EXPLICIT UPPER BOUNDS FOR $|L(1, \chi)|$
FOR PRIMITIVE CHARACTERS $\chi$

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Abstract

We give explicit upper bounds for $|L(1, \chi)|$ for primitive Dirichlet characters $\chi$. These bounds depend on the behaviour of $\chi$ at primes $p$ in a given set $S$ of primes.

1. Introduction

The aim of this paper is to prove the following fully explicit result we have long been trying to obtain; see [7–9, 13, 14, 16].

THEOREM 1 Let $S$ be a given finite set of pairwise distinct rational primes. Then, for any primitive Dirichlet character $\chi$ of conductor $q_\chi > 1$ we have

$$
\left| \prod_{p \in S} \left( 1 - \frac{\chi(p)}{p} \right) L(1, \chi) \right| \leq \frac{1}{2} \left| \prod_{p \in S} \left( 1 - \frac{1}{p} \right) \right| \times \left( \log q_\chi + \kappa_\chi + \omega \log 4 + 2 \sum_{p \in S} \frac{\log p}{p - 1} \right) + o(1),
$$

where

$$
\kappa_\chi = \begin{cases} 
\kappa_{\text{even}} = 2 + \gamma - \log(4\pi) = 0.046191 \cdots & \text{if } \chi(-1) = +1, \\
\kappa_{\text{odd}} = 2 + \gamma - \log \pi = 1.432485 \cdots & \text{if } \chi(-1) = -1,
\end{cases}
$$

where $\omega \geq 0$ is the number of primes $p \in S$ which do not divide $q_\chi$, and where $o(1)$ is an explicit error term which tends rapidly to zero when $q_\chi$ goes to infinity. Moreover, if $S = \emptyset$ or if $S = \{2\}$, then this error term $o(1)$ is always less than or equal to zero, and if none of the primes in $S$ divides $q_\chi$ then this error term $o(1)$ is less than or equal to zero for $q_\chi$ large enough.

It is worth pointing out that whereas our proof in [16] of this result in the case of even characters cannot be adapted to deal with the case of odd characters, our present proof works with even or odd characters (and in the case of even characters we obtain the same result as in [16], albeit with a worse error term; see Remark 8 below). We refer the reader to [1, 6, 18–20, 24, 26] and section 4 below for various applications of such explicit bounds on $L$-functions. Let us point out that these

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bounds are not the best possible theoretically (see, for example, [3, 21, 27]). However, as explained in [16], if such better bounds are made explicit, we end up with useless ones in a reasonable range for $q_x$. Therefore, applications of these better bounds to practical problems (see section 4 below) are not possible.

2. Notation
We set

$$A = A(x) = \begin{cases} 0 & \text{if } x(-1) = +1, \\ 1 & \text{if } x(-1) = -1. \end{cases}$$

Let $d_1 = p_1 p_2 \cdots p_\omega \geq 1$ denote the product of the $\omega = \omega(d_1) \geq 0$ primes in $S$ which do not divide $q_x$, let $d_2 \geq 1$ denote the product of the primes in $S$ which divide $q_x$, set $d = d_1 d_2 = \prod_{p \in S} p$ and let $\psi$ denote the (non-primitive for $d_1 > 1$) Dirichlet character modulo $q = q_d$ induced by $x$. Note that $d_2$ divides $q_x$, that $d$ divides $q_\psi$ and that

$$\prod_{p \in S} \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi) = L(1, \psi). \quad (1)$$

Hence, we want to find an upper bound on $|L(1, \psi)|$. We let $\mu$ and $\phi$ denote the Möbius and Euler totient functions. Whenever $D \geq 1$ is a positive square-free integer, we set $\tilde{\phi}(D) = 1$ if $D = 1$ and $\tilde{\phi}(D) = \prod_{p | D} (p - 2)$ if $D > 1$. Hence, $\sum_{\delta | D} \phi(\delta) = \phi(D)$ if $D \geq 1$ is square-free. Finally, we set $\epsilon_D = 1$ if $D = 1$ and $\epsilon_D = 0$ if $D > 1$.

