = g(f(e_a))_c = g(\sum_{b \in |Y|} A_{a,b} e_b)_c = (BA)_{a,c}$ by linearity and continuity of $g$. It follows, since $g \circ f$ is continuous, that $BA \in E_X \ll Z$, and we have seen that $M(g \circ f) = BA$.

An obvious consequence of the considerations above on continuous linear maps is the following
Proposition 13 Let \( f : E_X \to E_Y \) be linear and continuous. Then its transpose \( f^\perp : E_Y \to E_X \) (given by \( f^\perp(y') = \langle y' \circ f \rangle \)) defines a continuous linear map \( f^\perp : E_{Y^\perp} \to E_{X^\perp} \) whose matrix is given as usual by \( M^\perp(f) = M(f)^\perp \) where \( A^\perp_{(b,a)} = A_{(a,b)} \) for a matrix \( A \in K^{\mid X \mid \times \mid Y \mid} \).

Proof: The function \( f^\perp \) is given by \( f^\perp(y') = M(y' \circ f) \). It is clearly well defined, linear, and takes its values in \( E_{X^\perp} \). Let \( A = M(f) \in E_{X^\perp} \). Since clearly \( A^\perp \in E_{Y^\perp} \), it suffices to show that \( f^\perp = \widehat{A}^\perp \). So let \( y' \in E_{Y^\perp} \) and let \( x \in \mid X \mid \), we have

\[
\begin{align*}
    f^\perp(y')_a &= M(y' \circ f)_a \\
    &= \langle y', f(e_a) \rangle \\
    &= \sum_{b \in \mid Y \mid} A_{a,b} y'_b \\
    &= \widehat{A}^\perp(y')_a.
\end{align*}
\]

The map \( f^\perp \) is characterized by the following standard adjunction property:

\[
    \forall x \in E_X \quad \forall y' \in E_{Y^\perp} \quad \langle f(x), y' \rangle = \langle x, f^\perp(y') \rangle.
\]

One defines in that way a contravariant involutive endofunctor \( \mathcal{K} \rightarrow \mathcal{K}^\perp \) on \( K_K \).

Let \( X \) and \( Y \) be Köthe spaces and let \( \varphi : \mid X \mid \to \mid Y \mid \) be a bijection. We denote by \( \varphi^* \) the reindexing map \( K^{\mid Y \mid} \to K^{\mid X \mid} \) given by \( (\varphi^* y)_a = y_{\varphi(a)} \). The following lemma is easy but will be useful in the sequel.

Lemma 14 If, for each \( y \in K^{\mid Y \mid} \), one has \( y \in E_Y \) iff \( \varphi^* y \in E_X \), then \( \varphi^* \) is a linear homeomorphism from \( E_Y \) to \( E_X \).

Proof: It is clear that \( \varphi^* \) is linear, and since \( \varphi^{-1} \) satisfies the same condition as \( \varphi \), it will be sufficient to show that \( \varphi^* \) is continuous. So let \( x' \in E_{X^\perp} \) and let \( y' = (\varphi^{-1})^* x' \). We show that \( y' \in E_{Y^\perp} \); let \( y \in E_Y \). We know that \( \varphi^* y \in E_X \), so the sum \( \sum_{a \in \mid X \mid} \mid y_{\varphi(a)} x'_a \mid \) converges, that is, the sum \( \sum_{b \in \mid Y \mid} \mid y_b x'_{\varphi^{-1}(b)} \mid \) converges (to the same value). Since this holds for each \( y \in E_Y \), we have \( y' \in E_{Y^\perp} \). We have also shown that for each \( y \in E_Y \) we have \( N_{x'}(\varphi^* y) = N_{y'}(y) \), and so \( \varphi^* \) is continuous.

Of course, not all isomorphisms between Köthe spaces can be described as bijections between the webs (in sharp contrast with what happens in the “discrete” denotational semantics), but the “logical” isomorphisms which will be used for describing the categorical structures of the model under study will be of that particular shape.

2.3 Equi-continuous sets

We start a short digression about the notion of equi-continuous sets. These sets indeed can easily be used for describing the topology of the dual of a Köthe space, but unfortunately, this description does not seem to extend simply to more general function spaces, see section 2.6.

If \( X \) and \( Y \) are Köthe spaces, a subset \( H \) of \( \mathcal{L}(E_X, E_Y) \) is equi-continuous if, for any \( 0 \)-neighborhood \( V \) in \( E_Y \), the set

\[
    \bigcap_{f \in H} f^{-1}(V)
\]
is a 0-neighborhood in $E_X$. If $H \subseteq E_X$, we shall say that $H$ is equicontinuous if the set $\hat{H} = \{ \hat{x} | x \in H \} \subseteq E_{X^*}$ is an equicontinuous set of continuous linear forms on $E_{X^*}$.

**Lemma 15** A subset $H$ of $E_X$ is equicontinuous iff there exists $x \in E_X$ such that for all $y \in H$, $|y| \leq |x|$.

**Proof:** Indeed, $H$ is equicontinuous, by definition, iff there is a neighborhood $V'$ of 0 in $E_{X^*}$ such that

$$\forall x' \in V' \forall y \in H \quad |\langle y, x' \rangle| < 1$$

but this in turn is equivalent to requiring that there exists $x \in E_X$ such that for all $x' \in E_{X^*}$ and all $y \in H$, if $N_x(x') < 1$ then $|\langle y, x' \rangle| < 1$. Using the same kind of trick as in the proof of lemma 4, one shows that this is equivalent to

$$\forall x' \in E_{X^*} \forall y \in H \quad |\langle y, x' \rangle| < N_x(x')$$

which clearly implies (taking $x' = e_a$) that $|y_a| \leq |x_a|$ for all $a \in |X|$ and all $y \in H$. It is obvious that conversely, if $|y| \leq |x|$, then the condition above is fulfilled.

Given a Köthe space $X$, we denote by $\mathcal{E}_X$ the set of all the equicontinuous subsets $E_X$; it is clear that each of these sets is bounded (that is, for each $B \in \mathcal{E}_X$ and each continuous semi-norm $p$, the set $p(B)$ is bounded). We show that these sets are even relatively compact.

**Lemma 16** Let $x \in E_X$. Then $R(x)$ is a compact subset of $E_X$.

**Proof:** Let $D$ be the closed unit disk of $K$. Let $\varphi : D^{|X|} \rightarrow E_X$ be the function which maps $t \in D^{|X|}$ to $t_x = (t_ax_a)_{a \in |X|}$ which belongs to $E_X$ because the family $t$ is bounded. It is clear that $\varphi(D^{|X|}) = R(X)$, and since $D^{|X|}$ is a compact space (for the product topology), it will be sufficient to show that $\varphi$ is continuous.

So let $t \in D^{|X|}$ and let $x' \in E_{X^*}$. Since the series $\sum_{a \in |X|} |x_a x'_a|$ converges to, say, $S \in R^+$, there is a finite subset $A$ of $|X|$ such that $\sum_{a \in A} |x_a x'_a| \leq 1/4$. Let $U$ be the set of all the elements $s$ of $D^{|X|}$ such that

$$\forall a \in A \quad |s_a - t_a| < \varepsilon = \frac{1}{4(1 + S)},$$

this is a neighborhood of $t$ in $D^{|X|}$ (for the product topology) because $A$ is finite. If $s \in U$, we have

$$N_{x'}(\varphi(s) - \varphi(t)) = \sum_{a \notin A} |x'_a x_a (s_a - t_a)| + \sum_{a \in A} |x'_a x_a (s_a - t_a)|$$

$$\leq 2 \sum_{a \notin A} |x'_a x_a| + S \varepsilon < 1$$

by definition of $\varepsilon$ and so $\varphi$ is continuous.

As we have seen, this implies that each equicontinuous set is relatively compact. The converse does not hold in general. Consider the space $\ell^1 (\mathbb{N}^+)$, with canonical basis $(e_n)_{n \in \mathbb{N}^+}$. Then the set

$$\{0\} \cup \left\{ \frac{1}{n} e_n \mid n \in \mathbb{N}^+ \right\}$$

is compact since each neighborhood of 0 in $\ell^1 (\mathbb{N}^+)$ contains almost all the vectors $\frac{1}{n} e_n$ (remember that $\ell^1 (\mathbb{N}^+)$ is simply the usual Banach space). But clearly, there is no $x \in \ell^1 (\mathbb{N}^+)$ such that $\frac{1}{n} e_n \in R(x)$ for all $n \in \mathbb{N}^+$. 
Lemma 17 Let $X$ and $Y$ be Köthe spaces and let $f : E_Y \to E_X$ be a continuous linear map. If $H \in \mathcal{E}_Y$ then $f(H) \in \mathcal{E}_X$.

Proof: Let $A = |M(f)| \in E_X \to Y$ and let $x \in E_X$ be positive and such that $H \subseteq R(x)$. Then $f(H) \subseteq R(Ax)$ and so $f(H) \in \mathcal{E}_Y$.

2.4 Direct sums and products

Let $(X_j)_{j \in J}$ be a countable family of Köthe spaces. Let $K$ be the disjoint sum of the sets $|X_j|$. Let $E \subseteq K^*$ be defined by: $z \in E$ iff, for all $j \in J$, the restriction $\pi_j(z)$ of the family $z$ to $|X_j| \subseteq K$ belongs to $E_{X_j}$. Then $z' \in E^\perp$ iff $\pi_j(z') \in E_{X_j}^\perp$ for each $j \in J$ and, moreover, $\pi_j(z') = 0$ for almost all $j \in J$. Therefore, $(K, E)$ is a Köthe space that we denote by $\&_{j \in J} X_j$. The dual operation $\&_{j \in J} X_j = (\&_{j \in J} X_j^\perp)^\perp$ can be described as follows: $|\&_{j \in J} X_j|$ is the disjoint sum of the $|X_j|$’s, and $z \in E_{\&_{j \in J} X_j}$ iff $z \in E_{\&_{j \in J} X_j}$ and $\pi_j(z) = 0$ for almost all $j \in J$. It is clear then that $\&_{j \in J} X_j$ is the cartesian product of the spaces $X_j$ in the category $\mathcal{K}_K$, and that $\&_{j \in J} X_j$ is their direct sum. Given $x = (x(j))_{j \in J} \in E_{\&_{j \in J} X_j}$ and $x' = (x'(j))_{j \in J} \in E_{\&_{j \in J} X_j}$, we have

$$\langle x, x' \rangle = \sum_{j \in J} \langle x(j), x'(j) \rangle \quad \text{and} \quad N_{x'}(x) = \sum_{j \in J} N_{x'(j)}(x(j))$$

which are finite sums.

So the space $E_{\&_{j \in J} X_j}$ is canonically isomorphic to $\prod_{j \in J} E_{X_j}$, and its topology is the product topology. Observe also that, as usual in vector spaces, the finite direct sum $E_X \oplus Y$ and the finite direct product $E_X \& Y$ are identical.

In particular, $K^N$ is a Köthe space, with dual $K^{[N]}$, the space of almost everywhere vanishing sequences. As the infinite denumerable sum of 1 (the neutral element of the tensor product, see section 2.5) in the category $\mathcal{K}_K$, this latter space will correspond to the domain of “flat natural numbers". The space $K^{[N]}$ is not metrizable (we shall prove this fact, using a simple Cantor diagonalization argument), which shows that we cannot restrict our attention to Fréchet Köthe spaces, for instance (a Fréchet space is a complete metrizable lcs). Assume indeed that $K^{[N]}$ is metrizable, then its topology is induced by a denumerable family of neighborhood $(U_n)_{n \in \mathbb{N}}$ of 0. Therefore, there exists a family $(x'(n))_{n \in \mathbb{N}}$ of elements of $K^N$ such that $N_{x'(n)}(x) < 1 \Rightarrow x \in U_n$ for all $x \in K^{[N]}$. Clearly, we can assume that $x'(n)$ belong to $(\mathbb{R}_+^N$ for each $n \in \mathbb{N}$. Let $x' \in (\mathbb{R}_+)^N$ be given by $x'_n = x'(n) + 1$. Then there must exist $m \in \mathbb{N}$ such that $x \in U_m \Rightarrow N_{x'}(x) < 1$ for all $x \in K^{[N]}$, and this implies that $x'(m) \geq x'$ (for the product order), which leads to the usual contradiction, spelling out this inequation at index $m$.

Proposition 18 Let $X$ be a Köthe space and let $(Y_j)_{j \in J}$ be a family of Köthe spaces. Let $\varphi : |X \&_{j \in J} Y_j| \to |\&_{j \in J} (X \& Y_j)|$ be the obvious bijection (distributivity of the cartesian product over the disjoint sum). Then $\varphi^*$ is a linear homeomorphism from $E_{\&_{j \in J} (X \& Y_j)}$ to $E_X \&_{\&_{j \in J} Y_j}$.

The proof is a straightforward verification using lemma 14. This can also be considered as a piece of abstract non-sense resulting from the fact that $\&$ is the cartesian product and from the fact that the functor $Y \mapsto (X \& Y)$ has a left adjoint (tensor product, see below).

2.5 Tensor product

One defines of course $X \& Y$ by $X \& Y = (X \& Y^\perp)^\perp$. The dual operation $\&$ is given by $X_\& Y = X^\perp \& Y$. In other terms, $X_\& Y$ is the space of all sequences $P \in K^{[x] \times [y]}$ such that
the double series \( \sum_{a,b} P_{(a,b)} x'_a y'_b \) is converges absolutely for all \( x' \in E_X \) and \( y' \in E_Y \). For \( x \in E_X \) and \( y \in E_Y \), we define \( x \otimes y \in K^{[X \times [Y]} \) by \( (x \otimes y)_{(a,b)} = x_a y_b \), and then clearly \( E_X \otimes Y = \{ x \otimes y \mid x \in E_X \text{ and } y \in E_Y \} \).

Lemma 19 Let \( A \in K^{[X] \times [Y]} \). One has \( A \in E_{(X \otimes Y) - Z} \) iff for all \( x \in E_X \), \( y \in E_Y \) and \( z' \in E_{Z^\perp} \), the series

\[
\sum_{(a,b) \in [X \otimes Y] \neq Z} |A_{(a,b),c} x_a y_b z'_c|
\]

converges.

