On the mean value of $|L(1, \chi)|^2$ for odd primitive Dirichlet characters

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Abstract: Let $f > 1$ be given. Whereas a simple formula for the mean value of $|L(1, \chi)|^2$ for odd Dirichlet characters modulo $f$ is known, we explain why there is no hope of ever finding a simple formula for the mean value of $|L(1, \chi)|^2$ for primitive odd Dirichlet characters modulo $f$.

Key words: Dirichlet characters; $L$-functions.

1. Introduction. Let $f > 1$ be given. Let $X^-_f$ and $P^-_f$ denote the set of all the odd Dirichlet characters modulo $f$ and of all the primitive odd Dirichlet characters modulo $f$, respectively (see [1, Section 6.8] for the definition of Dirichlet characters, see [1, Section 8.7] for the definition of primitivity and recall that an odd Dirichlet character is a Dirichlet character $\chi$ which satisfies $\chi(-1) = -1$). Whenever $d > 0$ divides $f$ we let $\tilde{\psi} \in X^-_f$ denote the character induced by $\psi \in X^-_f$ (see [1, Chapter II]). Notice that $\chi \in X^-_f$ is not primitive if and only if there exist a prime $p$ dividing $f$ and $\psi \in X^-_{f/d}$ such that $\chi = \tilde{\psi}$ is induced by $\psi$, for any complex $s$ we get (use the inclusion-exclusion principle):

\begin{equation}
(1) \sum_{\chi \in P^-_f} |L(1, \chi)|^2 = \sum_{d|f} \mu(d) \sum_{\psi \in X^-_{f/d}} |L(s, \psi)|^2
\end{equation}

where $\mu$ and $\phi$ denote the Möbius and Euler totient functions (see [1, Chapter 2]) and $L(s, \psi)$ denotes the Dirichlet $L$-functions associated with $\chi$ (see [1, Chapter 11]). Notice that $\#X^-_{15} = \#X^-_{12} = 0$ and $\#X^-_{15} = \phi(f)/2$ whenever $f > 2$. We proved:

Theorem 1 (See [2], [3]). It holds

\begin{equation}
(2) \sum_{\chi \in X^-_f} |L(1, \chi)|^2 = \frac{\pi^2 \phi(f)}{12} \prod_{p|f} \left(1 - \frac{1}{p^2}\right) - \frac{\pi^2 \phi^2(f)}{4f^2}.
\end{equation}

We deduce:

Corollary 2 (See [5]). If $f > 1$ is square-full then it holds

\begin{equation}
(3) \sum_{\chi \in P^-_f} |L(1, \chi)|^2 = \frac{\pi^2 \phi^2(f)}{12f} \prod_{p|f} \left(1 - \frac{1}{p^2}\right).\n\end{equation}

Proof. If $d > 0$ is square-free and divides $f$ and $\psi \in X^-_{f/d}$ then $L(s, \tilde{\psi}) = L(s, \psi)$ (use the Euler products of both these terms (see [1, Section 11.5])). Hence, (1) yields

\begin{equation}
\sum_{\chi \in P^-_f} |L(1, \chi)|^2 = \sum_{d|f} \mu(d) \sum_{\psi \in X^-_{f/d}} |L(1, \psi)|^2
\end{equation}

and the desired result follows from Theorem 1.

It was conjectured (not in contradiction with (3)) that:

Conjecture 3 (See [MR 91j:11068] and [5]).

For any rational integer $f > 1$ we have:

\begin{equation}
(4) \sum_{\chi \in P^-_f} |L(1, \chi)|^2 = \frac{\pi^2 \phi(f) J(f)}{12 f} \prod_{p|f} \left(1 + \frac{1}{p} + 2\mu(f)\right)
\end{equation}

where $J(f) = \sum_{d|f} \mu(d) \phi(f/d)$ is the number of primitive characters modulo $f$.