3. Proof of Theorem 1
3.1. First step
Set

$$\theta(x, A, \psi) = \sum_{n \geq 1} A^n \psi(n) e^{-\pi n^2 x / q_x} = \frac{1}{2} \sum_{n \in \mathbb{Z}} A^n \psi(n) e^{-\pi n^2 x / q_x} \quad (x > 0) \quad (2)$$

(with the convention $0^0 = 1$) and

$$g(x, A) = \sum_{n \geq 1} A^n e^{-\pi n^2 x} \quad (x > 0). \quad (3)$$

Then for any $a > 0$ it holds that

$$\left( \frac{q \psi}{\pi} \right)^{(s+A)/2} \Gamma((s + A)/2) L(s, \psi) = \int_0^\infty \theta(x, A, \psi) x^{(s+A)/2} \frac{dx}{x} = \int_a^\infty \theta(1/x, A, \psi) x^{-(s+A)/2} \frac{dx}{x} + \int_a^0 \theta(x, A, \psi) x^{(s+A)/2} \frac{dx}{x} \quad (\Re(s) > 1). \quad (4)$$

If $\psi$ were primitive then we would choose $a = 1$, we would use the functional equation satisfied by $x \mapsto \theta(x, A, \psi)$ and we would finally proceed as in [9]. Since $\psi$ is not necessarily primitive we have to be a little more clever prior to applying the method used in [9].
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LEMMA 2 For \( x > 0 \),

\[
|\theta(x, A, \psi)| \leq \sum_{\delta | d} \delta^A \mu(\delta) g(\delta^2 x/q \psi, A)
\]

(5)

and

\[
|\theta(1/x, A, \psi)| \leq x^A \sqrt{x/d_1} \sum_{\delta_1 | d_1} \delta_1^A \phi(\delta_1) \sum_{\delta_2 | d_2} \delta_2^A \mu(\delta_2) g(\delta_1^2 \delta_2^2 x/q \psi, A).
\]

(6)

Proof. Since \( \psi(n) = 0 \) for \( \gcd(n, d) > 1 \), we have

\[
|\theta(x, A, \psi)| \leq \sum_{n \geq 1} n^A \exp(-\pi n^2 x/q \psi)
\]

\[
= \sum_{n \geq 1} \sum_{\delta | n \text{ and } \delta | d} \mu(\delta) n^A \exp(-\pi n^2 x/q \psi)
\]

\[
= \sum_{\delta | d} \delta^A \mu(\delta) \sum_{m \geq 1} m^A \exp(-\pi (m\delta)^2 x/q \psi),
\]

and (5) follows, by (3).

Now, set

\[
\theta(x, A, a, q) = \sum_{b \in \mathbb{Z}} (a + bq)^A e^{-\pi (a+ bq)^2 x/q} \quad (x > 0 \text{ and } q > 0)
\]

(with the convention \( 0^0 = 1 \)). The Poisson summation formula yields

\[
\theta(x, A, a, q) = \left(-i\right)^A x^A \sqrt{q \chi} \sum_{b \in \mathbb{Z}} b^A e^{2\pi iab/q \chi} e^{-\pi b^2/q \chi} \quad (x > 0)
\]

(see [2, (8) p. 63] for the case \( A = 0 \) and [2, (10) p. 70] for the case \( A = 1 \)). Using (2), we obtain

\[
\theta(x, A, \psi) = \frac{1}{2} \sum_{a \in \mathbb{Z}} \sum_{b \in \mathbb{Z}} (a + bq \psi)^A (a + bq \psi) e^{-\pi (a+ bq \psi)^2 x/q \psi}
\]

\[
= \frac{1}{2} \sum_{a = 1}^{q} \psi(a) \theta(x, A, a, q \psi) = \left(-i\right)^A x^A \sqrt{q \chi} \sum_{b \in \mathbb{Z}} b^A \tau_b(\psi) e^{-\pi b^2 / q \chi}
\]

and

\[
\theta(1/x, A, \psi) = (-i)^A x^A \sqrt{x/d_1} \sum_{b \geq 1} \frac{\tau_b(\psi)}{\sqrt{q \chi}} b^A e^{-\pi b^2 x/q \psi},
\]

where

\[
\tau_b(\psi) = \sum_{a = 1}^{q \psi} \psi(a) e^{2\pi iab/q \psi}.
\]