**Proof:** The condition is clearly necessary, let us show that it is sufficient, and so assume that it holds. Let \( t \in E_{(X \otimes Y)} \) and \( z' \in E_{Z^\perp} \), we must show that the series

\[
\sum_{(a,b) \in [X \otimes Y] \neq Z} |A_{(a,b),c} t_{(a,b),z'_c}| = \sum_{(a,b) \in [X \otimes Y]} |t_{(a,b)}| \sum_{c \in [Z]} |A_{(a,b),c} z'_c|
\]

converges. But for this, it is sufficient to show that the family \( \{ (\sum_{c \in [Z]} |A_{(a,b),c} z'_c|)_{(a,b) \in [X \otimes Y]} \} \) belongs to \( E_{(X \otimes Y)}^\perp = \{ x \otimes y \mid x \in E_X \text{ and } y \in E_Y \} \), and this holds iff the series

\[
\sum_{(a,b) \in [X \otimes Y]} |x_a y_b| \sum_{c \in [Z]} |A_{(a,b),c} z'_c|
\]

converges for all \( x \in E_X \) and \( y \in E_Y \). But this is precisely our hypothesis. \( \blacksquare \)

Given matrices \( A \in E_{X \otimes Y} \) and \( B \in E_{Z^\perp} \), we define a matrix \( A \otimes B = C \in K^{[X \times [Z]} \times ([Y] \times [T]) \) as follows: \( C_{(a,c),(b,d)} = A_{(a,b)} B_{(c,d)} \).

Lemma 20 \( A \otimes B \in E_{(X \otimes Z) - (Y \otimes T)} \).

**Proof:** Let \( x \in E_X \) and \( z \in E_Z \). Let \( M \in E_{(Y \otimes T)} \). We assume all these families to be positive. We make the same assumption concerning \( A \) and \( B \), since it suffices to show that \( |A \otimes B| = |A| \otimes |B| \in E_{(X \otimes Z) - (Y \otimes T)} \). Due to lemma 19, we just have to prove that the series

\[
\sum_{(a,c),(b,d) \in [(X \otimes Z) - (Y \otimes T)]} A_{(a,b)} B_{(c,d)} x_a z_c M_{(b,d)}
\]

converges. Denoting by \( f \) and \( g \) the continuous linear maps defined by \( A \) and \( B \), this sum is equal to

\[
\sum_{(b,d) \in [Y \otimes T]} M_{(b,d)} f(x) g(z) d
\]

and this latter sum converges since \( M \in E_{(Y \otimes T)^\perp} \), \( f(x) \in E_Y \) and \( g(z) \in E_T \). \( \blacksquare \)

Given \( f \in \mathcal{L}(E_X, E_Y) \) and \( g \in \mathcal{L}(E_Z, E_T) \), we denote by \( f \otimes g \) the continuous linear function \( E_X \otimes Z \to E_Y \otimes T \) defined by the matrix \( M(f) \otimes M(g) \).

Then one proves easily the following:

**Lemma 21** If \( x \in E_X \) and \( z \in E_Z \), then \( (f \otimes g)(x \otimes y) = f(x) \otimes g(z) \).
Due to lemma 3, the finite sums of tensors $x \otimes z$ constitute a dense subspace of $X \otimes Z$ (which is isomorphic to the algebraic tensor product $E_X \otimes E_Z$), and so this latter property completely determines the map $f \otimes g$. So we have shown that the operation $\otimes$ defined on Köthe spaces is functorial. That it defines a tensor product (associativity, MacLane pentagon and symmetry) with neutral element $1 = \{(\ast), K\}$ is easy to check, using lemma 14.

We denote by $\otimes$ the bilinear map $E_X \times E_Y \rightarrow E_{X \otimes Y}$ which maps $(x, y)$ to $x \otimes y$.

**Lemma 22** The canonical associativity bijection $\varphi : [X \leftrightarrow (Y \leftrightarrow Z)] \rightarrow [(X \otimes Y) \leftrightarrow Z]$ induces a linear homeomorphism $\varphi^*$ from $E(X \otimes Y) \leftrightarrow Z$ to $E_X \leftrightarrow (Y \leftrightarrow Z)$.

**Proof:** We show that $\varphi$ satisfies the conditions of lemma 14. Let $M \in E(X \otimes Y) \leftrightarrow Z$, and assume first that $M \in E(X \otimes Y) \rightarrow Z$. Let $x \in E_X$. For each $t \in E_X \otimes Z$, we must show that the sum

$$\sum_{(a,b,c) \in |X| \times |Y| \times |Z|} |M((a,b,c)x_a b_c)|$$

is finite, but for this it suffices to show that the family $(\sum_{a \in |X|} M((a,b,c)x_a b_c))$ belongs to $E_X \otimes Z$. Let $y \in E_Y$ and $z' \in E_{Z'}$, we just have to show that the sum

$$\sum_{(a,b,c) \in |X| \times |Y| \times |Z|} |M((a,b,c)x_a y_b z'_c)|$$

is finite, but this directly results from our assumption about $M$. Conversely, assume that $\varphi^* M \in E_{X \leftrightarrow (Y \leftrightarrow Z)}$. According to lemma 19, we just have to show that for all $x \in E_X$, $y \in E_Y$ and $z' \in E_{Z'}$, the sum

$$\sum_{(a,b,c) \in |X| \times |Y| \times |Z|} |M((a,b,c)x_a y_b z'_c)|$$

is finite, but this results from the fact that $y \otimes z' \in E_{(Y \leftrightarrow Z)}$. ■

By proposition 12, this linear homeomorphism $\varphi^*$ induces a (linear) bijection from $\mathcal{L}(E_X \otimes Y, E_Z)$ to $\mathcal{L}(E_X, E_Y \leftrightarrow Z)$. If $f : E_X \otimes Y \rightarrow E_Z$ is linear and continuous, the corresponding linear and continuous map $g : E_X \rightarrow (E_Y \leftrightarrow Z)$ maps $x$ to $g(x)$, the matrix of the linear and continuous map $f_x : E_Y \rightarrow E_Z$ defined by $f_x(y) = f(x \otimes y)$.

So the category $\mathcal{K}_K$ is monoidal symmetric closed. It is actually a $*$-autonomous category.

Let $f : E_Y \rightarrow E_X$ and $g : E_Z \rightarrow E_Y$ be linear continuous maps between Köthe spaces. Then we can define a linear continuous map $f \rightarrow g : E_Y \otimes Z \rightarrow E_Y \otimes T$ by its matrix (indexed by $(|X| \times |Z|) \times (|Y| \times |T|)$) as follows: $M(f \rightarrow g)_{(a,c),(b,d)} = M(f)_{b,c} M(g)_{a,d}$. This matrix belongs to $E_{(X \otimes Z) \otimes (Y \otimes T)}$ by lemma 20, indeed $M(f \rightarrow g) = (M(f) \otimes M(g))$. ■

**Lemma 23** The operation $(f, g) \mapsto f \rightarrow g$ is functorial (contravariant in its first argument and covariant in its second argument). Moreover, if $h \in \mathcal{L}(E_X, E_Z)$, one has $(f \rightarrow g)(M(h)) = M(g \circ h \circ f)$.

**Proof:** Functoriality results from the functoriality of the $\otimes$ operation on morphisms. Let $h \in \mathcal{L}(E_X, E_Z)$, we have $(f \rightarrow g)(M(h)) = M(f \rightarrow g)M(h)$ (product of matrices) by proposition 12, and an easy computation shows that $M(f \rightarrow g)M(h) = M(h)M(h)M(f)$. ■

**Proposition 24** The tensor product distributes over direct sums.

This results from proposition 18.

A central notion in the theory of locally convex spaces is the notion of bounded set. We conclude this section by observing that the universal bilinear map $\otimes : E_X \times E_Y \rightarrow E_{X \otimes Y}$ associated to this tensor product, though not continuous in general, nevertheless preserves bounded sets.
Definition 25 A subset $B$ of a lcs $E$ is bounded if, for any neighborhood $U$ of 0 in $E$, there exists $\lambda \in \mathbb{R}^+$ such that $B \subseteq \lambda U$ or, equivalently, if, for any continuous semi-norm $p : E \to \mathbb{R}^+$, the set \{${p(x) \mid x \in B}$\} is bounded.

Bounded sets are closed under taking subsets and under finite unions. A subset of a direct product of two lcs is bounded iff its two projections are. Moreover, continuous linear maps preserve boundedness (i.e. are bounded). The converse holds for instance in Banach spaces, but is false in general: an lcs is bornological if any bounded linear map from $E$ to a Banach space is continuous. Even in our setting, there are bounded linear forms on $(\ell^1)^\perp$ which are not continuous. Indeed, as a vector space, $(\ell^1)^\perp = \ell^\infty$ and one can check that a subset of $(\ell^1)^\perp$ is bounded if it is bounded for the norm of the Banach space $\ell^\infty$. So a bounded linear form on $(\ell^1)^\perp$ is just a continuous linear form on the Banach space $\ell^\infty$. Then it is a standard exercise to construct, using the Hahn-Banach theorem, a non-zero continuous linear form on $\ell^\infty$ which maps to 0 all the sequences which converge to 0, and such a linear form cannot be continuous on $(\ell^1)^\perp$; this latter space is not bornological.

The bilinear map $\otimes : E_X \times E_Y \to E_{X \otimes Y}$ cannot be continuous in general, since the bilinear form $E_X \times E_{X^\perp} \to K$ is not continuous as soon as $|X|$ is infinite, whereas the linear evaluation map $E_{X \otimes X^\perp} \to K$ is continuous by monoidal closeness. We show that nevertheless this map $\otimes$ is bounded, that is, it maps bounded sets to bounded sets (this condition is certainly not sufficient for a separately continuous bilinear map to factorize through the tensor product).

Assume that $\otimes$ is not bounded. Let $B$ and $C$ be bounded subsets of $E_X$ and $E_Y$ and let $M \in E_{(X \otimes Y)^\perp}$ be such that $\{N_M(x \otimes y) \mid (x,y) \in B \times C\}$ is not bounded. Then we can find in $B$ and $C$ two sequences $(x(n))_{n \in \mathbb{N}}$ and $(y(n))_{n \in \mathbb{N}}$ such that $N_M(x(n) \otimes y(n)) \geq 4^n$ for all $n \in \mathbb{N}$. For each $x \in |X|$, the sequence $(x(n))_{n \in \mathbb{N}}$ is bounded in $K$ since $B$ is bounded, and so the series $\sum_{n=0}^\infty |x(n)|_a 2^{-n}$ converges; let us denote by $x_a$ its sum. Then $x = (x_a)_{a \in |X|}$ belongs to $E_X$: let $x' \in E_{X^\perp}$ be positive, and let $A$ be an upper bound of $\{N_{x'}(z) \mid z \in B\}$, then $N_{x'}(2^{-n}x(n)) \leq 2^{-n}A$ and so $N_{x'}(x) \leq 2A$. Now let $y' \in K^{|Y|}$ be given by $y'_b = \sum_{a \in |X|} |M_{a,b}| x_a$, by our assumption about $M$, we have $y' \in E_{Y^\perp}$. Therefore, the sequence $N_{y'}(y(n))$ must be bounded. But $N_{y'}(y(n)) = N_M(x \otimes y(n)) \geq 2^{-n}N_M(x(n) \otimes y(n)) \geq 2^n$, contradiction.

2.6 A negative result about the topology of linear function spaces

In this section, we take $K = \mathbb{R}$ and hence $D = [-1,1]$.

We start with some simple combinatorial considerations. For $p \in \mathbb{N}$, let $(P_j)_{j=1,\ldots,4^p}$ be an enumeration of all the subsets of $\{1,\ldots,2p\}$ (we denote by $Q_j$ the complementary of $P_j$ in $\{1,\ldots,2p\}$) and let $A^{(p)}$ the $\mathbb{N} \times \mathbb{N}$-matrix given by

$$A^{(p)}_{i,j} = \begin{cases} 1 & \text{if } j \leq 4^p \text{ and } i \in P_j, \\ -1 & \text{if } j \leq 4^p \text{ and } i \in Q_j, \\ 0 & \text{otherwise.} \end{cases}$$

Given $t \in \mathbb{D}^\mathbb{N}$, we set

$$S^{(p)}(t) = \sum_{j=1}^{4^p} \sum_{i=1}^{2p} A^{(p)}_{i,j} t_i \in \mathbb{R}^+.$$

Observe first that, for all $t \in \mathbb{D}^\mathbb{N}$, one has $S^{(p)}(t) = S^{(p)}(|t|)$, due to the definition of $A^{(p)}$. Then, using the fact that the function $x \mapsto |a + x| + |a - x|$ is monotone on $\mathbb{R}^+$ (for any given $a \in \mathbb{R}$),
we obtain easily
\[ \forall t \in \mathbb{D}^\mathbb{N} \quad S^{[p]}(t) \leq \sum_{j=1}^{4p} \left| \sum_{i=1}^{2p} A^{[p]}_{i,j} \right| = S^{[p]} \cdot \]

But we have
\[ S^{[p]} = \sum_{j=1}^{4p} \left| \#P_j - \#Q_j \right| = \sum_{k=0}^{2p} \binom{2p}{k} \left| k - (2p - k) \right| = 4 \sum_{k=0}^{p-1} (p - k) \binom{2p}{k} \]

A standard computation\(^6\) (using e.g. the gamma function and the Stirling formula) shows now that
\[ \lim_{p \to \infty} \frac{S^{[p]}}{p^{4p}} = 0. \]

Using actually only finite-dimensional matrices, we have thus proven the following

**Lemma 26** For all \( \varepsilon > 0 \), there exists a matrix \( A \in \ell^1(\mathbb{N} \times \mathbb{N}) \setminus \{0\} \) such that, for all \( t \in \mathbb{D}^\mathbb{N} \),
\[ \sum_{j \in \mathbb{N}} \left| \sum_{i \in \mathbb{N}} A_{i,j} t_i \right| \leq \varepsilon \|A\|_1. \]

Let \( X \) and \( Y \) be Köthe spaces. The topology of uniform convergence over the equicontinuous subsets of \( E_X \) is defined on \( E_X \otimes Y \) by the following semi-norms
\[ \nu_{x,y}(A) = \sup_{u \in R(x)} N_y(A \cdot u) = \sup_{t \in \mathbb{D}^\mathbb{N}} \left| \sum_{j \in \mathbb{N}} \left| \sum_{i \in \mathbb{N}} A_{i,j} x_i y_j t_i \right| \right| \leq N_{x \otimes y}(A), \]

for \( x \in E_X \) and \( y' \in E_{Y^*} \). This topology \( \tau \) is always coarser than the topology of \( X \otimes Y \). When \( Y \) is finite-dimensional, the two topologies coincide. We show that, in some circumstances, the topology \( \tau \) is strictly coarser than the topology of \( X \otimes Y \).