This conjecture is false. Indeed, if $f = 15$ then $P^-_{15}$ is reduced to the character $n \mapsto \chi(n) = (n/15)$ (Jacobi’s symbol) for which $L(1, \chi) = 2\pi/\sqrt{15}$ (for the class number of the imaginary quadratic field $Q(\sqrt{-15})$ is equal to 2), the left hand side of (4) is equal to $4\pi^2/15$ while the right hand side of (4) is equal to $52\pi^2/15^2$. Not only is this conjecture false, but its falsity does not trivially come from any misprint in (4) for according to such a conjecture, $S^{-}(pq)$ defined below should be polynomial in $p$ and $q$ whenever $p$ and $q$ range over the positive rational primes, whereas we will prove:

Theorem 4. Let $p$ and $q$ denote distinct positive primes. Even though

\begin{equation}
S^{-}(pq) \overset{\text{def}}{=} \frac{(pq)^3}{\pi^2} \sum_{\chi \in P^-_{qr}} |L(1, \chi)|^2
\end{equation}

is a polynomial in $p$ and $q$, we have

\begin{equation}
\sum_{\chi \in P^-_f} |L(1, \chi)|^2 = \frac{\pi^2 \phi(f)}{12f} \prod_{p|f} \left(1 - \frac{1}{p^2}\right) \neq \frac{\pi^2 \phi^2(f)}{4f^2}.
\end{equation}

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is always a positive rational number, there does not exist any polynomial \( f(X, Y) \) such that for all pairs \((p, q)\) we have \( S^-(pq) = f(p, q)\).

Therefore, it seems that there is no hope of ever finding a neat explicit formula for the sums \( \sum_{X \in P_f} |L(1, \chi)|^2 \) which would be valid for any \( f > 1 \).


Theorem 5. Whenever \( d \geq 1 \) divides \( f > 1 \) we set

\[
T_\pm(f, d) \stackrel{\text{def}}{=} \sum_{1 \leq a \leq \frac{f}{d}} \sum_{b \leq a \atop \text{(mod } f/d\text{)}} \* \ab,
\]

(where \( \* \) stands for a summation ranging over indices relatively prime to \( f \)). We have

\[
S^-(f) \stackrel{\text{def}}{=} \frac{f^2}{\pi^2} \sum_{\chi \in P_f} |L(1, \chi)|^2 = \sum_{d|f} \mu(d) \phi(f/d) T_+(f, d),
\]

and \( S^-(f) \) is always a positive rational integer. Notice that if \( p \) and \( q \) denote distinct positive primes then (6) yields

\[
S^-(pq) = \phi(pq)T_+(pq, 1) - \phi(q)T_+(pq, p) - \phi(p)T_+(pq, q) + T_+(pq, pq).
\]

Proof.

\[
S^-(f) = f^2 \sum_{\chi \in P_f} |L(0, \chi)|^2
\]

(use the functional equation satisfied by \( L(s, \chi) \))

(see [1, Section 12.10])

\[
f^2 \sum_{d|f} \mu(d) \sum_{\psi \in X_f/d} |L(0, \psi)|^2
\]

(use 1 for \( s = 0 \))

\[
= \sum_{d|f} \mu(d) \psi \in X_f/d \sum_{a=1}^{\frac{f}{d}} \psi(a)^2
\]

(use [1, Section 12.13])

\[
= \frac{1}{2} \sum_{d|f} \mu(d) \phi(f/d) (T_+(f, d) - T_-(f, d))
\]

(for \( T_+(f, f) = T_-(f, f) \) and \( T_+(f, f/2) = T_-(f, f/2) \) whenever \( f \) is even)

where we have used

\[
\sum_{\psi \in X_f/d} \psi(a) \psi(b) = \sum_{\psi \in X_f/d} \psi(a) \overline{\psi(b)}
\]

\[
= \begin{cases} 
\frac{\phi(f/d)}{2} & \text{if } b \equiv a \pmod{f/d} \\
-\frac{\phi(f/d)}{2} & \text{if } b \equiv -a \pmod{f/d} \\
0 & \text{otherwise}
\end{cases}
\]

(provided that \( \gcd(a, f) = \gcd(b, f) = 1 \)). Now, since the canonical morphism \( s : (\mathbb{Z}/f\mathbb{Z})^* \rightarrow (\mathbb{Z}/(f/d)\mathbb{Z})^* \) is surjective, for any given \( a \) relatively prime to \( f \) we have

\[
\sum_{1 \leq a \leq f \atop b \leq a \atop (\text{mod } f/d)} a = \# \ker s \cdot \sum_{1 \leq a \leq f} a = f \phi^2(f) = 2 \phi(f/d)
\]

and

\[
T_-(f, d) = \sum_{1 \leq a \leq f} \sum_{b \leq a \atop (\text{mod } f/d)} a = (f - b)
\]

\[
= f^2 \phi^2(f) = T_+(f, d),
\]

which provides us with the desired result in using \( \sum_{d|f} \mu(d) = 0 \).