Now, for any \( b \in \mathbb{Z} \) it holds that \( \tau_b(\psi) = \mu(d_1) \chi(d_1) \mu(\delta_b) \phi(\delta_b) \tilde{\chi}(b) \tau_1(\chi) \), where \( \delta_b = \gcd(b, d_1) \)

and

\[
\tau(\chi) = \sum_{a = 1}^{q \chi} \chi(a) e^{2\pi iax/q \chi}
\]
(see [17, Lemma 5.4]). In particular, it holds that

$$|\tau_b(\psi)/\sqrt{q_x}| = \begin{cases} 0 & \text{if } \gcd(b, q_x) > 1, \\ \phi(\delta_b) & \text{if } \gcd(b, q_x) = 1. \end{cases}$$

(7)

To get (6) we finally notice that, by (3), we have

$$\sum_{\delta_1 | d_1} \delta_1^A \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \delta_2^A \mu(\delta_2) g(\delta_1^2 \delta_2^2 x/q, A) = \sum_{b \geq 1} a_b b^A e^{-\pi b^2 x/q}$$

with

$$a_b = \sum_{\delta_1 | d_1, \delta_2 | d_2} \phi(\delta_1) \mu(\delta_2)$$

$$= \left( \sum_{\delta_1 | \gcd(b, d_1)} \delta_1^A \tilde{\phi}(\delta_1) \right) \left( \sum_{\delta_2 | \gcd(b, d_2)} \mu(\delta_2) \right)$$

$$= \begin{cases} 0 & \text{if } \gcd(b, d_2) > 1, \\ \phi(\gcd(b, d_1)) & \text{if } \gcd(b, d_2) = 1 \end{cases}$$

(for \(\gcd(d_1, d_2) = 1\), and we use (7) to obtain \(0 \leq |\tau_b(\psi)/\sqrt{q_x}| \leq a_b \) for \(d_2\) divides \(q_x\)).

According to this Lemma 2, the integral representation (4) above is valid for all \(s\) in the complex plane, and for \(s = 1\) we obtain

$$\left( \frac{q\psi}{\pi} \right)^{\frac{1}{2}(1+A)} \Gamma \left( \frac{1+A}{2} \right) |L(1, \psi)| \leq \int_{1/\pi}^{\infty} |\theta(1/x, A, \psi)| x^{-(1+A)/2} \frac{dx}{x}$$

$$+ \int_{1/\pi}^{\infty} |\theta(x, A, \psi)| x^{(1+A)/2} \frac{dx}{x}$$

and the following.

**Proposition 3**

$$\left( \frac{q\psi}{\pi} \right)^{\frac{1}{2}(1+A)} \Gamma \left( \frac{1+A}{2} \right) |L(1, \psi)|$$

$$\leq \frac{1}{\sqrt{d_1}} \sum_{\delta_1 | d_1} \delta_1^A \phi(\delta_1) \sum_{\delta_2 | d_2} \delta_2^A \mu(\delta_2) \int_{1/a}^{\infty} x^{A/2} g(\delta_1^2 \delta_2^2 x/q \phi, A) \frac{dx}{x}$$

$$+ \sum_{\delta | \mu \delta} \delta^A \mu(\delta) \int_{a}^{\infty} x^{(1+A)/2} g(\delta^2 x/q \phi, A) \frac{dx}{x}.$$  