Let \( Y \) be the Köthe space given by \( |Y| = \mathbb{N} \) and \( E_Y = \ell^1 \), and let \( X = Y^* \), so that \( E_X \) is the space of all bounded elements of \( K^\mathbb{N} \). Then clearly \( E_X \otimes Y = \ell^1(\mathbb{N} \times \mathbb{N}) \). If the topology of \( E_X \otimes Y \) were the topology of uniform convergence over equicontinuous sets, there would exist a constant \( k > 0 \) and two vectors \( x \in E_X \) and \( y' \in E_{Y^*} \) such that, for all \( A \in E_X \otimes Y \), \( \|A\|_1 \leq k \nu_{x,y}(A) \). We can assume \( \|x\|_\infty \cdot \|y'\|_\infty \leq 1 \) (due to the presence in this statement of the multiplicative constant \( k \)), and so this statement is equivalent to the existence of \( k > 0 \) and of \( y' \in E_{Y^*} \) with \( \|y'\|_\infty \leq 1 \) such that
\[ \forall A \in \ell^1(\mathbb{N} \times \mathbb{N}) \quad \|A\|_1 \leq k \sup_{t \in \mathbb{D}^\mathbb{N}} \left| \sum_{j \in \mathbb{N}} \left| \sum_{i \in \mathbb{N}} A_{i,j} t_i \right| \right| \]

\(^{6}\)Hint: submit this exercise to your favorite computer algebra system!
and so finally, if the topology of $E_{X \rightarrow Y}$ were the topology of uniform convergence over equiconti-
uous sets, there would exist a constant $k > 0$ such that

$$\forall A \in \ell^1(N \times N) \quad \|A\|_1 \leq k \sup_{t \in D^N} \sum_{i \in \mathbb{N}} \left| \sum_{j \in \mathbb{N}} A_{i,j} t_i \right|.$$  

But this cannot be the case, since by lemma 26, one would be able to find $A \in \ell^1(N \times N)$, $A \neq 0$, such that

$$\forall t \in D^N \quad \sum_{j \in \mathbb{N}} \left| \sum_{i \in \mathbb{N}} A_{i,j} t_i \right| \leq \frac{\|A\|_1}{2k} \leq \frac{1}{2} \sup_{e \in D^N} \sum_{j \in \mathbb{N}} \left| \sum_{i \in \mathbb{N}} A_{i,j} s_i \right|$$

and this is impossible since $\sup_{t \in D^N} \sum_{j \in \mathbb{N}} \left| \sum_{i \in \mathbb{N}} A_{i,j} t_i \right| > 0$ as $A \neq 0$.

So we can conclude with the following negative result.

**Proposition 27** There are Köthe spaces $X$ and $Y$ such that the topology of $X \rightarrow Y$ is strictly finer
than the topology of uniform convergence over the equicontinuous subsets of $E_X$.

And we are left with the problem of finding a functional characterization of the topology of $E_{X \rightarrow Y}$,
but at least, a track is closed.

### 3 Exponentials and entire functions

If $I$ is a set, we denote by $\mathcal{M}(I)$ the set of all finite multisets of elements of $I$. Let us first introduce
some notations concerning finite multisets. If $\mu \in \mathcal{M}(I)$, we define its *support* $|\mu|$ as the set of all 
i \in I \text{ such that } \mu(i) \neq 0$. We define its *size* (or cardinality) as $\#\mu = \sum_{i \in I} \mu(i) \in \mathbb{N}$. We also define its *factorial* as $! = \prod_{i \in I} \mu(i)!$. We extend to $\mathcal{M}(I)$ all the ordinary operations on natural
numbers componentwise and denote by $[i_1, \ldots, i_n]$ the multiset of the elements $i_1, \ldots, i_n$ of $I$, taking
multiplicities into account. If $\mu, \nu \in \mathcal{M}(\mathbb{N})$ are such that $\nu \leq \mu$, we define the binomial coefficient

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu - \nu)!} = \prod_{i \in I} \binom{\mu(i)}{\nu(i)}$$

For $x \in \mathbb{K}^I$ and $\mu \in \mathcal{M}(I)$, we define $x^\mu \in \mathbb{K}$ as $x^\mu = \prod_{i \in \mathbb{N}} x_i^{\mu(i)}$. Since the multiset $\mu$ is finite,
this product makes sense (we adopt the usual convention that $0^0 = 1$).

With these notations, the usual binomial equation immediately generalizes as follows: for $x, y \in \mathbb{K}^I$
and $\mu \in \mathcal{M}(I)$, one has $(x + y)^\mu = \sum_{\nu \leq \mu} \binom{\mu}{\nu} x^\nu y^ {\mu - \nu}$.

Let now $S$ be a commutative monoid (with additive notations for the operations). If $\gamma \in \mathcal{M}(S)$,
we denote by $\Sigma(\gamma)$ the element of $S$ given by $\Sigma(\gamma) = \sum_{s \in S} s^\gamma$.

We define next the multinomial coefficients for multisets. Let $J$ be another index set. Let $\mu \in \mathcal{M}(I)$
and let $\alpha \in \mathcal{M}(I \times J)$. If the following property holds:

$$\forall i \in I \quad \sum_{j \in J} \alpha(i, j) = \mu(i)$$

then we define the multinomial coefficient

$$\binom{\mu}{\alpha} = \frac{\mu!}{\alpha!} \in \mathbb{N}.$$
The binomial coefficient \( \binom{n}{k} \) corresponds to the particular case \( J = \{1, 2\} \) and \( \alpha(i, 1) = \nu(i) \) and \( \alpha(i, 2) = \mu(i) - \nu(i) \).

Let \( X \) be a Köthe space and let \( x \in E_X \). We define then \( x^! \in K^{M(\|X\|)} \) by \( (x^!)_\mu = x^\mu \). The Köthe space \( !X \) is given by

\[
\|!X\| = M(\|X\|) \quad \text{and} \quad E!X = \{x^! \mid x \in E_X\}^{\perp \perp}.
\]

Our definition of an entire map is just the natural extension to this setting of the usual definition of an entire map between finite dimensional spaces (analytic map with an infinite radius of convergence at 0).

**Definition 28** Let \( X \) and \( Y \) be Köthe spaces. A function \( h : E_X \to E_Y \) is entire if there exists a continuous linear function \( f : E_X \to E_Y \) such that \( h(x) = f(x^!) \) for all \( x \in E_X \). Such a linear function \( f \) (and its matrix) will be called a power series defining \( h \). If \( f : E!X \to E_Y \) is a continuous linear function, we denote by \( F(f) \) the corresponding entire function from \( E_X \) to \( E_Y \).

We shall first show that, when a function is entire, it is defined by a unique power series. This is done as usual, with the help of derivatives.

Let \( X \) be a Köthe space and let \( f : E_X \to K \) be an entire map, with defining power series given by its matrix \( M \in E^{\perp \perp}_{\|X\|} \). This means that, for any \( x \in E_X \), one has

\[
f(x) = \sum_{\mu \in M(\|X\|)} M_\mu x^\mu,
\]

this series converging absolutely. Let \( s = (a_1, \ldots, a_n) \) be a finite sequence of pairwise distinct elements of \( |X| \). We define an injective continuous linear map \( \eta_s : K^n \to E_X \) by \( \eta_s(t_1, \ldots, t_n) = \sum_{i=1}^n t_i e_{a_i} \). Then \( f \circ \eta_s \) is an entire function (in the usual sense) defined on \( K^n \) by the following absolutely converging series, because the \( a_i \)’s are pairwise distinct:

\[
(f \circ \eta_s)(t_1, \ldots, t_n) = \sum_{k_1, \ldots, k_n \in N} M_{k_1[a_1]+\ldots+k_n[a_n]} t_1^{k_1} \cdots t_n^{k_n}.
\]

Let \( \mu \in M(\|X\|) \), and let \( s = (a_1, \ldots, a_n) \) be an enumeration, without repetitions, of \( |\mu| \) (so that \( n \) is the cardinality of \( |\mu| \), not the cardinality of \( \mu \)). Then \( M_\mu \) appears as the coefficient of \( t_1^{\mu(a_1)} \cdots t_n^{\mu(a_n)} \) in the development above of \( (f \circ \eta_s)(t_1, \ldots, t_n) \). Therefore

\[
M_\mu = \frac{1}{\mu!} \frac{\partial^{\# \mu} (f \circ \eta_s)}{\partial \mu^{(a_1)} \cdots \partial \mu^{(a_n)}}(0, \ldots, 0)
\]

and this shows that an entire function from \( E_X \) to \( K \) has only one defining power series. This immediately extends to the case of the general entire functions \( f : E_X \to E_Y \) (consider the entire functions \( x \mapsto f(x)_b \) for \( b \in |Y| \)).

**Proposition 29** If \( f : E_X \to E_Y \) is an entire function, there is exactly one linear continuous function \( \bar{f} : E_X \to E_Y \) such that \( f(x) = \bar{f}(x^!) \) for each \( x \in E_X \).

In particular, there is a canonical linear isomorphism between \( E^{\perp \perp}_{\|X\|} \) and \( \mathcal{A}(X) \), the linear space of all entire mappings from \( E_X \) to \( K \). We denote by \( \Theta_X : \mathcal{A}(X) \to E^{\perp \perp}_{\|X\|} \) this isomorphism (which maps an entire scalar-valued function to its defining power series).
3.1 Functorial action of the exponentials

Given a continuous linear map \( f : E_X \to E_Y \) of matrix \( A = M(f) \in E_X \otimes_Y \), we want to define a continuous linear map \( !f : E_{1X} \to E_{1Y} \) of matrix \( !A \in E_{1X} \otimes Y \). Rather than directly defining \( !f \), we define its transpose \( g = (!f)^\top : E_{(1X)^\bot} \to E_{(1Y)^\bot} \). For this, we identify \( E_{(1X)^\bot} \) with the space \( \mathcal{A}(X) \) of all entire mappings from \( E_X \) to \( K \). Given \( h \in \mathcal{A}(Y) \), we define \( g(h) : E_X \to K \) by

\[
\forall x \in E_X \quad g(h)(x) = h(f(x)) \in K \tag{2}
\]

and we shall prove that \( g(h) \) is entire. Let \( \nu \in \mathcal{M}(\{Y\}) \) and consider the element \( h \in \mathcal{A}(Y) \) which corresponds to the element \( e_\nu \in E_{\{Y\}^\bot} \), \( h \) is just the “monome” function \( h(y) = y^\nu \). By condition \( (2) \), the matrix \( !A \) must satisfy, for all \( x \in E_X \)

\[
\sum_{\mu \in \mathcal{M}(\{X\})} (!A)_{\mu,\nu} x^\mu = (f(x))^\nu = \prod_{b \in \{Y\}} \left( \sum_{a \in \{X\}} A_{a,b} x_a \right)^{\nu(b)} . \tag{3}
\]

We shall need now the following lemma.

**Lemma 30** Let \((u_i)_{i \in I}\) be a family of elements of \( K \) such that the series \( \sum_{i \in I} u_i \) converges absolutely and let \( k \in \mathbb{N} \). Then the series

\[
\sum_{\gamma \in \mathcal{M}(\{I\})} \left[ \frac{k}{\gamma} \right] u^\gamma
\]

converges absolutely and its sum is \( \left( \sum_{i \in I} u_i \right)^k \).

**Proof:** The equality holds when \( u \) vanishes almost everywhere; this is standard. One concludes by continuity of the map \( t \mapsto t^k \). \( \blacksquare \)

So we can write

\[
\left( \sum_{a \in \{X\}} A_{a,b} x_a \right)^{\nu(b)} = \sum_{\rho \in \mathcal{M}(\{X\}) \atop \# \rho = \# \nu(b)} \left[ \frac{\nu(b)}{\rho} \right] \prod_{a \in \{X\}} (A_{a,b} x_a)^{\rho(a)}
\]

\[
= \sum_{\rho \in \mathcal{M}(\{X\}) \atop \# \rho = \# \nu(b)} \left[ \frac{\nu(b)}{\rho} \right] \left( \prod_{a \in \{X\}} (A_{a,b})^{\rho(a)} \right) x^\rho.
\]

Let \( L(\nu) \) be the set of all multisets \( \sigma \) over the set \( \{X\} \times \{Y\} \) such that \( \sum_{a \in \{X\}} \sigma(a,b) = \nu(b) \) for each \( b \in \{Y\} \) (these multisets \( \sigma \) are necessarily finite and by \( \sigma_b \) we denote the element of \( \mathcal{M}(\{X\}) \) given by \( \sigma_b(a) = \sigma(a,b) \)).