\[ \Box \]

Lemma 6. Whenever \( q = np + 1 \) and \( p \) are prime, it holds \( S^-(p, q) = g(p, q) \) where \( g(X, Y) \stackrel{\text{def}}{=} X^2Y^2(X - 1)^2(Y - 1)^2/12 - 2X Y^2(X - 1)(Y - 1)^2/6 \).

Proof. Using

\[
T_+(f, 1) = \sum_{1 \leq a \leq f} a^2 = \frac{1}{3} f^2 \phi^2(f) + \frac{1}{6} f \prod_{p|f} (1 - p)
\]

we obtain \( T_+(pq, 1) = p^2q^2(p - 1)(q - 1)/3 + pq(p - 1)(q - 1)/6 \), and using

\[
T_+(f, f) = \left( \sum_{1 \leq a \leq f} a \right)^2 = \frac{1}{4} f^2 \phi^2(f)
\]

we obtain \( T_+(pq, pq) = p^2q^2(p - 1)^2(q - 1)^2/4 \). Now, writing \( a = A + qA' \) and \( b = A + qB' \) with \( 1 \leq A \leq q, 0 \leq A' \leq p - 1 \) and \( 0 \leq B' \leq p - 1 \), we get

\[
T_+(pq, p) = \sum_{A=1}^{q-1} \left( \sum_{1 \leq \# \gcd(A + qA', p) = 1}^{p-1} A + qA' \right)^2.
\]
Then, we notice that $p$ divides $A + qA'$ if and only if $A' \equiv -A \pmod{p}$, and we write $A = pQ + R$ with $1 \leq R \leq p$ and $Q \geq 0$. We get

$$T_+(pq,p) = \sum_{Q=0}^{n-1} \sum_{R=1}^{p} \left( \sum_{A^r \equiv -R \pmod{p}}^{-1} pQ + R + qA' \right)^2$$

$$= \sum_{Q=0}^{n-1} \sum_{R=1}^{p} \left( \sum_{A^r \equiv 0 \pmod{p}}^{-1} pQ + R + qA' \right)^2$$

$$= \sum_{Q=0}^{n-1} \sum_{R=1}^{p} \left( p(p-1)Q + (p + q - 1)R + \frac{p - 3}{2}pq \right)^2$$

In the same way,

$$T_+(pq,q) = \sum_{A=1}^{p-1} \left( \sum_{q \equiv 0 (A + pA', q) = 1}^{-1} A + pA' \right)^2$$

and $q$ divides $A + pA'$ if and only if $q$ divides $nA + npA'$, hence if and only if $A' \equiv nA \pmod{q}$. Since $0 \leq nA \leq n(p-1) < q$, then $q$ divides $A + pA'$ if and only if $A' = nA$, which yields $A + pA' = A + pmA = qA$. Hence,

$$T_+(pq,q) = \sum_{A=1}^{p-1} \left( -qA + \sum_{q \equiv 0}^{-1} (A + pA') \right)^2$$

$$= \sum_{A=1}^{p-1} \left( pq \frac{q-1}{2} \right)^2 = p^2q^2(p-1)(q-1)/4.$$

The Lemma follows from (7) and these four previous formulæ.

Now, we are in a position to prove the last assertion of Theorem 4: Lemma 6 would give $f(X,Y) = g(X,Y)$ (according to Dirichlet’s Theorem, for any prime $p$ there are infinitely many primes $q$ of the form $q = np + 1$, $n \geq 1$. Hence, for any prime $p$ we would have $f(p,Y) = g(p,Y)$. Now, since there are infinitely many primes $p > 2$ we would then obtain $f(X,Y) = g(X,Y))$. But this identity cannot hold for while $S^n(pq) = S^n(qp)$, this expression $g(p,q)$ is not symmetrical in $p$ and $q$.

References