3.2. Second step

Our next step is to use

$$e^{-t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) t^{-s} ds \quad (c > 0)$$
to obtain
\[
\int_B e^{-ax} x^\beta dx = \frac{B^\beta}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \Gamma(s - \beta) ds
\]
(for \(B > 0, \alpha > 0, \beta > 0\) and \(c > \beta\)), and
\[
\int_1^{x^{A/2}} g(\delta_1^2 \delta_2^2 x / q \psi, A) dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a^{s - \frac{1}{2} A}}{\Gamma(s - \frac{1}{2} A)} \Gamma(s) \xi(2s - A) ds
\]
and, for \(c > 1\),
\[
\int_1^{x^{(1+A)/2}} g(\delta_1^2 \delta_2^2 x / q \psi, A) dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a^{s - \frac{1}{2} (1+A) - s}}{\delta_2^2 (s - \frac{1}{2} (1 + A))} \Gamma(s) \xi(2s - A) ds.
\]
Hence, using Proposition 3,
\[
\sum_{\delta_1 | d_1} \delta_1^A \phi(\delta_1) \sum_{\delta_2 | d_2} \delta_2^A \mu(\delta_2)(\delta_1 \delta_2)^{2s-2s} = \left\{ \prod_{p | d_1} (1 + (p - 2)p A^{2s}) \right\} \left\{ \prod_{p | d_2} (1 - p A^{2s}) \right\}
\]
and
\[
\sum_{\delta | d} \delta^A \mu(\delta) \delta^{-2s} = \prod_{p | d} (1 - p A^{2s}),
\]
we finally obtain the following.

**Proposition 4** For \(c > 1\)
\[
\left| \left\{ \prod_{p \in \mathbb{P}} \left( 1 - \frac{\chi(p)}{p} \right) \right\} L(1, \chi) \right| = |L(1, \psi)| \leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h_{d_1, d_2}(s, A, a) ds, \tag{8}
\]
where
\[
h_{d_1, d_2}(s, A, a) = \left( \frac{q \psi}{\pi} \right)^{s - \frac{1}{2} (1+A)} \frac{\Gamma(s)}{\Gamma(\frac{1}{2} (1 + A))} \xi(2s - A) H_{d_1, d_2}(s, A, a)
\]
with
\[
H_{d_1, d_2}(s, A, a) = \left\{ \prod_{p | d_1} (1 - p A^{2s}) \right\} \times \left( \frac{a^{s - \frac{1}{2} A}}{\sqrt{d_1 (s - \frac{1}{2} A)}} \left\{ \prod_{p | d_1} (1 + (p - 2)p A^{2s}) \right\} + \frac{a^{s - \frac{1}{2} (A+1) - s}}{s - \frac{1}{2} (A + 1)} \left\{ \prod_{p | d_1} (1 - p A^{2s}) \right\} \right).
\]
Moreover, for \(\sigma_1 < 0 < 1 < \sigma_2\) there exists \(c \geq 0\) such that in the range \(\sigma_1 \leq \Re(s) \leq \sigma_2\) and \(|\Im(s)| \geq 1\) we have
\[
|h_{d_1, d_2}(s, A, a)| = O(|t| e^{-\pi |t|/2}), \tag{9}
\]
**Proof.** For the proof of (9), see [12, Lemma 12], for example.
3.3. Third step and completion of the proof of Theorem 1

According to (9), we can shift the line of integration \( \Re(s) = c > 1 \) in (8) leftward to the line \( \Re(s) = -1/2 \). We pick up residues at \( s = 1/2 \) and \( s = 0 \) if \( A = 0 \), and at \( s = 1, s = 1/2 \) and \( s = 0 \) if \( A = 1 \). Hence, for \( A = 0 \) (\( \Leftrightarrow \chi(-1) = +1 \)) we obtain

\[
|L(1, \psi)| \leq \text{Res}_{s=1/2}(h_{d_1,d_2}(s, 0, a)) + \text{Res}_{s=0}(h_{d_1,d_2}(s, 0, a)) + I_{d_1,d_2}(0, a),
\]

and for \( A = 1 \) (\( \Leftrightarrow \chi(-1) = -1 \)) we obtain

\[
|L(1, \psi)| \leq \text{Res}_{s=1}(h_{d_1,d_2}(s, 1, a)) + \text{Res}_{s=1/2}(h_{d_1,d_2}(s, 1, a)) + \text{Res}_{s=0}(h_{d_1,d_2}(s, 1, a)) + I_{d_1,d_2}(1, a),
\]

where

\[
I_{d_1,d_2}(A, a) = \frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} h_{d_1,d_2}(s, A, a) ds.
\]

In the following Lemmas 5 and 6 we compute these residues and in Lemma 7 we give an explicit upper bound for the error term \( I_{d_1,d_2}(A, a) \), thus proving Theorem 1.