The product \( \prod_{b \in \{Y\}} \left( \sum_{a \in \{X\}} A_{a,b} x_a \right)^{\nu(b)} \) (which is actually a finite product over \( \nu \)) can now be written as the following absolutely converging sum:

\[
\sum_{\sigma \in L(\nu)} \left( \prod_{b \in \{Y\}} \left[ \frac{\nu(b)}{\sigma_b} \right] x^{\sigma_b} \right) \left( \prod_{(a,b) \in \{X\} \times \{Y\}} A_{a,b}^{\sigma(a)} \right) = \sum_{\sigma \in L(\nu)} \left[ \nu \right] A^{\sigma} x^{\sum_{b \in \{Y\}} \sigma_b}_b.
\]
So finally
\[
\prod_{b \in |Y|} \left( \sum_{a \in |X|} A_{a,b} x_a \right)^{\nu(b)} = \sum_{\mu \in \mathcal{M}(|X|)} \left( \sum_{\sigma \in L(\mu, \nu)} \left[ \begin{array}{c} \nu \\ \sigma \end{array} \right] A^\sigma \right) x^\mu
\]
where by \(L(\mu, \nu)\) we denote the set of all multisets \(\sigma\) over \(|X| \times |Y|\) such that \(\sum_{a \in |X|} \sigma(a, b) = \nu(b)\) for each \(b \in |Y|\) and \(\sum_{b \in |Y|} \sigma(a, b) = \mu(a)\) for each \(a \in |X|\); such multisets \(\sigma\) are “Latin squares” with sums of columns prescribed by \(\mu\) and sums of lines prescribed by \(\nu\). Observe that a necessary condition for \(L(\mu, \nu)\) to be non empty is that \(\#\mu = \#\nu\), and then \(\#\sigma = \#\mu = \#\nu\) for each \(\sigma \in L(\mu, \nu)\).

This gives us an explicit formula for the matrix \(!A:\)
\[
(!!A)_{\mu, \nu} = \sum_{\sigma \in L(\mu, \nu)} \left[ \begin{array}{c} \nu \\ \sigma \end{array} \right] A^\sigma
\] (4)

Moreover, the above calculation shows that the series \(\sum_{\mu \in \mathcal{M}(|X|)} (!!A)_{\mu, \nu} x^\mu\) converges absolutely, for each \(x \in E_X\) and each \(\nu \in \mathcal{M}(|Y|)\). Indeed, we can assume that \(A\) and \(x\) are positive, and we have obtained the series \(\sum_{\mu \in \mathcal{M}(|X|)} (!!A)_{\mu, \nu} x^\mu\) as the finite product of converging series of positive terms \(\prod_{b \in |Y|} \left( \sum_{a \in |X|} A_{a,b} x_a \right)^{\nu(b)}\). Assume still \(A\) and \(x\) to be positive, and let \(R \in E_{(|Y|)^\perp}\) be also positive, we show that the series
\[
\sum_{(\mu, \nu) \in \mathcal{M}(|X|) \times \mathcal{M}(|Y|)} (!!A)_{\mu, \nu} x^\mu R_\nu
\]
converges. Let \(h \in \mathcal{A}(Y)\) be the entire function defined by \(R\) and let \(f : E_X \to E_Y\) be the linear function defined by \(A\). Then
\[
\infty > h(f(x)) = \sum_{\nu \in \mathcal{M}(|Y|)} R_\nu (f(x))^\nu = \sum_{\nu \in \mathcal{M}(|Y|)} R_\nu \prod_{b \in |Y|} \left( \sum_{a \in |X|} A_{a,b} x_a \right)^{\nu(b)} = \sum_{\nu \in \mathcal{M}(|Y|)} R_\nu \left( \sum_{\mu \in \mathcal{M}(|X|)} (!!A)_{\mu, \nu} x^\mu \right) \text{ by the calculation above} = \sum_{(\mu, \nu) \in \mathcal{M}(|X|) \times \mathcal{M}(|Y|)} (!!A)_{\mu, \nu} x^\mu R_\nu
\]
since we are dealing with series of positive terms. Due to the following lemma, this implies that \(!A \in E_{!X \rightarrow !Y}\).

**Lemma 31** Let \(X\) and \(Z\) be Köthe spaces and let \(M \in \mathbf{K}^{(|X|, |Z|)}\). Then \(M \in E_{!X \rightarrow !Z}\) iff for all \(x \in E_X\) and all \(z' \in E_{Z'}\), the series \(\sum_{\mu \in \mathcal{M}(|X|), c \in |Z|} |M(x) z'| c\) converges.

This is proved exactly like lemma 19, using the fact that \(E_{(|X|)^\perp} = \{x' | x \in E_X\}^\perp\).

Therefore, if \(A \in E_{X \rightarrow Y}\), \(!!A = ||A| \in E_{!X \rightarrow !Y}\) and so \(!A \in E_{!X \rightarrow !Y}\).
Moreover, the calculation above shows that, for any \( h \in \mathcal{A}(Y) \) and any linear and continuous \( f : E_X \rightarrow E_Y \), one has \( h(f(x)) = (((A)^+R)x^+) \) (product of matrices) for all \( x \in E_X \), where \( A \in E_X \otimes Y \) is the matrix of \( f \) and \( R \in E_{\langle Y \rangle}^+ \) is the matrix (the power series) of \( h \).

To summarize, denoting by \( \Theta_X \) the linear isomorphism from \( \mathcal{A}(X) \) to \( E_{\langle X \rangle}^+ \) which results from proposition 29 (that is, \( \Theta_X(h) = M(h) \)):

**Proposition 32** Let \( A \in E_X \otimes Y \). Then the matrix \( \! A \) defined by equation (4) belongs to \( E_{\langle X \rangle} \otimes Y \). Moreover, if \( h \in \mathcal{A}(Y) \), then

\[
((\! A)^+)(\Theta_Y(h)) = \Theta_X(h \circ A).
\]  
(5)

If \( f : E_X \rightarrow E_Y \) is linear and continuous, then \( h \circ f : E_X \rightarrow \mathbf{K} \) is entire and

\[
\tilde{h} \circ f = \tilde{h} \circ \! f
\]

where \( \! f \) is the linear continuous map from \( E_{\langle X \rangle} \) to \( E_{\langle Y \rangle} \) whose matrix is \( \! M(f) \). Moreover, the operation \( f \mapsto \! f \) is functorial.

**Lemma 33** Let \( X \) and \( Y \) be Köthe spaces. One has

\[
E_{\langle X \otimes Y \rangle}^+ = \{ x^+ \otimes y^+ \mid x \in E_X \text{ and } y \in E_Y \}^+
\]

**Proof:** Since, for \( x \in E_X \) and \( y \in E_Y \), one has \( x^+ \otimes y^+ \in E_{\langle X \otimes Y \rangle}^+ \), the inclusion \( \subseteq \) holds. Let us prove the converse, so let \( R \in \{ x^+ \otimes y^+ \mid x \in E_X \text{ and } y \in E_Y \}^+ \) and let \( M \in E_{\langle X \rangle} \) and \( N \in E_{\langle Y \rangle} \), we must show that the series

\[
T = \sum_{\mu \in \langle X \rangle, \nu \in \langle Y \rangle} |M_\mu N_\nu R_{\mu, \nu}|
\]

converges. First, for any fixed \( \nu \in \langle Y \rangle \), the series \( \sum_{\mu \in \langle X \rangle} |M_\mu R_{\mu, \nu}| \) converges as the family \( (R_{\mu, \nu})_{\mu \in \langle X \rangle} \) belongs to \( E_{\langle X \rangle}^+ \). Indeed, let \( x \in E_X \), we have to show that \( S = \sum_{\mu \in \langle X \rangle} |x^+ R_{\mu, \nu}| \) converges. But let \( y : \langle Y \rangle \rightarrow \mathbf{K} \) be the characteristic function of \( \nu \), which belongs to \( E_Y \) since it vanishes almost everywhere. Then \( y^+ = 1 \) and so the sum \( S \) is majorized by the sum \( \sum_{\mu \in \langle X \rangle} \sum_{\nu' \in \langle Y \rangle} |x^+ y^{\nu'} R_{\mu, \nu'}| \) which converges by our hypothesis about \( R \). So proving that \( T \) converges amounts to showing that the series (which is now well defined)

\[
\sum_{\nu \in \langle Y \rangle} |N_\nu| \sum_{\mu \in \langle X \rangle} |M_\mu R_{\mu, \nu}|
\]

converges. For this, it is sufficient to show that the family \( (\sum_{\mu \in \langle X \rangle} |M_\mu R_{\mu, \nu}|)_{\nu \in \langle Y \rangle} \) belongs to \( E_{\langle Y \rangle}^+ \). So let \( y \in E_Y \), we must show that

\[
T' = \sum_{\mu \in \langle X \rangle, \nu \in \langle Y \rangle} |M_\mu y^{\nu} R_{\mu, \nu}|
\]

converges. The same considerations, on the other side, reduce our problem to showing that, for each \( x \in E_X \) and each \( y \in E_Y \), the series \( \sum_{\mu \in \langle X \rangle, \nu \in \langle Y \rangle} |x^+ y^{\nu} R_{\mu, \nu}| \) converges, and this in turn results from our hypothesis about \( R \).

This exponential satisfies the following fundamental isomorphism.
Proposition 34 Let $X$ and $Y$ be Köthe spaces. There is a canonical linear homeomorphism between $!(X \otimes Y)$ and $!X \otimes !Y$.

Proof: We assume $|X| \cap |Y| = \emptyset$, just for simplifying the notation. The function $\varphi : !\!(X \otimes Y) \to !\!(X \otimes Y)$ which maps $(\lambda, \rho)$ to $\lambda + \rho$ is a bijection, with inverse $\psi$. Then $\psi^*$ is a linear homeomorphism $E_{!(X \otimes Y)^+} \to E_{((X \otimes Y)^+}$ using lemma 14. This results from lemma 33 and from the observation that, for $x \in E_X$ and $y \in E_Y$, denoting by $(x, y)$ the corresponding pair in $E_X \otimes E_Y \simeq E_X \times E_Y$, one has $(x, y)^\mu = x^\lambda y^\rho$, where $(\lambda, \rho) = \varphi(\mu)$, that is

$$(x, y)^1 = \psi^* (x^1 \otimes y^1).$$

3.2 The co-monadic structure of the exponential

For each space $X$, we consider the two particular matrices (called respectively dereliction and digging): $d^X \in K^{[X - X]}$ and $p^X \in K^{[X - ![X]}$ given by

$$(d^X)_{\mu, a} = \delta_{\mu, [a]} \quad \text{and} \quad (p^X)_{\mu, M} = \delta_{\mu, \Sigma(M)}.$$

We recall that, if $M \in ![X]$, then $\Sigma(M) \in ![X]$ is given by $\Sigma(M)(a) = \sum_{\nu \in ![X]} M(\nu) \nu(a)$.

Let $x \in E_X$ and let $x' \in E_{X -}$, we have

$$\sum_{\mu \in ![X], a \in [X]} [x^\mu x'_a (d^X)_{\mu, a}] = \sum_{a \in [X]} |x_a x'_a|$$

and hence $d^X \in E_{X - X}$.

Remark: This matrix $d^X$ allows to see any linear continuous map as an entire map. Indeed, let $f : E_X \to E_Y$ be linear and continuous. Then $f \circ \hat{d^X} : E_{X - X} \to E_Y$ is linear and continuous, and so defines an entire map $h : E_X \to E_Y$ by $h(x) = f(d^X(x'))$, that is $h(x) = f(x)$. We shall say that an entire map $h : E_X \to E_Y$ is linear when $h = f \circ \hat{d^X}$ for some linear and continuous $f$, and then one easily sees that $f = \hat{h}$. One checks also easily that an entire map $h$ is linear iff it is linear in the usual sense, that is $h(\lambda x) = \lambda h(x)$ and $h(x + y) = h(x) + h(y)$ for $\lambda \in K$ and $x, y \in E_X$.

Now, let $x \in E_X$ and let $R \in E_{([[X]]^+}$, we have

$$\sum_{\mu \in ![X], M \in [[X]]} |x^\mu R_M (p^X)_{\mu, M}| = \sum_{M \in [[X]]} \left| x^{\Sigma(M)} R_M \right| .$$

But $(x^1)^1 \in E_{[X]}$ and one has $((x^1)^1)_M = x^{\Sigma(M)}$ for each $M \in ![X]$ and so the sum above converges, by our hypothesis about $R$. Therefore, $p^X \in E_{X - ![X]}$.

Checking that $d^X$ is natural in $X$ is a straightforward verification. We check that $p^X$ is natural in $X$. Let $A \in E_{X - Y}$, we must show that the following equation of matrices holds:

$$p^Y (\!A) = (\![A]) p^X .$$

This amounts to showing that, for each $\mu \in ![X]$ and each $N \in [[X]]$, one has

$$(\!A)_{\mu, \Sigma(N)} = \sum_{M \in [[X]]} (\![A])_{M, N} .$$
This latter sum converges absolutely\(^7\) because it is the a coefficient of a matrix obtained as the product of two matrices of linear continuous maps. Let \( t \in E_X \). Equation (3) gives in that case

\[
\sum_{M \in [!X]} (|A|)_M \cdot t^M = \prod_{\nu \in [!Y]} \left( \sum_{\mu \in [!X]} (|A|)_{\mu, \nu} t^\mu \right)^{N(\nu)}
\]

Let \( x \in E_X \) and replace in this equation \( t \) by \( x^\rho \), we obtain

\[
\sum_{\rho \in [!X]} \left( \sum_{M \in [!X]} (|A|)_{M, N} \right) x^\rho = \prod_{\nu \in [!Y]} \left( \sum_{\mu \in [!X]} (|A|)_{\mu, \nu} x^\mu \right)^{N(\nu)}
\]

For each \( \nu \in [!Y] \), we have \( \sum_{\mu \in [!X]} (|A|)_{\mu, \nu} x^\mu = (Ax)^\nu \) by equation (3), and hence the right-hand side of equation (6) is equal to \( (Ax)^\Sigma(\nu) \), and so, applying once more equation (3), we arrive to

\[
\sum_{\rho \in [!X]} \left( \sum_{M \in [!X]} (|A|)_{M, N} \right) x^\rho = \sum_{\rho \in [!X]} (|A|)_{\rho, \Sigma(\nu)} x^\rho
\]

and we conclude by proposition 29.

To prove that, with the natural transformations \( d^X \) and \( p^X \), the functor \( X \mapsto ![X] \) is a comonad, one has to check the commutations of three diagrams corresponding to the following equations

\[
(Id^X)p^X = I_X \quad \text{and} \quad d^Xp^X = I_X
\]

and

\[
(Id^X)p^X = p^Xp^X
\]

We check the third equation; the first one is similar (but simpler) and the second one is trivial.

Let \( M \in ![X] \) and \( S \in ![[X]]. \) For \( P \in L(M, S) \) (with the notations introduced for equation (4)), one has \( (p^X)^P \neq 0 \) iff \( |P| \subseteq \{ (\rho, R) \in ![X] \times ![X] \mid \Sigma(R) = \rho \} = \mathcal{H} \), by definition of \( p^X \). Now let \( P \in M([!X] \times ![X]) \) be such that \( |P| \subseteq \mathcal{H} \). Then \( P \in L(M, S) \) holds iff, for all \( R \in ![X], \)

\[
S(R) = P(\Sigma(R), R)
\]

and, for all \( \rho \in ![X], \)

\[
M(\rho) = \sum_{R \in ![X]} P(\rho, R) = \sum_{\Sigma(R) = \rho} S(R)
\]

If \( M \) and \( S \) satisfy this condition, then \( L(M, S) \) has exactly one element \( P \) such that \( |P| \subseteq \mathcal{H}, \) and \( P \) is given by \( P(\rho, R) = S(R)\delta_{\rho, \Sigma(R)}. \) Moreover, if \( P \) and \( S \) satisfy \( S(R) = P(\Sigma(R), R) \) for all \( R \in ![X] \) and if \( |P| \subseteq \mathcal{H}, \) then \( \left[ \frac{\Sigma}{P} \right] = 1 \). Therefore, formula (4) gives in this case

\[
(Id^X)_M^S = \begin{cases} 1 & \text{if } \forall \rho \in ![X], M(\rho) = \sum_{\Sigma(R) = \rho} S(R) \\ 0 & \text{otherwise} \end{cases}
\]

\( ^7 \)It is not a finite sum in general, because for a given \( M \in ![X] \), one has \( \Sigma(M + k[0]) = \Sigma(M) \) for all \( k \in \mathbb{N}. \)
Given $\mu \in \|X\|$ and $S \in \|\|X\||$, we have therefore $((\mathbf{1} p^X) p^X)_{\mu,S} = \sum_{\Sigma(M) = \mu} (\mathbf{1} p^X)_{M,S} = \delta_{\mu,T(S)}$ where $T(S) \in \|X\|$ is given by

$$T(S)(a) = \sum_{\rho \in \|X\|} \rho(a) \sum_{\Sigma(R) = \rho} S(R) = \sum_{R \in \|\|X\||} (\Sigma(R))(a) S(R) = \sum_{R \in \|\|X\||} \left( \sum_{\rho \in \|X\|} \rho(a) R(\rho) \right) S(R) = \sum_{\rho \in \|X\|, R \in \|\|X\||} \rho(a) R(\rho) S(R) = \Sigma(\Sigma(S))(a),$$

but it is clear on the other hand that $((\mathbf{1} p^X) p^X)_{\mu,S} = \delta_{\mu,\Sigma(\Sigma(S))}$ and this concludes the proof.

### 3.3 The co-algebraic structure of the exponential

The diagonal continuous linear map $D_X : X \to X \otimes X$ maps $x \in E_X$ to $(x,x) \in E_X \times E_X \simeq E_{X \otimes X}$, and its matrix is given by $M(D_X)_{a,(i,b)} = \delta_{a,b}$. Then $\mathbf{1}(D_X)$ induces, through the isomorphism of proposition 3.4, a linear continuous map $\Delta_X : \|X\| \to \|X\| \otimes \|X\|$ (corresponding to contraction in logic). The matrix of this operator is easily seen to be given by

$$M(\Delta_X)_{\mu,(\lambda,\rho)} = \delta_{\mu,\lambda+\rho}$$

for $\mu, \lambda, \rho \in \|X\|$. This turns $\|X\|$ into a co-algebra, with neutral element $e_0$ (where $0$ denotes the empty multi-set), and $\Delta_X$ is associative and commutative, because $D_X$ is, trivially. Categorically, the neutral element must be considered as the linear continuous map $w_X : E_X \to E_1 = K$ given by $w_X(x) = x_0$ (corresponding to weakening in logic). In more algebraic notations, the action of $\Delta_X$ on an element $t$ of $E_{\|X\|}$ can be written

$$\Delta_X(t) = \sum_{\lambda, \rho \in \|X\|} t_{\lambda+\rho}(e_\lambda \otimes e_\rho).$$

### 3.4 The Hopf algebra structure of the exponential

We apply proposition 3.4, and the fact that finite sums and cartesian products of Köthe spaces coincide for endowing the space $\|X\|$ with an algebraic structure. Indeed, the co-diagonal morphism $X \oplus X \to X$ induces a linear continuous map $\alpha_X : X \otimes X \to X$ given by $\alpha_X(x,y) = x+y$. Its matrix is given by $M(\alpha_X)_{(i,a),b} = \delta_{a,b}$. Then $\mathbf{1}(\alpha_X)$ induces, through the isomorphism of proposition 3.4, a linear continuous map $m_X : \|X\| \otimes \|X\| \to \|X\|$. An easy computation, using formula (4), shows that, for $\lambda, \rho, \mu \in \|\|X\||$,

$$M(m_X)_{(\lambda,\rho),\mu} = \left( \begin{array}{c} \mu \\ \lambda \end{array} \right) \delta_{\lambda+\rho,\mu}.$$ 

The neutral element of this operation is $e_0 \in E_{\|X\|}$. The associativity of this operation immediately results from its definition, and from the associativity of addition. Commutativity of $m_X$ is
established in the same way. If we consider \( m_X \) as a bilinear map \( E_{!X} \times E_{!X} \to E_{!X} \), its action on 
\((t,s) \in E_{!X} \times E_{!X}\) can be written

\[
m_X(s,t) = \sum_{\lambda, \rho \in [X]} \left(\begin{array}{c} \lambda + \rho \\ \lambda \end{array}\right) s_\lambda t_\rho e_{\lambda + \rho} = \sum_{\mu \in [X]} \left(\sum_{\lambda \leq \mu} \left(\begin{array}{c} \mu \\ \lambda \end{array}\right) s_\lambda t_{\mu - \lambda}\right) e_\mu.
\]

Keeping implicit the linear homeomorphism between \(!X \otimes !X\) and \(!\langle X \& X \rangle\) and the linear isomorphism between \( E_{\langle X \rangle^+} \) and \( \mathcal{A}(X) \), the dual of \( m_X \) maps an element \( h \) of \( \mathcal{A}(X) \) to the element \( k \) of 
\( \mathcal{A}(X \& X) \) defined by \( k(x,y) = h(x + y) \).

The last structure map of a Hopf algebra is the antipode \( S_X \). Let \( n_X : E_X \to E_X \) be the continuous linear map defined by \( n_X(x) = -x \), its matrix is given by \( M(n_X)_{a,b} = -\delta_{a,b} \). The antipode is defined as \( S_X = !n_X \), and it is a linear continuous map from \(!X\) to itself. Its matrix is easily seen (again, using formula (4)) to be given by

\[
M(S_X)_{\mu, \nu} = (-1)^{\|\mu\|} \delta_{\mu, \nu}.
\]

In other words, \( S_X(t) = \sum_{\mu \in [X]} (-1)^{\|\mu\|} e_\mu \) for all \( t \in E_{!X} \).

Checking that \( (!X, m_X, \Delta_X, S_X) \) is a Hopf algebra consists in checking that \( (!X, m_X, \Delta_X) \) is a bi-algebra. This is done using straightforwardly the definitions of \( m_X, \Delta_X \) and the functoriality of \(!\). The last diagram to check expresses that \( S_X \) is an antipode, and this is also trivial.

4 A cartesian closed category of entire mappings

The standard co-Kleisli construction applied to the co-monad \( X \mapsto !X \) gives rise to a cartesian closed category \( \mathcal{K}_K^! \) which is defined as follows:

- an object in this category is a Köthe space;
- a morphism from \( X \) to \( Y \) in this category is a continuous linear map from \( E_{!X} \) to \( E_Y \);
- the identity from \( X \) to \( X \) in \( \mathcal{K}_K^! \) is \( \tilde{d}^X \in \mathcal{L}(E_{!X}, E_X) \);
- if \( f : X \to Y \) and \( g : Y \to Z \) are morphisms in \( \mathcal{K}_K^! \), their composite \( g \circ_A f \) in \( \mathcal{K}_K^! \) is given by 
  \[
g \circ_A f = g \circ_A !f \circ \tilde{p}^X \text{ where composition is taken in } \mathcal{K}_K:
\]

\[
E_{!X} \xrightarrow{\tilde{p}^X} E_{!X} \xrightarrow{!f} E_{!Y} \xrightarrow{g} E_Z
\]

So there is a bijective correspondence between the morphisms \( f : X \to Y \) in \( \mathcal{K}_K^! \) and the entire functions from \( E_X \) to \( E_Y \). We show first that the definition above of identity and composition is compatible with this correspondence. Remember that, if \( f : E_{!X} \to E_Y \) is linear and continuous, we denote by \( \mathcal{F}(f) : E_X \to E_Y \) the corresponding entire function (see definition 28).

As to the identity, given \( x \in E_X \), we have

\[
\mathcal{F}(\tilde{d}^X)(x) = d^X(x) = \sum_{a \in [X]} \left(\sum_{\mu \in [X]} (d^X)_\mu(a(x')) e_\mu\right) e_a = \sum_{a \in [X]} (x')_a e_a = x
\]
since $(x')_a = x_a$.

As to composition, we must show that (with the notations above), $\mathcal{F}(g \circ A f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. This results immediately from the following

Lemma 35 If $x \in E_X$, then $(!f)(\widehat{p^X}(x')) = f(x')!$. 

Proof: Denoting by $A$ the matrix of $f$ (so that $A \in E_{!X \times Y}$), we have

$$
(!f)(\widehat{p^X}(x')) = \sum_{M \in !|X|} x^\Sigma([M]) (!f)(e_M)
= \sum_{M \in !|X|} x^\Sigma([M]) \left( \sum_{\nu \in !|Y|} (!A)_{M, \nu} e_\nu \right)
= \sum_{\nu \in !|Y|} \left( \sum_{M \in !|X|} (!A)_{M, \nu} x^\Sigma([M]) \right) e_\nu
= \sum_{\nu \in !|Y|} \left( \sum_{M \in !|X|} (!A)_{M, \nu} (x')^M \right) e_\nu
= \sum_{\nu \in !|Y|} f(x')^\nu e_\nu \text{ by equation (3)}
= f(x')!
$$

and we are done. 

The cartesian product of $X$ and $Y$ in $K^1_X$ is $X \times Y$. If $f : E_{X \times Y} \rightarrow E_Z$ is entire, its “transpose” (curryification) is $f^c : E_X \rightarrow E_{Y \times Z}$ where $f^c(x) = M(\tilde{g})$ where $g : E_Y \rightarrow E_Z$ is the entire function defined by $g(y) = f(x, y)$. The function $f^c$ itself is entire, and we have $M(f^c)_{\mu + \nu, 0} = M(\tilde{f})_{\mu, \nu} c$ where $\mu + \nu$ is the multiset on $|X| + |Y|$ obtained by juxtaposing $\mu \in !|X|$ and $\nu \in !|Y|$. There is also an entire evaluation map $ev : E_{!(X \times Y) \otimes X} \rightarrow E_Y$ given by $ev(f, x) = f(x)$; its power series (identifying the canonically isomorphic spaces $!(X \otimes Y) \otimes !X$) and $!(X \otimes Y) \otimes !X$ is given by the matrix

$$
M(ev)_{\nu, \mu, b} = \delta_{\nu, (|X| + 0)}.
$$

4.1 Computing derivatives

The space $!X$ possesses another important structure which is a map $\partial_X : E_X \rightarrow E_{!X}$ given by

$$
\partial_X(x)_\mu = \begin{cases} x_a & \text{if } \mu = [a] \text{ for some } a \in |X| \\ 0 & \text{otherwise} \end{cases}
$$

It is clear that $\partial_X \in \mathcal{L}(E_X, E_Y)$, with matrix $M(\partial_X)_{\alpha, \mu} = \delta_{[\alpha], [\mu]}$. This map is natural in $X$ and one has

$$
\widehat{d^X} \circ \partial_X = \text{Id}_X.
$$

Given an entire function $f : E_X \rightarrow E_{Y^2}$, $\tilde{f} \circ \partial_X$ is the linear continuous map from $E_X$ to $E_Y$ whose matrix is given by $M(f \circ \partial_X)_{a, b} = f_{[a], [b]}$; that is, $\tilde{f} \circ \partial_X$ is the derivative of $f$ at 0. More
generally, one can define a map $\partial^1_X$ as the composite of the two maps

$$!X \otimes X \xrightarrow{!X \otimes \partial^1_X} !X \otimes !X \xrightarrow{m_X} !X$$

and an easy computation using equation (4) shows that, for $\mu, \nu \in |!X|$ and $a \in |X|$, one has $(\partial^1_X)_{\mu, \nu} = \nu(a)\delta_{\mu + |a|, \nu}$. Therefore, if $f : E_X \rightarrow E_Y$ is entire, the composite $\tilde{f} \circ \partial^1_X$ is a continuous linear map $!X \otimes X \rightarrow Y$ whose transpose $Df : !X \rightarrow (X \otimes Y)$ (using the monoidal closeness of $\mathcal{K}_K$) is the derivative of $f$ (it is an entire map from $E_X$ to $E_X \otimes Y$), in the sense of the derivatives of power series. In particular, for all $y' \in E_{Y^2}$ and $u \in E_X$ the map $t \mapsto \langle f(x + tu), y' \rangle$ from $K$ to $K$ is entire (as the composite of three entire maps) and one checks easily that

$$\langle Df(x) \cdot u, y' \rangle = \frac{d\langle f(x + tu), y' \rangle}{dt}(0)$$

where, if $A \in E_X \otimes Y$ and $u \in E_X$, we recall that we denote by $A \cdot u$ the application of the matrix $A$ to the vector $u$.