**Lemma 5** The function \( s \mapsto h_{d_1,d_2}(s, A, a) \) has a double pole at \( s = (1 + A)/2 \) and we have

\[
\text{Res}_{s=\frac{1}{2}(A+1)}(h_{d_1,d_2}(s, A, a)) = \frac{\phi(d)}{2d} \left( \log(q^\prime/\pi) + 2\gamma + \frac{\Gamma'}{\Gamma} \left( \frac{1 + A}{2} \right) + 2 \sum_{p | d} \log p - 1 + 2^{\omega+1} \sqrt{a/d_1} - \log a \right).
\]

Hence, \( \text{Res}_{s=\frac{1}{2}(A+1)}(h_{d_1,d_2}(s, A, a)) \) is minimal for

\[
a = d_1/4^\omega,
\]

in which case we have

\[
\text{Res}_{s=\frac{1}{2}(A+1)}(h_{d_1,d_2}(s, A, d_1/4^\omega)) = \frac{1}{2} \left\{ \prod_{p \in S} \left( 1 - \frac{1}{p} \right) \left( \log q_X + \kappa_X + \omega \log 4 + 2 \sum_{p \in S} \log p \right) \right\}.
\]

**Proof.** To obtain equation (10) use \( \zeta(1 + \epsilon) = 1/\epsilon + \gamma + O(\epsilon) \). To deduce the last assertion, use \( (\Gamma'/\Gamma)(1) = -\gamma \) and \( (\Gamma'/\Gamma)(\frac{1}{2}) = -\gamma - \log 4 \).

**Lemma 6** 1.) If \( A = 1 \) then \( s = (1 + A)/2 = 1 \), \( s = A/2 = \frac{1}{2} \) and \( s = 0 \) are the only poles of \( h_{d_1,d_2}(s, A, a) \) in the half-plane \( \Re(s) > -1 \) and we have

\[
\text{Res}_{s=1/2}(h_{d_1,d_2}(s, 1, a)) = -\varepsilon_{d_2} \frac{\pi}{2\sqrt{q_X}} \frac{\phi(d_1)}{d_1}
\]

and

\[
\text{Res}_{s=0}(h_{d_1,d_2}(s, 1, a)) = \frac{\pi}{12d_1q_X} \left( \frac{2}{\sqrt{ad_1}} \mu(d_1) \phi(d_1) + a \right) \mu(d) \phi(d).
\]
2.) If $A = 0$ then $s = (1 + A)/2 = \frac{1}{2}$ and $s = A/2 = 0$ are the only poles of $h_{d_1,d_2}(s, A, a)$ in the half-plane $\Re(s) > -1$ and we have

$$\text{Res}_{s=0}(h_{d_1,d_2}(s, 0, d_1/4^w)) = \begin{cases} 
- \frac{1}{2\sqrt{q_X}}(\log q_X - \kappa_{\text{even}}) & \text{if } d_1 = d_2 = 1, \\
- \frac{1}{2\sqrt{q_X}} \frac{\phi(d_1)}{d_1} \left( \log(4\pi q_X/4^w) - \gamma + 2 \sum_{p|d_1} \frac{\log p}{p-1} \right) & \text{if } d_1 > 1, d_2 = 1, \\
- \frac{1}{\sqrt{q_X}} \frac{\phi(d_1)}{d_1} \log d_2 & \text{if } d_2 \text{ is prime,} \\
0 & \text{otherwise.}
\end{cases}$$

Proof. Recall that $\zeta(0) = -1/2$, $\zeta(-1) = -1/12$ and $\zeta'(0) = -\frac{1}{2} \log(2\pi)$; see [25, Theorem 3]. Hence, we obtain

$$\text{Res}_{s=0}(h_{d_1,d_2}(s, 0, a)) = \epsilon d \frac{a}{d_1 q_X} - \frac{1}{2\sqrt{q_X}} \frac{\phi(d_1)}{d_1} \times \begin{cases} 
\log(4\pi a q_X/d_1) - \gamma + 2 \sum_{p|d_1} \frac{\log p}{p-1} & \text{if } d_2 = 1, \\
2 \log d_2 & \text{if } d_2 \text{ is prime,} \\
0 & \text{otherwise.}
\end{cases}$$

The desired result follows.