Observe also that when $f : E_X \rightarrow E_Y$ is linear, then we have

$$\forall x \in E_X \quad Df(x) = M(f)$$

(8)

due to the easily checked commutation of the following diagram

$$
\begin{array}{ccc}
!X \otimes X & \xrightarrow{\partial^1_X} & !X \\
\downarrow w_X \otimes X & \downarrow d^X & \downarrow 1 \otimes X \\
1 \otimes X & \longrightarrow & X
\end{array}
$$

where the map $1 \otimes X \rightarrow X$ is the isomorphism expressing the neutrality of 1 for the tensor product (in some sense, this diagram generalizes equation (7)).

Of course, we can compute $n$-th derivatives for all $n \in \mathbb{N}$, and this is done by precomposing $\tilde{f}$ with the linear continuous map $\partial^n_X : !X \otimes X \otimes^n \rightarrow !X$ obtained by iterating $\partial^1_X$ in the obvious way: $\partial^{n+1}_X = \partial_X \circ (\partial^n_X \otimes X)$. Again, an easy induction shows that

$$\langle \partial^n_X \rangle_{\mu, a_1, \ldots, a_n, \nu} = \nu(a_1 \ldots a_n) \delta_{\mu + |a_1 \ldots a_n|, \nu}.$$ 

(9)

If $f : E_X \rightarrow E_Y$ is entire, its $n$-th derivative $D^n f$ is the transpose of the linear continuous map $\tilde{f} \circ \partial^n_X : E_X \otimes X \otimes^n \rightarrow E_Y$, considered as an entire map from $E_X$ to $E_X \otimes^n \otimes Y$.

**Taylor formula.** One proves then the Taylor formula: for each $x \in E_X$,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0)(x \otimes^n).$$

(10)

Let $A \in E_X \otimes Y$ be the power series defining $f$, we have

$$f(x) = \sum_{\mu \in |X|, b \in |Y|} A_{\mu, b} x^\mu e_b$$

$$= \sum_{n=0}^{\infty} \left( \sum_{\mu = n, b} A_{\mu, b} x^\mu e_b \right)$$
but for an element $\mu$ of $|X|$ such that $\#\mu = n$, there are exactly $n!/\mu!$ tuples $(a_1, \ldots, a_n)$ such that $[a_1, \ldots, a_n] = \mu$. Moreover, for such a tuple, we have $x^\mu = (x^{\otimes n})_{(a_1, \ldots, a_n)}$. Applying equation (9) and the definition above of $D^n f$, we see that (with $\mu$ and $(a_1, \ldots, a_n)$ as before, and with $b \in |Y|$):

$$(D^n f(0))_{(a_1, \ldots, a_n), b} = \mu! A_{\mu, b},$$

so

$$f(x) = \sum_{n=0}^{\infty} \sum_{\substack{\{a_1, \ldots, a_n\} \subseteq |X^{\otimes n}| \atop b \in |Y|}} \frac{[a_1, \ldots, a_n]!}{n! \times [a_1, \ldots, a_n]!} \frac{1}{(D^n f(0))_{(a_1, \ldots, a_n), b}(x^{\otimes n})_{(a_1, \ldots, a_n)}}$$

and we conclude.

The Taylor sum (10) converges absolutely for each $x$ (this means that for each $y' \in E_{Y^\perp}$, the series of positive terms $\sum_{n=0}^{\infty} N_y(\frac{1}{n!} D^n f(0)(x^{\otimes n}))$ converges). Indeed,

$$\frac{1}{n!} D^n f(0)(x^{\otimes n}) = \sum_{\#\mu = n, \ b} A_{\mu, b} x^\mu e_b$$

so that $N_y\left(\frac{1}{n!} D^n f(0)(x^{\otimes n})\right) \leq \sum_{\#\mu = n, \ b} |y'(A_{\mu, b} x^\mu)|$ and we conclude since we know that $\sum_{\mu, b} |y'(A_{\mu, b} x^\mu)| < \infty$ because $f$ is entire.

One can say better, namely that the the series $\sum_{n} f_n$ converges absolutely in $E_{|X^\alpha Y}$, where $f_n$ is the entire (indeed “polynomial”) map defined by $f_n(x) = \frac{1}{n!} D^n f(0)(x^{\otimes n})$.

**Partial derivatives.** When $f : E_X \to E_Z$ is entire, we also denote by

$$\frac{d f}{d x} : E_X \to E_X \otimes z \quad \text{and} \quad \frac{d^n f}{d x^n} : E_X \to E_X^{\otimes n} \otimes z$$

its derivatives.

Consider now an entire map of two variables $f : E_{X \otimes Y} \to E_Z$. Using the cartesian closseness of the category of Köthe spaces and entire mappings, we can transpose this map into an entire map $f_1 : E_X \to E_{Y \otimes z}$ which in turn can be derived, giving rise to the entire map

$$\frac{d f_1}{d x} : E_X \to E_X \otimes !Y \otimes z$$

(we have used the monoidal closeness of $K$, which, when considered as a linear continuous map $\frac{d f}{d x} : E_X \to E_{X \otimes !Y} \otimes z$, can be transposed into a linear continuous map (up to the symmetry of $\otimes$) $E_{X^{\otimes Y}} \to E_X \otimes z$ giving rise to an entire mapping

$$\frac{\partial f}{\partial x} : E_{X^{\otimes Y}} \to E_X \otimes z,$$

the partial derivative of $f$ with respect to the first parameter. One defines of course similarly

$$\frac{\partial f}{\partial y} : E_{X \otimes Y} \to E_Y \otimes z.$$

Then the derivative

$$\frac{d f}{d (x, y)} : E_{X \otimes Y} \to E_{X \otimes Y} \otimes z.$$
can be expressed as follows, using these partial derivatives:

\[
\frac{df}{d(x,y)}(x,y) \cdot (u,v) = \frac{\partial f}{\partial x}(x,y) \cdot u + \frac{\partial f}{\partial y}(x,y) \cdot v \in E_Z
\]  

(11)

for \( x,u \in E_X \) and \( y,v \in E_Y \). Of course, this can be generalized in the obvious way to entire functions depending on a finite number of parameters. 

**Chain rule.** Consider now two entire mappings \( f : E_X \to E_Y \) and \( g : E_Y \to E_Z \

Then the usual chain rule holds, that is, the derivative of \( g \circ f \) satisfies

\[
\frac{d(g \circ f)}{dx}(x) \cdot u = \frac{dg}{dy}(f(x)) \cdot \left( \frac{df}{dx}(x) \cdot u \right)
\]  

(12)

**Contraction and evaluation.** Let \( f : E_X \times X \to E_Z \) be an entire map of two parameters in the same space \( E_X \) (the first generic parameter will be denoted by “\( x \)”, the second by “\( y \)”). The map \( g : E_X \to E_Z \) defined by \( g(x) = f(x,x) \) is entire: \( \tilde{g} \) is obtained by composing \( \tilde{f} \) with \( !D_X : E_X \to E_I[X \times X] \), a linear continuous map, i.e. by composing \( f \) with the diagonal \( X \to (X \times X) \) in \( K \). Using formulae (11) and (12), we obtain easily

\[
\frac{dg}{dx}(x) \cdot u = \frac{\partial f}{\partial x}(x,x) \cdot u + \frac{\partial f}{\partial y}(x,x) \cdot u
\]  

(13)

which will be useful for deriving proofs containing contractions.

Using the matricial characterization above of the evaluation map \( ev : E_I[X \hookrightarrow Y \times X] \to E_Y \), one shows easily that the two partial derivatives of this entire map are given by

\[
\frac{\partial ev}{\partial f}(f,x) \cdot h = h(x)
\]  

(14)

and

\[
\frac{\partial ev}{\partial x}(f,x) \cdot u = \frac{df}{dx}(x) \cdot u
\]  

(15)

for \( f,h \in E_X \hookrightarrow Y \) and \( x,u \in E_X \) (in the latter formula, we identify an element of \( E_X \hookrightarrow Y \) with an entire map from \( E_X \) to \( E_Y \)). The first partial derivative does not depend on \( f \); this corresponds to the fact that evaluation is linear in its first argument. Observe also that the second derivative is linear in its first argument as well.

5 Towards an intrinsic theory

All the spaces introduced so far were equipped with **explicit bases** which, from the viewpoint of classical functional analysis, seems to be a major drawback of our approach. The goal of this section is to show that, at least in principle, these bases are not essential.

We shall say that a topological vector space \( E \) is an intrinsic Köthe space if there is a Köthe space \( X \) and a linear homeomorphism \( \varphi : E \to E_X \). Such a linear homeomorphism \( \varphi \) will be called a **chart** of \( E \). The goal of this section is to show that the constructions defined so far can be performed on intrinsic Köthe spaces, and not only on Köthe spaces. This means in some sense that they are “base independent”.
Let $E$ be a topological linear space and let $X$ be a Köthe space. Let $\varphi : E \to E_X$ be a linear homeomorphism. Through this isomorphism, $E$ inherits a bornology $\mathcal{E}_\varphi$ from the equicontinuous bornology $\mathcal{E}_X$ of $E_X$:

$$\mathcal{E}_\varphi = \{ \varphi^{-1}(G) \mid G \in \mathcal{E}_X \}.$$ 

Consider now some other linear homeomorphism $\psi : E \to E_Y$ to a Köthe space. We claim that $\mathcal{E}_\psi \subseteq \mathcal{E}_\varphi$, and so, by symmetry, $\mathcal{E}_\psi = \mathcal{E}_\varphi$. Let $H \subseteq E$ be such that $H \in \mathcal{E}_\psi$, that is, $\psi(H) \in \mathcal{E}_Y$. Let $\theta$ be the linear homeomorphism $\varphi \circ \psi^{-1} : E_Y \to E_X$. Then $\varphi(H) = \theta(\psi(H))$ and we conclude by proposition 17.

So an intrinsic Köthe space $E$ possesses a canonical equicontinuous bornology, that we can denote by $\mathcal{E}_E$, and which is equal to $\mathcal{E}_\varphi$ where $\varphi$ is any chart of $E$. Moreover, $E$ is clearly locally convex, Hausdorff, separable and complete and also, $E$ possesses an absolute basis (it is shown in [Jar81] that these properties are equivalent to saying that $E$ is linearly homeomorphic to a Köthe sequence space $\Lambda(P)$, but a priori, they are not sufficient for implying that $E$ is an intrinsic Köthe space in our sense).

### 5.1 Topological dual

Using this bornology, we can define the topological dual $E'$ of $E$ as a topological vector space: $E'$ is the linear space of all continuous linear forms on $E$, and a subset $U'$ of $E'$ is a neighborhood of 0 iff there exists $G \in \mathcal{E}_E$ such that, for any $f \in E'$, if $|f(x)| < 1$ for each $x \in G$, then $f \in U'$ (that is, the topology of $E'$ is the topology of uniform convergence on the elements of $\mathcal{E}_E$).

**Lemma 36** The map $\varphi' : E' \to E_X'$ which maps each $u' \in E'$ to $M(u' \circ \varphi^{-1})$ is a linear homeomorphism from $E'$ to $E_X'$.

So $E'$ is an intrinsic Köthe space and possesses therefore a canonical bornology $\mathcal{E}_{E'}$, and the reader can check that a subset $G'$ of $E'$ belongs to this bornology iff it is an equicontinuous set of linear forms. Moreover, this operation $E \mapsto E'$ is functorial; let indeed $f : E \to F$ be linear and continuous. Its usual algebraic transpose $f' : F' \to E'$, which maps $v' \in F'$ to $v' \circ f \in E'$, is linear. Its continuity results (for instance) from the characterization above of the dual topology, and from the fact that continuous linear maps preserve equicontinuous sets (proposition 17).

Last, the canonical linear map $\eta : E \to E''$ which maps $u \in E$ to the linear form $\eta(u)$ on $E'$ given by $\eta(u)(u') = u'(u)$ is easily seen to make the following diagram commutative

\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & E_X = E_X' \\
\eta \downarrow & & \downarrow \\
E'' & \xrightarrow{\varphi''} & E_X''
\end{array}
\]

so that $\eta = \varphi''^{-1} \circ \varphi$ is a linear homeomorphism.

If $u \in E$ and $u' \in E'$, then we denote by $\langle u, u' \rangle$ the application of $u'$ to $u$, and one checks easily that if $\varphi : E \to E_X$ is a linear homeomorphism of $E$ to some Köthe space, then $\langle u, u' \rangle = \langle \varphi(u), \varphi'(u') \rangle$. 
The cartesian product of two intrinsic Köthe spaces is an intrinsic Köthe space, endowed with the product topology, which is also the direct sum of the two spaces.

5.2 Linear function space

Now we study the space $\mathcal{L}(E, F)$ of continuous linear maps from $E$ to $F$, two intrinsic Köthe spaces. This is clearly a linear space, and we must endow it with a topology. Let $\varphi : E \to E_X$ and $\psi : F \to E_Y$ be linear homeomorphisms to some Köthe spaces $X$ and $Y$. We shall say that a subset $\mathcal{U}$ of $\mathcal{L}(E, F)$ is $\varphi\psi$-open if the set $\mathcal{U}_{\varphi, \psi} = \{M(\psi \circ f \circ \varphi^{-1}) | f \in \mathcal{U}\}$ is open in the Köthe space $E_X \rightarrow Y$. Since the map $f \mapsto M(\psi \circ f \circ \varphi^{-1})$ is a linear isomorphism between $\mathcal{L}(E, F)$ and $E_X \rightarrow Y$, we turn in that way $\mathcal{L}(E, F)$ into a topological vector space, whose topology depends \textit{a priori} on $\varphi$ and $\psi$. So let $\sigma : E \to E_Z$ and $\tau : F \to E_T$ be some other linear homeomorphisms to Köthe spaces. Let $\theta = \varphi \circ \sigma^{-1}$, it is a linear homeomorphism from $E_Z$ to $E_X$. One also sets $\eta = \tau \circ \psi^{-1}$, it is a linear homeomorphism from $E_Y$ to $E_T$.

By lemma 23, the map $\theta \circ \eta$ is a linear homeomorphism from $E_X \rightarrow Y$ to $E_Z \rightarrow T$. For $f \in \mathcal{L}(E, F)$, we have, by lemma 23 again,

$$(\theta \circ \eta)(M(\psi \circ f \circ \varphi^{-1})) = M(\tau \circ f \circ \sigma^{-1})$$

and therefore $(\theta \circ \eta)(\mathcal{U}_{\varphi, \psi}) = \mathcal{U}_{\sigma, \tau}$, hence $\mathcal{U}_{\sigma, \tau}$ is open (as $\theta \circ \eta$ is open as an homeomorphism), so $\mathcal{U}$ is $\sigma\tau$-open. Symmetrically, if $\mathcal{U}$ is $\sigma\tau$-open then $\mathcal{U}$ is $\varphi\psi$-open. So we have defined a canonical topology on $\mathcal{L}(E, F)$, which depends only on the topologies of $E$ and $F$ (since these topologies determine the class of linear homeomorphisms $\varphi$ and $\psi$ to some Köthe spaces $E_X$ and $E_Y$ and since the topology we have defined on $\mathcal{L}(E, F)$ does not depend on a particular choice of $\varphi$ and $\psi$ as we have shown). Moreover, by construction, $\mathcal{L}(E, F)$ is an intrinsic Köthe space, since the function

$$\mathcal{L}(\varphi, \psi) : \mathcal{L}(E, F) \rightarrow E_X \rightarrow Y$$

$$f \mapsto M(\psi \circ f \circ \varphi^{-1})$$

is a linear homeomorphism (for the canonical topology of $\mathcal{L}(E, F)$) as soon as $\varphi : E \rightarrow E_X$ and $\psi : F \rightarrow E_Y$ are linear homeomorphisms: this holds by definition of the topology of $\mathcal{L}(E, F)$. Of course, $K$ is an intrinsic Köthe space, and the topological vector spaces $E'$ and $\mathcal{L}(E, K)$ are equal.

We turn now the operation $(E, F) \mapsto \mathcal{L}(E, F)$ into a functor. Let $E_1$ and $F_1$ be intrinsic Köthe spaces, and let $f : E_1 \rightarrow E$ and $g : F \rightarrow F_1$ be linear continuous maps. If $h \in \mathcal{L}(E, F)$, then $g \circ h \circ f \in \mathcal{L}(E_1, F_1)$, and we set

$$\mathcal{L}(f, g)(h) = g \circ h \circ f$$

Then the diagram

$$\begin{array}{ccc}
\mathcal{L}(E, F) & \xrightarrow{\mathcal{L}(f, g)} & \mathcal{L}(E_1, F_1) \\
\mathcal{L}(\varphi, \psi) \downarrow & & \downarrow \mathcal{L}(\varphi_1, \psi_1) \\
E_X \rightarrow Y & \xrightarrow{(\varphi \circ f \circ \varphi_1^{-1}) \circ (\psi_1 \circ g \circ \psi^{-1})} & E_{X_1} \rightarrow Y_1
\end{array}$$

(16)

\footnote{The situation here is not as good as for the topological dual: we don’t know yet any way of defining the topology of $\mathcal{L}(E, F)$ directly in terms of the topologies (and the homologies) of $E$ and $F$; in particular, the topology of uniform convergence on all equicontinuous sets is too coarse in general, see proposition 27.}
(where \( \varphi_1 : E_1 \to E_{X_1} \) and \( \psi_1 : F_1 \to E_{Y_1} \) are linear homeomorphisms) is commutative by lemma 23, and therefore, in particular, \( \mathcal{L}(f, g) \) is continuous. In particular, when \( F = F_1 = K \) and \( g = \text{Id} \), one has of course \( \mathcal{L}(f, K) = \iota f \).

5.3 Tensor product

Due to the considerations above on topological duals of intrinsic Köthe spaces, \( \mathcal{L}(E, F) \) comes equipped also with an intrinsic bornology which allows to define its topological dual as an intrinsic Köthe space. Therefore, we define in general

\[
E \otimes F = \mathcal{L}(E, F')',
\]

and this is an intrinsic Köthe space: if \( \varphi : E \to E_X \) and \( \psi : F \to E_Y \) are linear homeomorphisms then \( \varphi \otimes \psi = \mathcal{L}(\varphi', \psi')' \) is a linear homeomorphism from \( E \otimes F \to E_{X \otimes Y} \).

Let \( E_1 \) and \( F_1 \) be intrinsic Köthe spaces, and let \( f : E \to E_1 \) and \( g : F \to F_1 \) be linear continuous maps. We define a map \( f \otimes g : E \otimes F \to E_1 \otimes F_1 \) by letting \( h \in \mathcal{L}(E_1, F_1) \), we set

\[
\langle (f \otimes g)(t), h \rangle = \langle t, h \circ f \rangle.
\]

In other terms, we have set \( f \otimes g = \mathcal{L}(f, y) \), and so \( f \otimes g \) takes actually its values in \( E_1 \otimes F_1 \) and is a linear continuous map. Let \( \varphi_1 : E_1 \to E_{X_1} \) and \( \psi_1 : F_1 \to E_{Y_1} \) be linear homeomorphisms to some Köthe spaces. Then the following diagram commutes; this is a consequence of the commutation of diagram 16.

\[
\begin{array}{ccc}
E \otimes F & \xrightarrow{f \otimes g} & E_1 \otimes F_1 \\
\varphi \otimes \psi & \downarrow & \varphi_1 \otimes \psi_1 \\
E_{X \otimes Y} & \xrightarrow{(\varphi_1 \circ f \circ \varphi^{-1}) \otimes (\psi_1 \circ g \circ \psi^{-1})} & E_{X_1 \otimes Y_1}
\end{array}
\]

This diagram, together with the corresponding properties of the tensor product of Köthe spaces, can be used for proving the various categorical properties of the tensor product (associativity, symmetry, MacLane pentagon). Given \( u \in E \) and \( v \in F \), the element \( u \otimes v \) of \( E \otimes F = \mathcal{L}(E, F')' \) is defined by \( \langle u \otimes v, h \rangle = \langle h(u), v \rangle \), and we have of course \( (f \otimes g)(u \otimes v) = f(u) \otimes g(v) \).

One can also prove, using the diagram above, that the category of intrinsic Köthe spaces and linear continuous maps is a \( \ast \)-autonomous monoidal closed category.

Direct sums and products of intrinsic Köthe spaces are defined as usual on topological vector spaces, and the direct product is endowed with the product topology.

5.4 Exponentials

Let \( E \) be an intrinsic Köthe space, and let \( h : E \to K \) be a function. If \( \varphi : E \to E_X \) and \( \psi : F \to E_Y \) are linear homeomorphisms, and if \( h \circ \varphi \) is entire (that is, belongs to \( A(X) \)), then \( h \circ \psi = h \circ \varphi \circ (\varphi^{-1} \circ \psi) \) is also entire by proposition 32, since \( \varphi^{-1} \circ \psi : E_Y \to E_X \) is linear and continuous.

So it is reasonable to say that a function \( h : E \to K \) is entire if, for some (and therefore for each) linear homeomorphism \( \varphi : E \to E_X \) to a Köthe space, the function \( h \circ \varphi^{-1} \) belongs to \( A(X) \). We denote by \( A(E) \) the linear space of all entire functions from \( E \) to \( K \). For any linear homeomorphism \( \varphi : E \to E_X \) to a Köthe space, we have a linear isomorphism \( \hat{A}(\varphi) : A(E) \to E_{(X)} \) given by

\[
\hat{A}(\varphi)(h) = \Theta_X(h \circ \varphi^{-1}).
\]
Through this isomorphism, $\mathcal{A}(E)$ inherits from $E_{(\!(X)\!)}^\perp$ a linear topology for which $\mathcal{A}(\varphi)$ becomes a linear homeomorphism (the inverse image by $\mathcal{A}(\varphi)$ of the topology of $E_{(\!(X)\!)}^\perp$). This topology does not depend on the particular linear homeomorphism $\varphi$ we have used. Indeed, let $\psi : E \to E_Y$ be another linear homeomorphism and let $\theta = \varphi \circ \psi^{-1} : E_Y \to E_X$; this is a linear homeomorphism, and by proposition 32, the following diagram is commutative

\[
\begin{array}{ccc}
E_{(\!(X)\!)}^\perp & \xrightarrow{\mathcal{A}(\varphi)} & (\!(\theta)\!)^\perp \\
E & \xrightarrow{\mathcal{A}(\psi)} & E_{(\!(Y)\!)}^\perp
\end{array}
\]

where $(\!(\theta)\!)^\perp$ is a linear homeomorphism. So $\mathcal{A}(E)$ has a canonical topology, and, endowed with that topology, $\mathcal{A}(E)$ is an intrinsic Köthe space.

Let $E$ and $F$ be intrinsic Köthe spaces and let $f : F \to E$ be linear and continuous. If $h \in \mathcal{A}(E)$, then $h \circ f \in \mathcal{A}(F)$, by proposition 32 again. The map $\mathcal{A}(f) : \mathcal{A}(E) \to \mathcal{A}(F)$ defined in that way makes the following diagram commutative

\[
\begin{array}{ccc}
\mathcal{A}(E) & \xrightarrow{\mathcal{A}(f)} & \mathcal{A}(F) \\
\mathcal{A}(\varphi) & \xrightarrow{(\!(\varphi \circ f \circ \psi^{-1})\!)^\perp} & \mathcal{A}(\psi) \\
E_{(\!(X)\!)}^\perp & \xrightarrow{(\!(\varphi \circ f \circ \psi^{-1})\!)^\perp} & E_{(\!(Y)\!)}^\perp
\end{array}
\]

and so is a linear continuous map. Therefore $E \mapsto \mathcal{A}(E)$ is a contravariant endofunctor on the category of intrinsic Köthe spaces and linear continuous maps. The dual functor $E \mapsto \mathcal{A}(E)'$ (and $f \mapsto \mathcal{A}(f)'$) will also be denoted by $E \mapsto !E$ and $f \mapsto !f$.

The various structural maps of the exponential can also be defined in an intrinsic way. As an example, we give an intrinsic version of the digging morphism $p^E : !E \to !!E$. Given $\varphi : E \to E_X$ a linear homeomorphism, we define $p^E$ as the dual of the unique map $q$ making the following diagram commutative:

\[
\begin{array}{ccc}
\mathcal{A}(E) & \xrightarrow{q} & \mathcal{A}(E) \\
\mathcal{A}(\varphi) & \xrightarrow{(p^X)^\perp} & \mathcal{A}(\varphi) \\
E_{(\!(X)\!)}^\perp & \xrightarrow{(p^X)^\perp} & E_{(\!(X)\!)}^\perp
\end{array}
\]

This map $q$, which, by definition, is linear and continuous (since $\mathcal{A}(\mathcal{A}(\varphi)'$) and $\mathcal{A}(\varphi)$ are linear homeomorphisms), does not depend on $\varphi$: this results from the functoriality of the operation $X \mapsto !X$ and from the naturality of $X \mapsto p^X$. Given $\Phi \in \mathcal{A}(\mathcal{A}(E)'$, one can check that the entire
function \( q(\Phi) : E \to K \) is defined as follows. For \( u \in E \), let \( \delta_u \in A(E)' \) be given by \( \delta_u(h) = h(u) \) (the Dirac functional). That \( \delta_u \) indeed belongs to \( A(E)' \) can be proved using a chart of \( E \). Then, for \( u \in E \), we have \( q(\Phi)(u) = \Phi(\delta_u) \).

We consider now the central isomorphism \(!X \otimes !Y = !X \otimes !Y\) and look for an intrinsic version of this isomorphism. Given \( E \) and \( F \), two intrinsic Köthe spaces, this isomorphism is the dual of a map \( \mathcal{A}(E \times F) \to \mathcal{L}(\mathcal{A}(E)', \mathcal{A}(F)) \) which maps any entire map \( h : E \times F \to K \) to the linear and continuous map \( \Phi_h : \mathcal{A}(E)' \to \mathcal{A}(F) \) given by \( \Phi_h(S)(v) = \langle S, h_v \rangle \) where \( h_v : E \to K \) is the entire map defined by \( h_v(u) = h(u, v) \). That \( h_v \) is entire is clear, therefore, for \( S \in \mathcal{A}(E)' \), \( \langle S, h_v \rangle \in K \) is well defined. Moreover, the map \( v \mapsto h_v \) is entire by cartesian closeness of \( K' \), and so the map \( \Phi_h(S) : F \to K \) is entire for all \( S \in \mathcal{A}(E)' \). One can check that the diagram

\[
\begin{array}{ccc}
\mathcal{A}(E)' & \xrightarrow{\Phi_h} & \mathcal{A}(F) \\
\mathcal{A}(\varphi)' \downarrow & & \uparrow \mathcal{A}(\psi) \\
E_{(X)} & \xrightarrow{k} & E_{(Y)'}
\end{array}
\]

commutes, where \( \varphi : E \to E_X \) and \( \psi : F \to E_Y \) are charts for \( E \) and \( F \). In this diagram, the map \( k \) is defined using \( h \) and the isomorphism between \(!X \otimes !Y\) and \( IX \otimes IY\) (and of course the charts \( \varphi \) and \( \psi \)), and so \( \Phi_h \) is linear and continuous. Last one checks that the map \( h \mapsto \Phi_h \) is linear and continuous using another diagram involving charts. Conversely, given \( \Phi \in \mathcal{L}(\mathcal{A}(E)', \mathcal{A}(F)) \), one defines an entire map \( h : E \times F \to K \) by \( h(u, v) = \Phi(\delta_u)(v) \) where \( \delta_u \in \mathcal{A}(E)' \) is the Dirac functional at \( u \). This establishes the announced intrinsic isomorphism between \( \mathcal{A}(E \times F) \) and \( \mathcal{L}(\mathcal{A}(E)', \mathcal{A}(F)) \), this latter space being canonically isomorphic to \( \mathcal{A}(E)' \otimes \mathcal{A}(F)' \). Now let \( S \in !E \) and \( T \in !F \), through this isomorphism, the action of \( S \otimes T \) on an element \( h \) of \( \mathcal{A}(E \times F) \) is given by

\[
\langle S \otimes T, h \rangle = \langle \Phi_h(S), T \rangle
\]

with the notations above. Of course, a similar formula holds, where the rôles of \( S \) and \( T \) are interchanged.