**Lemma 7**

$$|I_{d_1,d_2}(A, a)| \leq \begin{cases} 
0.246 \frac{R_{d_1,d_2}(A, a)}{q_X} & \text{if } A = 0, \text{ that is, if } \chi(-1) = +1, \\
0.124 \frac{1}{\sqrt{q_X}} R_{d_1,d_2}(A, a) & \text{if } A = 1, \text{ that is, if } \chi(-1) = -1,
\end{cases}$$

where

$$R_{d_1,d_2}(A, a) = \left( \prod_{p|d} (p^{A+1} + 1) \right) \left( \frac{\prod_{p|d_1} \left( \frac{p^{A+1}(p-2)}{p^{A+1}+1} \right)}{a^{\frac{A+1}{2}} \sqrt{d_1}} \right) + a^{\frac{1}{2}} (A+2).$$

**Proof.** Recalling that

$$|\Gamma(-\frac{1}{2} + it)| = | - \frac{1}{2} + it |^{-1} \sqrt{\pi \cosh(\pi t)},$$

$$|\Gamma(1 - 2it)| = \sqrt{\frac{2\pi t}{\sinh(2\pi t)}} = \sqrt{\frac{\pi t}{\sinh(\pi t) \cosh(\pi t)}}.$$
and that \( \zeta(s) = \pi^{-1}(2\pi)^s \sin(\pi s/2)\Gamma(1-s)\zeta(1-s) \), we obtain that for \( s = -\frac{1}{2} + it \) we have

\[
\left| \left( \frac{q^\phi}{\pi} \right)^{s-\frac{1}{2}(A+1)} \frac{\Gamma(s)}{\Gamma\left( \frac{1 + A}{2} \right)} \zeta(2s - A) \right| = \begin{cases} 
\frac{1}{\pi q^\phi} \sqrt{\frac{\pi t}{\sinh(\pi t)}} |\zeta(2 - 2it)| & \text{if } A = 0, \\
\left| 1 - it \right| \sqrt{\frac{i \sinh(\pi t)}{\cosh^2(\pi t)}} |\zeta(3 - 2it)| & \text{if } A = 1.
\end{cases}
\]

Note that for \( \Re(s) = -\frac{1}{2} \) we have that \( |s - \frac{1}{2} A| = |\frac{1}{2}(A + 1) - it| \) and \( |s - \frac{1}{2}(A + 1)| = |\frac{1}{2}(A + 2) - it| \geq |\frac{1}{2}(A + 1) - it| \) and so we obtain

\[
|H_{d_1, d_2}(-\frac{1}{2} + it, A, a)| \leq \frac{\prod_{p \mid d_1} (p^{A+1} - 1)}{\frac{1}{2}(A + 1) - it} \left( \frac{\prod_{p \mid d_1} \left( \frac{p^{A+1} - 1}{p^{A+1} + 1} \right)}{a^\frac{1}{2}(A+1) \sqrt{d_1}} + a^\frac{1}{2}(A+2) \right)
\]

Hence we finally find

\[
\left| I_{d_1, d_2}(A, a) ds \right| \leq \begin{cases} 
\frac{I_{\text{even}}}{q^\chi} R_{d_1, d_2}(A, a) & \text{if } A = 0 \iff \chi(-1) = +1, \\
\frac{I_{\text{odd}}}{q^{3/2}} R_{d_1, d_2}(A, a) & \text{if } A = 1 \iff \chi(-1) = -1,
\end{cases}
\]

where

\[
I_{\text{even}} = \frac{2\zeta(2)}{\pi^2} \int_0^\infty \sqrt{\frac{\pi t}{(1 + 4t^2) \sinh(\pi t)}} = 0.245114 \ldots
\]

and

\[
I_{\text{odd}} = \frac{\zeta(3)}{\pi^{3/2}} \int_0^\infty \sqrt{\frac{i \sinh(\pi t)}{\cosh^2(\pi t)}} = 0.123314 \ldots,
\]

and the proof is complete.

**Remark 8** In the case that \( \chi \) is even \( s \mapsto h_{d_1, d_2}(s, A, a) = h_{d_1, d_2}(s, 0, a) \) is analytic in the half-plane \( \Re(s) < 0 \). Hence, by shifting the vertical line of integration \( \Re(s) = -1/2 \) leftwards to any vertical line of integration \( \Re(s) = -m \), we obtain that for any \( m > 0 \) we have \( I_{d_1, d_2}(0, a) = O(q^{-m}) \), that is, the error term \( o(1) \) in Theorem 1 is \( O(q^{-m}) \) for any \( m > 0 \). In fact, our previous proof of Theorem 1 in the case of even characters in [16] shows that this error term is \( O(q^{-1} e^{-c_0 s}) \) for some \( c_0 > 0 \) depending on \( S \) only.

**Remark 9** One of the nice features of our previous proof of Theorem 1 (in the case of even characters) given in [16] is that it can be easily generalized to the case of (unramified at all the infinite places) primitive characters on ray class groups of number fields; see [11, Theorem 3].
REMARK 10 Using a completely different method Ramaré has lately obtained a result similar to our Theorem 1, albeit with a slightly better constant $\kappa_\chi$, namely $\kappa_\chi = 0$ if $\chi(-1) = +1$ and $\kappa_\chi = 5 - 2 \log 6 = 1.416481 \ldots$ if $\chi(-1) = -1$; see [22, 23]. However, contrary to ours (see [12, Theorem 7]), it is not clear how to generalize his proof to the case of primitive characters on ray class groups of number fields.

4. Some applications of these bounds

4.1. Upper bounds for relative class numbers

COROLLARY 11 Let $q \equiv 5 \pmod 8$, $q \neq 5$, be a prime, let $\chi_q$ denote any one of the two conjugate odd quartic characters of conductor $q$ and let $h_q^-$ denote the relative class number of the imaginary cyclic quartic field $N_q$ of conductor $q$. Then

$$h_q^- \leq \frac{q}{40\pi^2}(\log q + 2 + \gamma - \log(\pi/16))^2,$$

which implies $h_q^- < q$ for $q \leq 6350867$. Moreover, if $\chi_q(3) \neq -1$ ($\Leftrightarrow 3(q-1)/4 \neq 1 \pmod q$), then

$$h_q^- \leq \frac{q}{100\pi^2}(\log q + 2 + \gamma - \log(\pi/192))^2,$$

which implies $h_q^- < q$ for $q \leq 5 \times 10^{10}$.

Proof. We have

$$h_q^- = \frac{q}{2\pi^2}|L(1, \chi_q)|^2.$$

Assume that $d_1 > 1$, $d_2 = 1$, $a = d_1/4^\omega$ and $A = 1$. Then

$$R_{d_1, d_2}(A, a) = R_{d_1, 1}(1, d_1/4^\omega) = \frac{4^\omega}{d_1^{3/2}} \left( \prod_{p|d_1} (p^2(p-2)+1) \right) + \frac{d_1^{3/2}}{8^\omega} \left( \prod_{p|d_1} (p^2+1) \right)$$

by (12), and

$$\left| \prod_{p|d_1} \left( 1 - \frac{\chi(p)}{p} \right) L(1, \chi) \right| \leq \frac{1}{2} \left[ \prod_{p|d_1} \left( 1 - \frac{1}{p} \right) \right]$$

$$\times \left( \log q + \kappa_{\text{odd}} + \omega \log 4 + \sum_{p|d_1} \frac{\log p}{p-1} \right) + R(q, d_1),$$

where

$$R(q, d_1) = -\frac{\pi}{2\sqrt{q}} \phi(d_1) + \frac{\pi}{12q} \left( 2^{\omega+1} \left( \frac{\phi(d_1)}{d_1} \right)^2 + \frac{\mu(d_1)}{4^\omega} \phi(d_1) \right) + \frac{0.124}{q^{3/2}} R_{d_1, 1}(1, d_1/4^\omega).$$