Remember that \(!E \) is the topological dual of \( \mathcal{A}(E) \), so it is a kind of space of “distributions” where the space of smooth functions with compact support would have been replaced by a space of entire mappings. With this intuition in mind, we can adopt a more convenient notation for the application of an element \( S \) of \( \mathcal{A}(E)' \) to an element \( h \) of \( \mathcal{A}(E) \): let us write \( S_x(h(x)) \) instead of \( \langle S, h \rangle \). This notation, where \( x \) is a bound variable “of type \( E \)”, is due to Schwartz. Equation 17 can be written more conveniently

\[
(S \otimes T)(x, y)(h) = T_y(S_x(h(x, y)))
\]

and of course we also have \( (S \otimes T)(x, y)(h) = S_x(T_y(h(x, y))) \).

From this distribution viewpoint, the map \( m_E : !E \otimes !E \to !E \) (the intrinsic version of the map \( m_X \) defined for Köthe spaces in section 3.4) can be seen as defining a convolution product. Considering \( m_E \) as a bilinear map \( !E \times !E \to !E \), one can check that indeed

\[
\langle m_E(S, T), h \rangle = \langle S \otimes T, k \rangle \quad \text{where} \quad k(u, v) = h(u + v),
\]

that is, with the notation above, \( m_E(S, T)_u(h(u)) = S_x(T_y(h(x + y))) \) (and also in the other way round) and this is exactly the definition of a convolution product of distributions. The neutral
element for this operation is the inverse image of \( e_0 \in E_{1,1} \) through the chart \( A(\varphi) : !E \to E_{1,1} \) (where \( \varphi \) is a chart of \( E \)), which is easily seen to be the Dirac functional at 0, \( \delta_0 : h \mapsto h(0) \), as usual.

**Example:** Take \( K = \mathbb{R} \) and \( E = K \) (the implicit chart of \( E \) is the obvious map \( E \to E_1 \) where 1 is defined by \( |1| = \{ * \} \)). Given a continuous map \( f : \mathbb{R} \to \mathbb{R} \) with compact support, and \( h \in A(\mathbb{R}) \), we can set

\[
 f^\vee(h) = \int_{\mathbb{R}} h(x)f(x)dx
\]

defining a linear form \( f^\vee : A(\mathbb{R}) \to \mathbb{R} \). Let \( M \in \mathbb{R}^+ \) be an upper bound of \( |f(x)| \) for \( x \in \mathbb{R} \) and let \( K \in \mathbb{R}^+ \) be such that the support of \( f \) is included in \([-K, K] \). Then

\[
 \left| \int_{\mathbb{R}} x^n f(x)dx \right| \leq \frac{2MK}{n+1} K^n \leq CK^n
\]

for a constant \( C \in \mathbb{R}^+ \), which depends only on \( f \). So setting \( A_n = \int_{\mathbb{R}} x^n f(x)dx \) for each \( n \in \mathbb{N} \), we define a family of real numbers \( (A_n)_{n \in \mathbb{N} - \{1\}} \) which belongs to \( E_{0,1} \). The corresponding element of \( A(\mathbb{R})^\vee \) is easily seen to be \( f^\vee \). Moreover, using Weierstrass theorem (any continuous map can be uniformly approximated by a sequence of polynomials on a compact subset of \( \mathbb{R} \)), one shows easily that the map \( f \mapsto f^\vee \) is injective. Assume indeed that \( f(x_0) \neq 0 \) for some \( x_0 \in [-K, K] \), we may assume without loss of generality that \( f(x_0) = 1 \) and that \( x_0 = 0 \). Let \( \delta > 0 \) be such that \( f(x) \geq 1/2 \) for \( x \in [-\delta, \delta] \) (\( f \) is continuous). Assume also \( K \) chosen such that \( K \geq \delta \). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be the continuous function defined by

\[
 \varphi(x) = \begin{cases} 
 0 & \text{if } x \notin [-\delta, \delta] \\
 \frac{\delta + x}{\delta} & \text{if } -\delta \leq x \leq 0 \\
 \frac{x - \delta}{\delta} & \text{if } 0 \leq x \leq \delta 
\end{cases}
\]

and for each \( \varepsilon > 0 \), let \( P_\varepsilon : \mathbb{R} \to \mathbb{R} \) be a polynomial function such that \( |\varphi(x) - P_\varepsilon(x)| \leq \varepsilon \) for all \( x \in [-K, K] \). Let \( \varepsilon > 0 \) (with \( \varepsilon \leq 1/2 \)) and let \( M \) be an upper bound of \( |f(x)| \) (remember that \( f \) has a compact support). Then we have, denoting by \( x^+ \) the real max\((x, 0)\),

\[
 \left| \int_{\mathbb{R}} P_\varepsilon(x)f(x)dx \right| \geq \int_{-\delta}^{\delta} P_\varepsilon(x)f(x)dx \geq \int_{-\delta}^{\delta} |P_\varepsilon(x)f(x)|dx - \int_{-\delta}^{\delta} |P_\varepsilon(x)f(x)|dx \\
\geq \frac{1}{2} \int_{-\delta}^{\delta} (\varphi(x) - \varepsilon)^+dx - 2KM\varepsilon \\
\geq \delta(1 - \varepsilon)^2 - 2KM\varepsilon
\]

but for \( \varepsilon \) sufficiently small, we have \( \delta(1 - \varepsilon)^2 - 2KM\varepsilon > 0 \), so there is a polynomial \( P \) such that \( f^\vee(P) \neq 0 \).

So one can say that \( !\mathbb{R} \) contains the space of continuous functions with compact support as a subspace\(^9\). Given now \( f \) and \( g \) two such functions, one can check that \( m_{!\mathbb{R}}(f^\vee, g^\vee) = (f \ast g)^\vee \) where \( f \ast g \) is the usual convolution product of functions:

\[
 (f \ast g)(x) = \int_{\mathbb{R}} f(x - y)g(y)dy ,
\]

\(^9\)This space \( !\mathbb{R} \) contains actually all distributions with compact support: given a distribution \( T \) with compact support \( K \subseteq \mathbb{R} \), we can define \( T(h) \) for \( h \in A(\mathbb{R}) \) by \( T(h) = T(\varphi h) \) where \( \varphi \) is any smooth map with compact support which is constantly equal to 1 on \( K \), and this mapping from the distributions with compact support into \( !\mathbb{R} \) is injective by Weierstrass theorem again.
which is a continuous function with compact support. Indeed, for \( h \in \mathcal{A}(\mathbb{R}) \), we have:

\[
\langle w_\mathbb{R}(f^\vee, g^\vee), h \rangle = g_\mathbb{R}^\vee \left( \int \mathbb{R} f(y) h(x+y) \, dy \right) = g_\mathbb{R}^\vee \left( \int \mathbb{R} f(y) h(u) \, du \right) = \int \mathbb{R} f(x) \left( \int \mathbb{R} f(u-x) h(u) \, du \right) \, dx = \int \mathbb{R} \left( \int \mathbb{R} f(x) \, dx \right) h(u) \, du.
\]

6 Some examples

The goal of this section is to give the interpretation of a few terms (proofs) as continuous linear or as entire mappings, and to compute a few derivatives of the interpretations of these terms.

The space corresponding to the type of booleans is \( \text{Bool} = 1 \oplus 1 \), so that \( E_{\text{Bool}} \) is (isomorphic to) \( K^2 \), its generic element will be denoted by \( xe_t + ye_r \), taking \( |\text{Bool}| = \{ t, f \} \). Consider the following term of (e.g.) boolean PCF, with one boolean parameter \( b \):

\[
\text{if } b \text{ then } (\text{if } b \text{ then } \text{false} ) \text{ else } (\text{if } b \text{ then } \text{false} ),
\]

the semantics of this term will be an entire (indeed, polynomial) map from \( E_{\text{Bool}} = K^2 \) to itself. For computing this semantics, one considers first the linearized version of this term, with two boolean parameters \( b \) and \( c \), which tests if its two arguments are equal:

\[
\text{if } b \text{ then } (\text{if } c \text{ then } \text{false} ) \text{ else } (\text{if } c \text{ then } \text{false} ),
\]

and which is interpreted by a linear map from \( E_{\text{Bool}} \otimes \text{Bool} \) to \( E_{\text{Bool}} \) whose matrix \( A \) is given by

\[
A_{(a_1, a_2), d} = \begin{cases} 
1 & \text{if } a_1 = a_2 \text{ and } d = t, \text{ or } a_1 \neq a_2 \text{ and } d = f \\
0 & \text{otherwise}.
\end{cases}
\]

Then we compose this map with the contraction, or more precisely with the linear map \( C : !\text{ Bool} \to (\text{ Bool } \otimes \text{ Bool}) \) obtained by composing the contraction map \( !\text{ Bool} \to (\text{ Bool } \otimes !\text{ Bool}) \) with the tensorisation of the dereliction \( !\text{ Bool} \to \text{ Bool} \) with itself. This matrix is given by \( C_{\mu, (a_1, a_2)} = \delta_{[a_1, a_2], \mu} \). Therefore, the kernel of the linear map \( g \) defined by \( AC \) is spanned by the vectors \( e_\mu \) with \(#\mu \neq 2\), and \( g \) is defined by

\[
g(ue_t[t] + ve_r[r] + ye_t[\ell]) = (u + v)e_t + 2we_r.
\]

The entire map \( f : E_{\text{Bool}} \to E_{\text{Bool}} \) which interprets our initial program is then given by

\[
f(xe_t + ye_r) = g((xe_t + ye_r)^\perp) = (x^2 + y^2)e_t + 2xye_r.
\]

It is absolutely remarkable that the interpretation of a perfectly standard syntactical object like the program we have just considered (which actually can also be considered as a proof in linear logic) is a matrix which has a coefficient different from \( 0 \) and \( 1 \). This means that, by interpreting syntax by “sets with scalar coefficients” and not simply by sets, we obtain new informations about syntax. This is also an effect of non-uniformity: observe that, in this example, the coefficient 2 corresponds to a component of \( f \) which would not exist in a uniform setting.

As a second example, we consider the map

\[
2 : E_{\{1\}}^\perp \to K \\
f \mapsto f(f(0))
\]

which is a version of the “2 Church numeral”. In this definition, we identify the space \( E_{\{1\}}^\perp \) with the space of entire functions defined on \( K \) with values in \( K \). This map 2 is entire by cartesian
closeness of $\mathcal{K}^1_K$, and we want to compute its derivative. Let $X = (\mathbb{N})^1$, if $f \in E_X$ is given on $K$ by $f(x) = \sum_n A_n x^n$, then $2(f) = \sum_n A_n (A_0)^n$.

Let $T : E_{X \& X} \to K$ be the “linearized” (this function actually is linear only in its first argument) version of $2$, given by $T(f, g) = f(g(0))$, which is the composite of the evaluation map $\text{ev} : E_X \& I \to E_1$ and $H : E_{X \& X} \to E_{X \& I}$, this latter map being linear and given by $H(f, g) = (f, g(0))$. The chain rule gives therefore, for $f, g, h, k \in E_X$:

$$\frac{dT}{df(g)}(f, g) \cdot (h, k) = \frac{d\text{ev}}{d(u, x)}(f, g(0)) \cdot H(h, k) = h(g(0)) + \frac{df}{dx}(g(0)) \cdot k(0)$$

by formulae (14) and (15), so

$$\frac{d2}{df} (f) \cdot h = h(f(0)) + \frac{df}{dx}(f(0)) \cdot h(0)$$

by formula (13). For instance, when $f(0) = 0$, we have $\frac{d2}{df}(f) \cdot h = (1 + f'(0))h(0)$ (here, the linear application operation $A \cdot x$ is just scalar multiplication).

We leave to the reader the following computation. Let $X$ be a Köthe space, and consider now the “true” Church numeral $2$, namely the functional

$$2 : E_{X \to X} \to E_{(X \to X) \to X}$$

$$f \mapsto f \circ A \circ f$$

So $2$ is entire by cartesian closeness of $\mathcal{K}^1_K$, and $\frac{d2}{df}$ is an entire map from $E_{X \to X}$ to $E_{[(X \to X) \to X] \to (X \to X)}$; show that this map is given by

$$\left( \frac{d2}{df}(f) \cdot h \right) (x) = h(f(x)) + \frac{df}{dx}(f(x)) \cdot h(x).$$

For instance, $\frac{d2}{df}(\text{Id})$ maps any entire map $h : E_X \to E_X$ to $2h$.

An essential tool in programming is the iteration principle, which corresponds to the “for” loop of Pascal. As already explained, the type of natural numbers can be considered as a $\mathbb{N}$-indexed sum of the space $1$, that is $|N| = \mathbb{N}$ and $E_N = K^{[N]}$. So in the present setting, a “natural number” is a finite linear combination of standard natural numbers, with coefficients in $K$ (we had the same phenomenon with booleans, which were combinations of $t$ and $f$). Given a Köthe space $X$, for each $k \in \mathbb{N}$, there is an entire map

$$I_k : E_{[(X \to X) \& X]} \to E_X$$

$$(f, x) \mapsto f^k(x)$$

where $f^k = f \circ A \cdots \circ A \circ f$ (k times). That $I_k$ is entire results from the cartesian closeness of $\mathcal{K}^1_K$. Of course, $I_k$ can be considered as a linear continuous map from $E_{[(X \to X) \& X]}$ to $E_X$. By definition of $N$ as a direct sum, we can define a linear continuous map $I_{\mathbb{N}} : E_N \to E_{[(X \to X) \& X]}$ as follows:

$$I_{\mathbb{N}} (n, f, x) = \sum_{k \in \mathbb{N}} n_k f^k(x),$$

this sum being finite, since an element $n$ of $E_N$ satisfies $n_k = 0$ for almost all values of $k \in \mathbb{N}$; this is the iteration functional in $\mathcal{K}_K$. Using a standard trick involving a cartesian product, this
iteration functional can be used for defining a primitive recursion functional, so that Gödel system $T$ can be interpreted in the category $\mathcal{K}_K$, a powerful functional programming language enjoying strong normalization (see [GLT89]). Observe that a priori $\mathcal{K}_K$ is not a model of PCF, as obviously entire functions do not always have fixpoints: consider the map $x \mapsto x + 1$ from $K$ to $K$.

References