Now, choosing $d_1 = 2$ for which $\omega = 1$, noticing that

$$R(q, 2) = -\frac{\pi}{4\sqrt{q}} + \frac{3\pi}{48q} + \frac{0.124}{q^{3/2}} \sqrt{q} \leq 0$$
for $q > 5$ and noticing that $p \equiv 5 \pmod{8}$ yields $\chi_p(2) \in \{\pm i\}$, we obtain the desired first result.
In the same way, choosing $d_1 = 6$ for which $\omega = 2$, noticing that
\[
R(q, 6) = -\frac{\pi}{6\sqrt{q}} + \frac{73\pi}{864q} + \frac{0.124}{q^{3/2}} - \frac{144}{\sqrt{6}} \leq 0
\]
for $q \geq 13$ and noticing that $p \equiv 5 \pmod{8}$ yields $\chi_p(2) \in \{\pm i\}$, we obtain the desired second result.

Remark 12 Using Corollary 11 to alleviate the amount of required relative class number computation, M. Jacobson and the author are now trying to solve the open problem hinted at in [10]: determine the least (or at least one) prime $q \equiv 5 \pmod{8}$ for which $h_K - q > q$. Indeed, according to this corollary, in the range $q < 5 \times 10^{10}$ we may assume that $\chi_q(3) = +1$, which amounts to eliminating three-quarters of the primes $q$ in this range.

4.2. Lower bounds for relative class numbers
The second application of Theorem 1, and the main reason why we have long strived to obtain this result, is to obtain good lower bounds for relative class numbers of imaginary abelian number fields. Using Theorem 1 with $S = \emptyset$, we obtain the following.

Theorem 13 (See [15]) Let $K$ be an imaginary abelian number field of degree $2n > 2$ and root discriminant $\rho_K = d_K^{1/(2n)}$. Set $v_n = (n/(n - 1))^{n-1} \in [2, e)$. Assume that $K$ contains no imaginary quadratic subfield. Then
\[
h_K^- \geq \frac{1}{nv_n e} \left(\frac{\sqrt{\rho_K}}{\pi (\log \rho_K + \kappa_1)}\right)^n,
\]
where $\kappa_1 = 2 + \gamma - \log(4\pi) = 0.046\ldots$. In particular, for each entry $2n$ in Table 1, we have $h_K^- > 1$ as soon as $\rho_K \geq \rho_{2n}$.

However, using Theorem 1 with $S = \{2\}$, we obtain better results.

Theorem 14 (See [15]) Let $K$ be an imaginary abelian number field of degree $2n > 2$ and root discriminant $\rho_K = d_K^{1/(2n)}$. Set $v_n = (n/(n - 1))^{n-1} \in [2, e)$. Assume that $K$ contains no imaginary quadratic subfield. Let $k$ denote the maximal totally real subfield of $K$. Let $e$, $f$ and $g = n/e f$ denote the index of ramification of the prime 2 in $k$, the inertia degree of 2 in $k$ and the number of prime ideals of $k$ above 2, respectively. Set
\[
\kappa_{n,f,g} = \frac{(g - 1)k_1 + (n - fg)k_2 + (fg - g)k_3}{n} \leq \kappa_3 \leq 3,
\]
Table 2 (compare with Table 1)

<table>
<thead>
<tr>
<th>2n</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_{2n} )</td>
<td>4233</td>
<td>2344</td>
<td>1530</td>
<td>1124</td>
<td>513</td>
<td>303</td>
<td>205</td>
<td>180</td>
</tr>
</tbody>
</table>

where \( \kappa_1 = 2 + \gamma - \log(4\pi) = 0.046 \ldots \), \( \kappa_2 = 2 + \gamma - \log \pi = 1.432 \ldots \) and \( \kappa_3 = 2 + \gamma - \log(\pi/4) = 2.818 \ldots \). and set \( C_{n,f,g} = \frac{2(1+2^{-f})^{-g/n}}{\{4/3, 2\}} \). We have

\[
\hat{h}_K^r \geq \frac{1}{\nu_{n,e}} \left( \frac{C_{n,f,g} \sqrt{\rho_K}}{\pi (\log \rho_K + \kappa_{n,f,g})} \right)^n.
\]

In particular, for each entry \( 2n \) in Table 2, we have \( \hat{h}_K^r > 1 \) as soon as \( \rho_K \geq \rho_{2n} \).

References


